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On the smooth locus in flat linear degenerations of flag varieties

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Abstract We use some moment graph techniques, recently introduced by Lanini and Pütz, to provide a description of T-fixed points in the smooth locus of flat linear degenerations of flag varieties, generalizing a result proved by Cerulli Irelli, Feigin and Reineke for the Feigin degeneration. Moreover, we propose a different combinatorial criterion linking the smoothness at a fixed point to transitive tournaments.

1. Introduction

The *Complete Flag Variety* $\mathcal{F}l(\mathbb{C}^{n+1})$ is a fundamental and extensively studied object, playing a relevant role in many different areas of mathematics. It is a smooth irreducible algebraic variety of dimension $\binom{n+1}{2}$, classically realized as the variety of n-tuples of nested subspaces $(V_1, \dots, V_n)$ of $\mathbb{C}^{n+1}$ such that $\dim V_i = i$.

From a slightly different point of view, each subspace V_i of an n-tuple $(V_1, \dots, V_n)$ can be thought as a subspace of its own copy W_i of $\mathbb{C}^{n+1}$ and consequently it is possible to identify the complete flag variety with the closed subvariety of points $(V_1, \dots, V_n)$ in the product of Grassmann varieties $\mathrm{Gr}_1(W_1) \times \cdots \times \mathrm{Gr}_n(W_n)$, that satisfy the relations $\mathrm{id}_i V_i \subset V_{i+1}$ for every i, where id_i denotes the identity map from W_i to W_{i+1}.

It is possible to generalize this construction considering any $n-1$-tuple $\mathbf{f} := (f_1, \dots, f_{n-1})$ of linear endomorphisms of $\mathbb{C}^{n+1}$ and prescribing that $f_i V_i \subset V_{i+1}$.

Definition. *For a fixed family* $\mathbf{f} = (f_1, \dots, f_{n-1})$ *of linear endomorphisms of* $\mathbb{C}^{n+1}$*, the* Linear Degeneration of the Flag Variety associated to the family of maps $\mathbf{f}$ *is the algebraic variety of* n*-tuples of subspaces* $(V_1, \dots, V_n)$ *such that* $\dim V_i = i$ *and* $f_i V_i \subset V_{i+1}$. *We denote it by* $\mathcal{F}l^{\mathbf{f}}(\mathbb{C}^{n+1})$.

We also refer to $\mathcal{F}l^{\mathbf{f}}(\mathbb{C}^{n+1})$ as an $\mathbf{f}$-degeneration of the complete flag variety. The ubiquitous appearing of linear degenerations in many different context of mathematics as representation theory, algebraic geometry and combinatorics, motivated a growing interest for these objects, that are widely investigated in a great amount recent papers (see [8], for an introductory survey about linear degenerations and their role in mathematics, and [5], for a complete and more technical exposition on this topic).

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One of the crucial points in studying linear degenerations is that they appear as a special case of Quiver Grassmannians, for certain representations of the equioriented quiver of type A_n. More precisely, to the family of linear endomorphisms $\mathbf{f} = (f_1, \dots, f_{n-1})$ it is possible to associate an A_n-representation considering

$$M_{\mathbf{f}} := \quad \mathbb{C}^{n+1} \xrightarrow{f_1} \mathbb{C}^{n+1} \xrightarrow{f_2} \dots \xrightarrow{f_{n-2}} \mathbb{C}^{n+1} \xrightarrow{f_{n-1}} \mathbb{C}^{n+1}.$$

An n-tuple of subspaces $(V_1, \dots, V_n)$ with $\dim V_i = i$ and such that $f_i V_i \subset V_{i+1}$ corresponds to a subrepresentation of $M_{\mathbf{f}}$ with dimension vector $\mathbf{e} = (1, \dots, n)$ and consequently the linear degeneration associated to $\mathbf{f}$ can be identified with the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M_{\mathbf{f}})$. This interpretation of linear degenerations of flag variety from a "Quiver Representation Theory"- point of view has the merit of highlighting how relevant geometric properties of these objects can be recovered using a combination of both combinatorial tools and results of algebraic geometry.

Using their realization as quiver Grassmannians, it is possible to endow linear degenerations with suitable torus actions, allowing very often to obtain deep results about their topology and explicit computations concerning their homological invariants.

As an example, using these techniques in [6] it is proved that the Euler characteristics of the Feigin degenerations –namely, degenerations of $\mathcal{F}l(\mathbb{C}^{n+1})$ such that each f_i has rank n and the kernels of $f_1, \dots, f_{n-1}$ are in direct sum– are equal to the median Genocchi numbers.

Moreover, in [7] a decomposition of Feigin degenerations into disjoint affine cells, homogeneous with respect to the action of a suitable algebraic group, is achieved and smoothness of points in a cell can be verified looking at certain sequences of integers (*CI-F-R Condition*, Definition 6.1). The merit of this approach lies in the fact that the CI-F-R condition is purely combinatorial and can be verified algorithmically. In addition, in [7] it is shown that these integer sequences satisfy certain recursive identities. As a consequence, the authors prove that the Euler characteristic of the smooth locus in Feigin degenerations is given by the Schröder numbers, and they derive explicit closed formulae for the Poincaré polynomial of this locus. Aiming to study the smooth loci of a broader class of linear degenerations, it is therefore natural and interesting to seek a suitable generalization of these combinatorial tools, as such a generalization may provide deeper insight into the geometry of these varieties.

Recently, actions of some bigger tori on linear degenerations were introduced and studied in [13, 14], as a special case of torus actions on Cyclic Quiver Grassmannians.

As a consequence of constructions contained in [13], every $\mathbf{f}$-degeneration $\mathcal{F}l^{\mathbf{f}}(\mathbb{C}^{n+1})$ can be equipped with the action of a suitable torus T, such that the T-action has a finite number of fixed points and a finite number of 1-dimensional orbits. It also implies that $\mathcal{F}l^{\mathbf{f}}(\mathbb{C}^{n+1})$ has a Białynicki-Birula decomposition into rational cells, induced by a generic T-cocharacter. The degeneration $\mathcal{F}l^{\mathbf{f}}(\mathbb{C}^{n+1})$ is then a *GKM Variety* (Definition 4.1) and its T-equivariant cohomology can be determined using the *Moment Graph* (Definition 4.6). Moreover the moment graph encodes some relevant information about the dimension of tangent space at T-fixed points.

An explicit description of the moment graph in purely combinatorial terms is provided in [13, 14]. More precisely, each fixed point for the T action on $\mathcal{F}l^{\mathbf{f}}(\mathbb{C}^{n+1})$ can be associated to a family $S = (S_1, \dots S_n)$ of subsets of $\{1, \dots, n+1\}$, called *Admissible Sequence* (Definition 4.14) and edges in the moment graph correspond to combinatorial operations, called *Mutations* (Definition 4.20), on certain quivers associated to T- fixed points.

Inspired by one of the many open questions proposed by Lanini and Pütz [13, Section 8, Question 8E], the main purpose of this paper is to use GKM theory to provide combinatorial smoothness criteria for T-fixed points in flat linear degenerations, i.e. linear degenerations that are equidimensional varieties of dimension $\binom{n+1}{2}$. Some of these criteria provide a natural generalization of the CI-F-R Condition —which describes the smooth locus of the Feigin degeneration, a special case of flat irreducible degeneration— to all flat linear degenerations. Moreover, our criteria also highlights a deeper link between smoothness and combinatorics of mutations.

As a byproduct of our results, we obtain a new description of the smooth locus in PBW linear degenerations (Definition 6.14) as union of orbits by the action of a suitable algebraic group.

Organization of the Paper: In Section 2 linear degenerations of the flag variety are presented in a more geometric flavour as fibers of a suitable $GL_{n+1}(\mathbb{C})^n$-equivariant map from the *Universal Degenerate Flag Variety* to the total space $\mathrm{Hom}(\mathbb{C}^{n+1}, \mathbb{C}^{n+1})^{n-1}$ of representations of the equioriented quiver A_n. In this geometrical setting, *Flat Degenerations* (Definition 2.1) can be introduced in a very natural way and they can be classified completely up to $GL_{n+1}(\mathbb{C})^n$-action.

Section 3 contains all the relevant information about quiver representations that we are going to use to prove our results. In particular we focus on the construction of *Coefficient Quiver* $Q(M)$ (Definition 3.7) associated to a representation M of the equioriented quiver of type A_n.

In Section 4 we summarize the results proved in [13, 14] about the structure of GKM varieties of linear degenerations. In particular we recall how the structure of the moment graph can be described using mutations between *Successor Closed Subquivers* (Definition 4.12) of the coefficient quiver $Q(M_{\mathbf{f}})$.

In Section 5 we prove a formula for the dimension of tangent space at a fixed point p_S, that involves certain combinatorial properties of the admissible sequence $S = (S_1, \dots S_n)$. More precisely, for each fixed point p_S we define a set $\mathrm{Sing}(p_S)$, that encodes in a combinatorial way certain homological information about indecomposables appearing in p_S, when identified with a subrepresentation of $M_{\mathbf{f}}$. In particular we prove that $|\mathrm{Sing}(p_S)|$ equals the dimension of $\mathrm{Ext}^1(p_S, M_{\mathbf{f}}/p_S)$ and we obtain that

$$\dim T_{p_S} \mathcal{F}l^{\mathbf{f}}(\mathbb{C}^{n+1}) = \frac{n(n+1)}{2} + |\mathrm{Sing}(p_S)|$$

as a consequence of Euler Formula.

Section 6 is devoted to presenting our smoothness criteria for T-fixed points in flat degenerations. We identify a combinatorial property of admissible sequences, the *Generalized CI-F-R Condition* (Definition 6.1), that generalizes the smoothness condition provided in [7] for Feigin degeneration and has again the merit of being algorithmically checkable. Moreover, to each fixed point p_S we attach an oriented graph, the *Oriented Mutation Graph* $\widetilde{G}_S$ (Definition 6.8). This graph encodes relevant information about vertices adjacent to p_S in the moment graph. In the case of classical flag variety, where the cell decomposition corresponds to the Bruhat decomposition and every p_S corresponds to an element of the symmetric group, all the resulting oriented mutation graphs are acyclic orientations of a complete graph, and they encode information about the inversion sets of the permutations associated with the fixed points. However, this phenomenon does not occur in general –even if the linear degeneration under consideration is a Schubert variety– suggesting that the combinatorics of mutation graphs is very rich and has some deeper links with the geometric properties of the corresponding cells. These connections with the geometry of flat linear degenerations are made clearer in our main result, where we show that smoothness at p_S can be directly checked looking at the objects we introduce:

Theorem (Smoothness Criteria). *Let p_S be a T-fixed point in a flat linear degeneration of $\mathcal{F}l(\mathbb{C}^{n+1})$, associated to an admissible sequence S. The following conditions are equivalent:*

(1) *The point p_S is smooth;*
(2) *The set $\mathrm{Sing}(p_S)$ is empty;*
(3) *The admissible sequence S has the Generalized CI-F-R Condition;*
(4) *The Oriented Mutation graph $\widetilde{G}_S$ is a transitive tournament, i.e. an acyclic orientation of a complete graph;*
(5) *The unoriented graph underlying $\widetilde{G}_S$ is the complete graph over $n+1$ vertices.*

These criteria do not hold for linear degenerations that are not flat, and some counterexamples are provided at the end of Section 6. Furthermore, we discuss how our criteria can be used to describe the smooth locus in PBW degenerations. Finally, Section 7 contains the proofs of our results.

2. A geometric picture

In this section we recall some well known geometrical properties of linear degenerations. We refer to [5] for a complete discussion about this subject. Consider the two algebraic varieties

$$U := \mathrm{Hom}(\mathbb{C}^{n+1}, \mathbb{C}^{n+1})^{n-1} \qquad Z := \mathrm{Gr}_1(\mathbb{C}^{n+1}) \times \cdots \times \mathrm{Gr}_n(\mathbb{C}^{n+1}).$$

The set

$$Y = \{(\mathbf{f}, \mathbf{V}) \mid \mathbf{f} = (f_1, \ldots, f_{n-1}) \in U,\, \mathbf{V} = (V_1, \ldots, V_n) \in Z,\, f_i V_i \subset V_{i+1}\}$$

is a smooth closed subvariety of $U \times Z$ and is known as the *Universal Linear Degeneration of the Flag Variety.* The group $G := \mathrm{GL}_{n+1}(\mathbb{C})^n$ acts on U by the rule

$$(g_1, \ldots, g_n) \cdot (f_1, \ldots, f_{n-1}) := (g_2 f_1 g_1^{-1}, \ldots, g_n f_{n-1} g_{n-1}^{-1})$$

and on Z componentwise, by the classical matrix action on $\mathbb{C}^{n+1}$. So G acts on the product variety $U \times Z$ too, and the subvariety Y can be endowed of a structure of G-variety in a way that the two projections $\pi_1 : Y \to U$ and $\pi_2 : Y \to Z$ result to be G-equivariant morphisms.

The space U can be thought as a parameter space for Y: to any $n-1$-tuple of morphisms $\mathbf{f} = (f_1, \ldots, f_{n-1})$, one associates a closed subvariety $\pi_1^{-1}(\mathbf{f})$ of Y. For a fixed $\mathbf{f}$, this variety is isomorphic to the degeneration $\mathcal{F}l^{\mathbf{f}}(\mathbb{C}^{n+1})$. Moreover, the G-equivariance of π_1 implies that the isomorphism class of each degeneration is uniquely determined by the sequence $\{r_{i,j}(\mathbf{f})\}$ of ranks of composition maps $f_i \circ \cdots \circ f_j$.

Geometric properties of linear degenerations of the flag variety can be checked directly by looking at properties of π_1, extensively investigated in [5]. In particular the authors prove that there exist two maximal subsets $U_{irr,\,flat} \subset U_{flat} \subset U$ such that

- the restriction of π_1 to $\pi_1^{-1}(U_{flat})$ is a flat morphism of algebraic varieties;
- the fiber $\pi_1^{-1}(\mathbf{f})$ is an irreducible algebraic variety for every $\mathbf{f} \in U_{irr,\,flat}$

The complete flag variety $\mathcal{F}l(\mathbb{C}^{n+1})$ appears in $\pi_1^{-1}(U_{flat})$ as the fiber of the element $\mathbf{id} = (\mathrm{id}, \ldots, \mathrm{id})$. Flatness then implies that for every $\mathbf{f} \in U_{flat}$ the irreducible components of $\pi_1^{-1}(\mathbf{f})$ are equidimensional of dimension $n(n+1)/2$. Moreover, in [5] it is proved that this property characterizes exactly the linear degenerations appearing as fibers over U_{flat}.

Definition 2.1. *An $\mathbf{f}$-degeneration $\mathcal{F}l^{\mathbf{f}}(\mathbb{C}^{n+1})$, where $\mathbf{f} \in U_{flat}$ (resp. $\mathbf{f} \in U_{irr,flat}$), is called a* Flat *(resp.* Flat Irreducible*)* Degeneration *of the Flag Variety.*

Sets $U_{irr,\,flat}$ and U_{flat} are completely characterized in [5] in terms of rank sequences $\{r_{i,j}(\mathbf{f})\}$.

2.1. Classification of flat degenerations. Let us fix a basis $B = \{v_1, \dots, v_{n+1}\}$ of $\mathbb{C}^{n+1}$ and $R \subset \{1, \dots, n+1\}$, the projection operator $\pi_R : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ (with respect to B) is the linear operator on $\mathbb{C}^{n+1}$ defined by the rule:

$$\pi_R(v_i) = \begin{cases} 0 & \text{if } i \in R, \\ v_i & \text{otherwise} \end{cases}$$

For any family $\mathbf{R} = (R_1, \dots, R_{n-1})$ of subsets of $\{1, \dots, n+1\}$ we define the degenerate flag variety $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$ associated to $\mathbf{R}$ as the $\mathbf{f}$-degeneration of $\mathcal{F}l(\mathbb{C}^{n+1})$ obtained considering the family of endomorphisms $\mathbf{f} = (f_1, \dots, f_{n-1})$ such that $f_i = \pi_{R_i}$ for every i.

Each G-orbit in Y has a representative of the form $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$, so in the remaining of the paper we reduce our analysis to studying these special degenerations.

Using elementary linear algebra, orbit representatives for flat degeneration are described in [5].

Theorem 2.2 (cf. [5, Remark 2]). *Let $\mathbf{R} = (R_1, \dots, R_n)$ be a family of subsets of $\{1, \dots, n+1\}$. The degeneration $\mathcal{F}l^{\mathbf{f}}(\mathbb{C}^{n+1})$ is flat if and only if its orbit has a representative of the form $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$, where $\mathbf{R}$ satisfies the following conditions for all $h \in \{1, \dots, n\}$:*

(1) $|R_h| \leqslant 2$;

(2) $|R_h \cup R_{h+1}| \leqslant 3$

Moreover, $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$ is a flat irreducible degeneration if and only if $|R_h| \leqslant 1$ for all h.

Example 2.3 (The Feigin Degeneration). An $\mathbf{f}$-degeneration such that each f_i has rank n and such that the kernels of $f_1, \dots, f_{n-1}$ are in direct sum is a *Feigin Degeneration*. Up to base change every Feigin degeneration is of the form $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$, where $\mathbf{R}$ is the family of subsets $(\{2\}, \dots, \{n-1\})$. It is the toy model for our results. We summarize here some of its remarkable properties (see [5, 6, 7] for some more complete and precise statements):

- Feigin Degeneration is a singular, normal, irreducible, local complete intersection, projective variety of dimension $n(n+1)/2$;
- It can be endowed with a structure of GKM-variety for the action of a suitable algebraic torus T, i.e. the T action has a finite number of fixed points and of 1-dimensional orbits;
- The torus action induces a cellular decomposition of the Feigin Degeneration into a finite number of disjoint cells. Each cell contains exactly one fixed point;
- There exists an action of an algebraic group $\mathfrak{A}$ on the Feigin degeneration for which each cell is precisely the orbit of the corresponding T-fixed point under the $\mathfrak{A}$-action.

Because of cellular decomposition of Feigin degeneration into orbits of fixed points, it is natural to ask for a criterion to determine if a fixed point is smooth. Such a criterion is proved in [7]. We will recall it in Section 6, as a starting point for our more general results.

3. Quiver representations

In this section we summarize some results about representations of a generic quiver Q. For a complete reference on this subject we refer to [1, 12]. Moreover we recall how representations of quivers of Type A are related to linear degenerations of flag varieties. Let Q be a finite quiver with set of vertices Q_0 and set of oriented edges Q_1. If $\alpha \in Q_1$, we denote by $s(\alpha)$ and $t(\alpha)$ the source and the target of α, respectively.

Definition 3.1. *A representation (M, F) of a quiver $Q = (Q_0, Q_1)$ with base field $\mathbb{C}$ is the datum of:*

- *a family of finite dimensional complex vector spaces $M = (M_i)_{i \in Q_0}$,*
- *a sequence of maps $F = (f_\alpha)_{\alpha \in Q_1}$ such that $f_\alpha \in \mathrm{Hom}_{\mathbb{C}}(M_{s(\alpha)}, M_{t(\alpha)})$.*

Often we omit the family F in the notation if the context is clear. To each representation M is attached a dimension vector $\mathbf{d} = (d_i)_{i \in Q_0}$ such that $d_i = \dim_{\mathbb{C}} M_i$. A subrepresentation N of M is a collection of subspaces $N_i \subset M_i,\ i \in Q_0$ compatible with the maps f_α, i.e. $f_\alpha(N_{s(\alpha)}) \subseteq N_{t(\alpha)}$.

Definition 3.2. *Let M be a representation of a quiver Q. Let $\mathbf{e} = (e_i)_{i \in Q_0}$ be a vector of non negative integers. The Quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(Q, M)$ is the algebraic variety of subrepresentations of M with dimension vector $\mathbf{e}$.*

Example 3.3. Consider the quiver Q such that $Q_0 = \{1, \dots, n\}$ and $Q_1 = \{(i, i+1) | 1 \leqslant i < n\}$. We will refer to this quiver as the equioriented quiver of type A_n.

(1) The assignment $M_i = \mathbb{C}^{n+1}$ for all $i \in Q_0$ and $\varphi_\alpha = \mathrm{id}$ for all $\alpha \in Q_1$ defines a representation M of A_n. Each subrepresentation of M of dimension vector $(1, \dots, n)$ corresponds to a point in $\mathcal{F}l(\mathbb{C}^{n+1})$.
(2) Fix family of endomorphisms $\mathbf{f} = \{f_1, \dots, f_n\}$ of $\mathbb{C}^{n+1}$ and consider the representation $M_{\mathbf{f}}$ as described in the introduction. If we set $\mathbf{e} = (1, \dots, n)$, the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(A_n, M_{\mathbf{f}})$ is isomorphic to the linear degeneration $\mathcal{F}l^{\mathbf{f}}(\mathbb{C}^{n+1})$.

In our context we always work with representation of equioriented quivers of type A so we omit the quiver Q in the quiver Grassmannian notation.

We denote by $\mathrm{Rep}(Q)$ the category of representations of the quiver Q over $\mathbb{C}$. A general fact about representations of acyclic quivers is that $\mathrm{Rep}(Q)$ is hereditary, i.e. $\mathrm{Ext}^{\geqslant 2}(M, N) = 0$ for every $M, N \in \mathrm{Rep}(Q)$. As a consequence, if M and N are representations of dimension vectors $\mathbf{d}$ and $\mathbf{e}$ respectively, the following formula holds:

$$\dim_{\mathbb{C}} \mathrm{Hom}_Q(M, N) - \dim_{\mathbb{C}} \mathrm{Ext}^1(M, N) = \langle \mathbf{d}, \mathbf{e} \rangle, \tag{3.1}$$

where $\langle - , - \rangle$ denotes the Euler Form associated to the quiver Q.

Remark 3.4. If Q is the equioriented quiver of type A_n, the Euler Form is defined by the formula

$$\langle \mathbf{d}, \mathbf{e} \rangle = \sum_{i=1}^{n} d_i e_i - \sum_{i=1}^{n-1} d_i e_{i+1}.$$

As shown in Example 3.3, each linear degeneration is isomorphic to a quiver Grassmannian, so it is natural to use techniques coming from representation theory of quivers of type A to deduce information about linear degenerations of flag variety. We recall now some relevant results about quiver representations that we are going to use extensively in the next sections. A fundamental theorem in quiver representation theory concerns the indecomposable constituents of a generic representation M:

Theorem 3.5. *Let Q be a finite quiver with no loops and let M be a finite dimensional representation of Q. The M is the direct sum of uniquely determined indecomposable summands.*

In particular, we need a precise description of $\mathrm{Hom}_Q(M, N)$ and $\mathrm{Ext}^1(M, N)$ when M, N are indecomposable representations of quiver A_n.

Remark 3.6. By Gabriel's Theorem, the indecomposable representations of the equioriented quiver A_n are all of the form $U_{i,j}$, for $1 \leqslant i \leqslant j \leqslant n$, where $U_{i,j}$ is the representation

$$0 \to \ldots \to 0 \to \mathbb{C} \xrightarrow{\text{id}} \ldots \xrightarrow{\text{id}} \mathbb{C} \to 0 \to \ldots \to 0,$$

supported on the vertices with indices between i and j.

Some explicit formulae for dimension of $\operatorname{Hom}_Q(U_{i,j}, U_{h,k})$ and $\operatorname{Ext}^1(U_{i,j}, U_{h,k})$ hold:

$$\dim_{\mathbb{C}} \operatorname{Hom}_Q(U_{i,j}, U_{h,k}) = \begin{cases} 1 & \text{if } h \leqslant i \leqslant k \leqslant j \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

$$\dim_{\mathbb{C}} \operatorname{Ext}^1(U_{i,j}, U_{h,k}) = \begin{cases} 1 & \text{if } i+1 \leqslant h \leqslant j+1 \leqslant k \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

3.1. Coefficient quivers. Consider now a representation (M, F) of a quiver Q. For every $i \in Q_0$ fix a basis $B^i = \{v_k^i\}_{k=1}^{n+1}$ of M_i and set $B = \cup B^i$.

Definition 3.7 (Coefficient Quiver). *The coefficient quiver $Q(M,B)$ of M with respect the basis B is defined by the following data:*

- *$Q(M,B)$ has $|B|$ vertices labelled by the elements of B,*
- *there is an arrow between v_k^i and v_h^j if and only if there exists an edge $\alpha \in Q_1$ between i and j and the coefficient of v_h^j in $f_\alpha(v_k^i)$ is non zero.*

It is always possible to choose the basis B such that to each connected component of $Q(M,B)$ corresponds an indecomposable summand of M (see [12, Theorem 1.11]). In the following sections we always suppose that the chosen basis B used to construct the coefficient quiver $Q(M,B)$ has this property and we omit B from the notation.

Example 3.8. Fix a basis $\{v_1, \ldots, v_{n+1}\}$ of the $\mathbb{C}$-vector space $\mathbb{C}^{n+1}$. Consider the representation $M_{\mathbf{f}}$ of the quiver A_n defined by the data $M_i = \mathbb{C}^{n+1}$ and $f_{(i,i+1)} = \text{id}$ for every i. The coefficient quiver $Q(M_{\mathbf{f}})$ can be displayed as the union of $n+1$ linear segments of length n. Each linear segment corresponds to an indecomposable summand of M, isomorphic to $U_{1,n}$. In Figure 1 it is displayed the described coefficient quiver for $n = 3$.

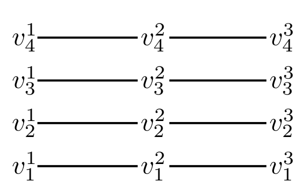


Figure 1. The coefficient quiver described in Example 3.8.

Example 3.9. Let $\mathbf{R} = (R_1, \ldots, R_n)$ be a family of subsets of $\{1, \ldots, n+1\}$. In the same setting of Example 3.8, consider the representation $M^{\mathbf{R}}$ of the quiver A_n defined by the data $M_i^{\mathbf{R}} = \mathbb{C}^{n+1}$ and $f_{(i,i+1)} = \pi_{R_i}$. Then the coefficient quiver of the representation $M^{\mathbf{R}}$ can be obtained from the coefficient quiver of Example 3.8 deleting the edges between v_j^i and v_j^{i+1} if $j \in R_i$. In Figure 2 is displayed the coefficient quiver of the representation $M^{\mathbf{R}}$ for $\mathbf{R} = (\{2\}, \{3\})$ and $n = 3$. We omitted the labeling of vertices in $Q(M^{\mathbf{R}})$, that must be considered as in Figure 1.

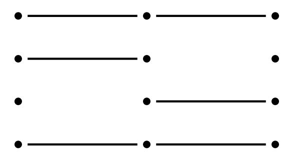


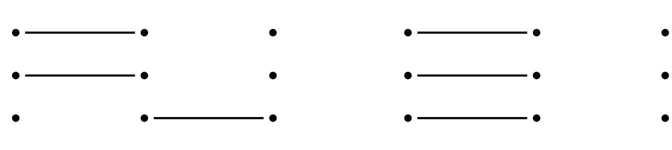
Figure 2. The coefficient quiver of $M^{\mathbf{R}}$ for $n = 3$ and $\mathbf{R} = (\{2\}, \{3\})$.

From here to the remain of the paper we deal only with representations of the equioriented quiver of type A_n. As mentioned before, we suppose that the coefficient quiver $Q(M)$ of a representation M is always displayed with respect to a basis B such that its connected components correspond to indecomposable summands of M. In particular, up to base change, we can suppose that there is an arrow between v_i^j and v_h^k **only if** $h = i$ and $k = j + 1$. In this case we say that the (possibly disconnected) subquiver of $Q(M)$ spanned by the vertices associated to $\{v_i^1 \dots v_i^n\}$ is *the i-th row of* $Q(M)$. We will also consider suitable subquivers of these rows. For integers $a \leqslant b$, we denote by $[a, b]^i_{Q(M)}$ the (possibly disconnected) *segment* in the i-th row of $Q(M)$ spanned by the vertices $\{v_i^a, \dots, v_i^b\}$. When $a = b$, we use the shorter notation $[a]^i_{Q(M)}$. Moreover, if Q' is a full subquiver of $Q(M)$ and $[a, b]^i_{Q(M)} \subset Q'$, we denote by $[a, b]^i_{Q'}$ the segment $[a, b]^i_{Q(M)}$ viewed as a subquiver of Q'.

Remark 3.10. We recall that, up to base change, every linear degeneration of the flag variety $\mathcal{F}l(\mathbb{C}^{n+1})$ is $GL_{n+1}(\mathbb{C})^n$-conjugate to one of the form $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$. Such a degeneration is isomorphic to the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M^{\mathbf{R}})$, where $\mathbf{e} = (1, \dots, n)$ and $M^{\mathbf{R}}$ is the representation described in Example 3.9. As a consequence, to each $GL_{n+1}(\mathbb{C})^n$-orbit O it is possible to attach a coefficient quiver Q_O, unique up to base change. By abuse of notation, if X is a degeneration in the orbit O, we will refer to Q_O as *the coefficient quiver of* X. Moreover, without loss of generality we can suppose that the vertices of i-th column of Q_O are labelled from bottom to top by the vectors $v_1^i, \dots v_{n+1}^i$. We adopt this convention in all of the figures appearing in the remain of the paper so we omit the vertex labeling.

Example 3.11. The coefficient quiver in Figure 2 is the coefficient quiver of Feigin degeneration for $n = 3$.

Example 3.12. Set $\mathbf{R}_1 = (\{1, 2\}, \{3, 4\})$ and $\mathbf{R}_2 = (\varnothing, \{2, 3, 4\})$. For a fixed basis $\{v_1, v_2, v_3, v_4\}$ of $\mathbb{C}^4$, the degenerations $\mathcal{F}l^{\mathbf{R}_1}(\mathbb{C}^4)$ and $\mathcal{F}l^{\mathbf{R}_2}(\mathbb{C}^4)$ are not flat because $\mathbf{R}_1$ and $\mathbf{R}_2$ do not satisfy the flatness conditions of Theorem 2.2. Their coefficient quivers are displayed in Figure 3.



(A) (B)

Figure 3. Coefficient quivers of two non flat degenerations of $\mathcal{F}l(\mathbb{C}^4)$

4. Torus actions and cohomology

In recent papers [13, 14], it is proved that every linear degeneration of the flag variety can be endowed with a structure of GKM variety, under the action of a suitable algebraic torus T. These results hold in the more general context of cyclic quiver Grassmannians and as a consequence of this more general point of view, it is possible to achieve a description of the moment graph in a purely combinatorial way using quiver representation theory. We use these techniques to prove our smoothness criteria. In this section we recall some basic facts about GKM varieties and the main results contained in [13] and [14].

4.1. GKM varieties. Let X be a complex projective algebraic T-variety, i.e. X is equipped by an action of an algebraic torus $T \simeq (\mathbb{C}^*)^r$ (for some integer r). We denote by $H_T^*(X)$ the T equivariant cohomology ring of X (with coefficients in $\mathbb{Q}$).

Definition 4.1. *A T-variety X is a GKM variety if:*

(1) *The number of T-fixed points and of 1 dimensional T-orbits in X is finite*
(2) *The usual cohomology of X can be recovered by $H_T^*(X)$ by scalar extension:*

$$H^*(X) \simeq H_T^*(X) \otimes_{H_T^*(pt)} \mathbb{Q}.$$

Remark 4.2. It is proved in [10] that if conditions (1) and (2) hold, the T-action on X is locally linearizable, i.e. the closure of each 1-dimensional orbit is T-equivariantly isomorphic to $\mathbb{P}^1(\mathbb{C})$. In particular the closure of a 1-dimensional orbit E contains exactly two T-fixed points x_E and y_E and the torus acts on $\bar{E}$ by a character α_E (uniquely defined up to a sign depending on the isomorphism $\bar{E} \simeq \mathbb{P}^1(\mathbb{C})$).

Remark 4.3. In our context, condition (2) is equivalent to require that rational cohomology of X vanishes in odd degrees. (cf. [13, Section 1])

Definition 4.4. *Let X be a T variety and χ a cocharacter of T. We say that χ is a* generic cocharacter (for the T action on X) *if $X^{\chi(\mathbb{C}^*)} = X^T$.*

Existence of a generic cocharacter implies the existence of a nice cell decomposition of X.

Definition 4.5 (Białynicki-Birula decomposition). *Let X be a complex projective algebraic variety equipped with an action of the one dimensional algebraic torus $\mathbb{C}^*$. Denote by $X_1, \dots, X_n$ the components of $X^{\mathbb{C}^*}$ and set*

$$C_i = \{x \in X \mid \lim_{z \to 0} z \cdot x \in X_i\}.$$

Then X is the disjoint union of $C_1, \dots, C_n$.

Definition 4.6 (Moment Graph). *Let X be a T variety. Its Moment Graph is the graph $G(X,T)$ defined by the following data:*

- *The vertices of $G(X,T)$ are indexed by the set of fixed points X^T,*
- *There is an (oriented) edge from x to y if there exists a 1-dimensional orbit E such that $x, y \in \overline{E}$ and $\lim_{z\to 0} \chi(z) \cdot p = x$ for $p \in E$.*

One of the main results in Goresky-Kottwitz-MacPherson theory (cf. [10]) assert that the T-equivariant cohomology of a GKM variety X can be explicitly computed starting from the moment graph $G(X,T)$ and from the set of T-characters α_E describing the torus action on 1-dimensional orbits (cf. Remark 4.2).

We say that a Białynicki-Birula cell C_i is *rational* if C_i is rationally smooth at every $x \in C_i$, i.e. for every $x \in C_i$, the top rational cohomology group $H^{2\dim_{\mathbb{C}} C_i}(C_i, C_i \smallsetminus \{x\})$ has rank 1 and $H^i(C_i, C_i \smallsetminus \{x\}) = 0$ for every $i \neq 2\dim_{\mathbb{C}} C_i$.

Definition 4.7 (BB filterable variety). *A T variety is BB filterable if:*

BB1) *the set X^T is finite,*

BB2) *X admit a BB decomposition in rational cells, induced by the action a suitable generic cocharacter χ of T.*

In [13] it is proved that, under the action of a suitable algebraic torus T, cyclic quiver Grassmannians have a structure of BB-filterable varieties. As a consequence of [13, Theorem 1.14] a projective BB filterable T-variety is GKM and consequently all linear degenerations of the flag variety are GKM varieties.

4.2. Torus actions on linear degenerations. In this section we recall the construction of the torus action studied in [13, 14]. This action is inspired by the one used in [4] to compute the Euler characteristic of certain quiver Grassmannians. Moreover, in [13] it is proved that a generic cocharacter χ for this action exists and the associated $\mathbb{C}^*$-action induces a Białynicki-Birula decomposition.

Let $M^{\mathbf{R}}$ be the representation of the quiver A_n associated to $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$ as described in Example 3.9. Each indecomposable summand of $M^{\mathbf{R}}$ is of the form $U_{h,k}$ (Remark 3.6) and can be identified with a connected component of some row r_i of $Q(M^{\mathbf{R}})$. In our notation we stress the fact that an indecomposable U, isomorphic to $U_{h.k}$, is identified with a connected component of r_i denoting it by $U^i_{h,k}$.

Suppose now that $M^{\mathbf{R}}$ has d_0 indecomposable summands. Label them with integers $\{1, \dots, d_0\}$ and consider the torus $T = (\mathbb{C}^*)^{d_0}$. Observe that a labeling choice defines an ordering of the indecomposable components. If $U^i_{h,k}$ is the j-th indecomposable summand, we define the T action on $U^i_{h,k}$ by the following formula: let $\gamma = (\gamma_1, \dots, \gamma_{d_0})$ be a generic element of T; then, for every $t \in \{h, \dots, k\}$ set

$$\gamma \cdot v^t_i = \gamma_j v^t_i. \tag{4.1}$$

Example 4.8. In this example we explicitly describe a torus action on the Feigin degeneration for $n = 3$. Consider the coefficient quiver in Figure 2. It has 6 connected components, corresponding to the indecomposable decomposition

$$M^{\mathbf{R}} = U^1_{1,3} \oplus U^2_{1,1} \oplus U^2_{2,3} \oplus U^3_{1,2} \oplus U^3_{3,3} \oplus U^4_{1,3}.$$

If we order the indecomposable components of $Q(M^{\mathbf{R}})$ as in Figure 4, the action of $\gamma = (\gamma_1, \dots, \gamma_6) \in T = (\mathbb{C}^*)^6$ on the j-th component is given by scalar multiplication by γ_j, as indicated in the last column. Observe that any other ordering of the indecomposable components would induce a different T-action.

We now recall how to define a generic cocharacter for the action defined by Formula 4.1.

Definition 4.9 (Attractive Grading, [13, Definition 3.10]). *An attractive grading on $Q(M)_0$ is a map $\mathrm{wt} : Q(M)_0 \to \mathbb{Z}$ that satisfies the following properties:*

(1) *For every $i \in Q_0$ it holds that $\mathrm{wt}(v^i_k) > \mathrm{wt}(v^i_h)$, whenever $k > h$;*

(2) *For every edge $\alpha \in Q_1$ there exists $d(\alpha) \in \mathbb{Z}$ such that $\mathrm{wt}(v^{t(\alpha)}_k) > \mathrm{wt}(v^{s(\alpha)}_h) + d(\alpha)$ whenever there is an oriented edge $v^{s(\alpha)}_h \to v^{t(\alpha)}_k$ in $Q(M)$.*

Fix an attractive grading on the vertices of $Q(M^{\mathbf{R}})$. If $U^i_{h,k}$ is the j-th indecomposable summand of $M^{\mathbf{R}}$, let us denote by b_j the starting vertex of the connected component associated to $U^i_{h,k}$ in $Q(M^{\mathbf{R}})$. The map $\chi : \mathbb{C}^* \to T$ defined by the assignment

$$z \in \mathbb{C}^* \to (z^{\mathrm{wt}(b_1)}, \dots, z^{\mathrm{wt}(b_{d_0})}) \tag{4.2}$$

defines a cocharacter of T. The following Theorems are proved in [13]:

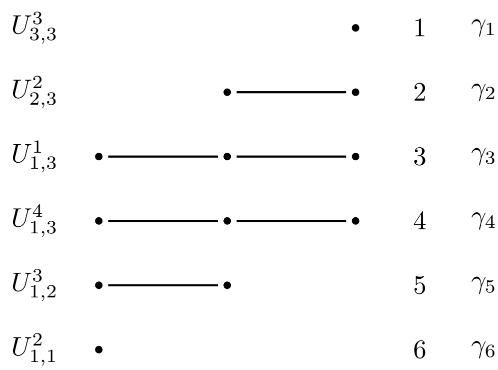


FIGURE 4. A T action on Feigin Degeneration for $n=3$. The indecomposable components are listed in the first column, ordered from top to bottom.

THEOREM 4.10 ([13, Theorem 5.14]). *Let* wt *be an attractive grading,* $T = (\mathbb{C})^{d_0}$ *the torus acting on* $\mathrm{Gr}_{\mathbf{e}}(M^{\mathbf{R}})$ *by Formula* 4.1 *and* χ *the cocharacter defined by Formula (*4.2*). Then* χ *is a generic cocharacter for the action of* T *on* $\mathrm{Gr}_{\mathbf{e}}(M^{\mathbf{R}})$

THEOREM 4.11 ([13, Proposition 6.4]). *The quiver Grassmannian* $\mathrm{Gr}_{\mathbf{e}}(M^{\mathbf{R}})$ *is a BB-filterable projective* T*-variety. Moreover, the* T*-action induces on* $\mathrm{Gr}_{\mathbf{e}}(M^{\mathbf{R}})$ *a structure of* GKM *variety.*

Under these hypothesis, the moment graph for the T action on $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$ can be recovered in an explicit combinatorial way. In [13, Section 6.2] it is proved that fixed points for the T action can be identified with suitable subquivers of the coefficient quiver $Q(M^{\mathbf{R}})$.

DEFINITION 4.12 (Successor Closed Subquiver, cf. [13, Definition 6.7]). *A successor closed subquiver of* $Q(M^{\mathbf{R}})$ *is a full subquiver* Q' *of* $Q(M^{\mathbf{R}})$ *such that*

- *if* $v \in Q'_0$ *and* $v = s(\alpha)$ *for some* $\alpha \in Q(M^{\mathbf{R}})_1$*, then* $t(\alpha) \in Q'_0$
- $|Q'_0 \cap B^i| = i$ *for all* $i \leqslant n$.

THEOREM 4.13 (cf. [13, Theorem 6.15]). *The* T*-fixed points in* $\mathrm{Gr}_{\mathbf{e}}(M^{\mathbf{R}})$ *are in bijection with successor closed subquivers of* $Q(M^{\mathbf{R}})$.

Successor closed subquivers of $Q(M^{\mathbf{R}})$ can be described using certain sequences of subsets.

DEFINITION 4.14 ($\mathbf{R}$-Admissible Sequence). *A family* $S = (S_1, \dots, S_n)$ *of subsets* $S_i \subset \{1 \dots, n+1\}$ *is* $\mathbf{R}$*-admissible sequence if* $|S_i| = i$ *for every* $i \leqslant n$ *and* $S_i \subset S_{i+1} \cup R_i$.

The condition $S_i \subset S_{i+1} \cup R_i$ implies that any $\mathbf{R}$-admissible sequence S define a successor closed subquiver of $Q(M^{\mathbf{R}})$, and *vice versa* every successor closed subquiver of $Q(M^{\mathbf{R}})$ is associated to a unique $\mathbf{R}$-admissible sequence. Indeed, if $S = (S_1, \dots, S_n)$ is an $\mathbf{R}$-admissible family, it suffices to consider the full subquiver $Q' \subset Q(M^{\mathbf{R}})$ whose set of vertices is $Q'_0 = \bigcup_{i=1}^n \{v^i_{j_1}, \dots, v^i_{j_i}\}$ with $\{j_1, \dots, j_i\} = S_i$.

We denote the subquiver associated to S by Q_S and by p_S the corresponding point in $\mathrm{Gr}_{\mathbf{e}}(M^{\mathbf{R}})^T$. Furthermore, recall that by definition of quiver Grassmannian, the point p_S can be identified with a subrepresentation of $M^{\mathbf{R}}$. We are going to denote this representation by M_S.

EXAMPLE 4.15. We recall that for Feigin Degeneration for every i we have $R_i = \{i+1\}$. A fixed point p_S is then associated to an $\mathbf{R}$-admissible sequence $S = (S_1, \dots, S_n)$

such that $S_i \subset S_{i+1} \cup \{i+1\}$. In Figure 5 two subquivers of the coefficient quiver of the Feigin degeneration for $n = 3$ are displayed. Both of them are encoded by families of subsets of $\{1, 2, 3, 4\}$. The family S associated to the subquiver in Figure 5a satisfies the condition of being an $\mathbf{R}$-admissible sequence and, in fact, the associated subquiver is successor closed. Conversely the subquiver in Figure 5b is not successor closed, because $v_1^2 \in Q_{S'}$ but $f_{2,3}(v_1^2) = v_1^3 \notin Q_{S'}$; indeed, the sequence S' is not $\mathbf{R}$-admissible.

(a) $S = (\{4\}, \{4, 3\}, \{4, 2, 1\})$ (b) $S' = (\{1\}, \{1, 4\}, \{4, 3, 2\})$

Figure 5. Two subquivers and their associated families of subsets

Aiming to clarify how combinatorics of admissible sequences plays a very relevant role in studying the singularity of fixed points, we think it could be very useful to remark now that for each $\mathbf{R}$-admissible sequence S we have three distinct objects: a fixed point for the torus action p_S, a subrepresentation M_S of $M^{\mathbf{R}}$ of dimension vector $(1, \dots, n)$ and a successor closed subquiver Q_S of $Q(M^{\mathbf{R}})$. We recall now how these three objects are linked.

A crucial fact to prove our smoothness criteria links M_S to the dimension of tangent space to $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$ at p_S:

Theorem 4.16 (cf. [3, Proposition 6]).

$$\dim \operatorname{Hom}(M_S, M^{\mathbf{R}}/M_S) = \dim T_{p_S}(\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1}))$$

Remark 4.17. The indecomposable summands of quotient representation $M^{\mathbf{R}}/M_S$ corresponds to the connected component of the Coefficient Quiver $Q(M^{\mathbf{R}}) \setminus Q_S$. In particular, since M_S has dimension vector $\mathbf{dim\, M_S} = (1, \dots, n)$, then $M^{\mathbf{R}}/M_S$ has dimension vector $\mathbf{dim\, M^R/M_S} = (n, \dots, 1)$ and then

$$\langle \mathbf{dim\, M_S}, \mathbf{dim\, M^R/M_S} \rangle = \frac{n(n+1)}{2}.$$

This implies, as a consequence of Formula (3.1) and of Theorem 4.16, that if $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$ is flat, a point p_S is smooth if and only if $\operatorname{Ext}^1(M_S, M^{\mathbf{R}}/M_S) = 0$.

Let us denote by $SC(M^{\mathbf{R}})$ the set of successor closed subquivers of $Q(M^{\mathbf{R}})$. It is proved in [13] that an attractive grading wt on the vertices of the coefficient quiver $Q(M^{\mathbf{R}})$ induces an ordering on the elements of $SC(M^{\mathbf{R}})$ that is isomorphic to the partial order on fixed points in the moment graph of $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$, induced by the action of the cocharacter defined by Formula (4.2). It is possible to give a combinatorial description of this ordering in terms of *mutations* of a successor closed subquiver Q_S.

We remark that, by our choice of the basis B, each connected component of $Q(M^{\mathbf{R}})$ is a linear segment, i.e. the valence of each vertex is smaller or equal than 2.

Definition 4.18 (Movable part of Q_S, cf. [13, Definition 6.8]). *A movable part of a linear segment $L \subset Q_S$ is a connected subquiver $L' \subset L$ such that L' has the same starting vertex of L.*

In other words, a movable part is a connected segment $[a, b]^i_{Q_S}$ of Q_S such that $[a-1, b]^i_{Q_S}$ is not a connected linear segment of Q_S.

Example 4.19. In Figure 6 several subquivers of the row r are displayed. The row r should be thought of as a union of linear segments belonging to a row of a certain coefficient quiver; these segments are shown as thick red lines. The segment enclosed by the blue dotted line in Figure 6*a* and 6*b* are movable parts, since they have the same starting point as one of the segments of r. Conversely, the subquivers enclosed by dotted lines in Figures 6*c* and 6*d* are not movable parts. Indeed, the linear segment in Figure 6*c* does not have the same starting point as the thick red segment containing it. Moreover, the subquiver in Figure 6*d* is not connected and therefore cannot be a movable part.

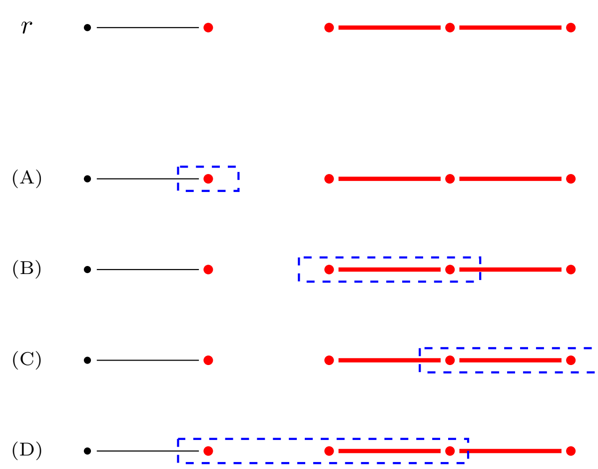


Figure 6. Examples and not-examples of movable parts

In [13, Section 6] the authors describe combinatorially the covering relations in the moment graph in the following way: let Q_S and $Q_{S'}$ be successor closed subquivers of $Q(M^{\mathbf{R}})$, we have that Q_S is covered by $Q_{S'}$ if and only if there exists a *fundamental mutation*[13, Definition 6.8] from $Q_{S'}$ to Q_S, i.e. it is possible to obtain Q_S from $Q_{S'}$ moving down *exactly one* movable part of $Q_{S'}$ with respect to a fixed attractive ordering on the elements of B.

One of the aims of this paper is to link mutations with the property of a fixed point in a flat degeneration of being singular. In particular, we are interested only to the valence of vertices in the moment graph, so we relax the definition of mutations as presented in [13]:

Definition 4.20 (Mutation). *Let $Q_S \in SC(M^{\mathbf{R}})$, a mutation of Q_S is the operation of moving* exactly one *movable part from its row to another row, obtaining again a successor closed subquiver of $Q(M^{\mathbf{R}})$.*

If $Q_S, Q_{S'} \in SC(M^{\mathbf{R}})$, we say that *there is a mutation from $Q_{S'}$ to Q_S* if Q_S is obtained from $Q_{S'}$ by moving exactly one movable part to another row. In other words, if $Q_S, Q_{S'}$ are successor closed subquivers of $Q(M^{\mathbf{R}})$, then there is a mutation from $Q_{S'}$ to Q_S if and only if $Q_S \setminus Q_{S'}$ and $Q_{S'} \setminus Q_S$ are both movable parts of the same length (i.e. are spanned by the same number of vertices). In terms of connected sub-segments of the coefficient quiver $Q(M^{\mathbf{R}})$ we say that Q_S is obtained from $Q_{S'}$ by a mutation if there exists two connected segment $I = [a,b]^i_{Q_{S'}}$ and $J = [a,b]^j_{Q(M^{\mathbf{R}})}$ such that $Q_S = (Q_{S'} \cup J) \setminus I$.

Example 4.21. In Figure 7, some successor-closed subquivers of the coefficient quiver of the Feigin degeneration for $n = 4$ are displayed. They correspond to the sequences

$$S = (\{3\}, \{3, 5\}, \{1, 4, 5\}, \{1, 3, 4, 5\}),$$

$$S' = (\{3\}, \{3, 4\}, \{1, 4, 5\}, \{1, 3, 4, 5\}),$$

$$S'' = (\{4\}, \{3, 4\}, \{1, 4, 5\}, \{1, 3, 4, 5\}),$$

respectively.

The successor-closed subquiver $Q_{S'}$ is obtained from Q_S by moving the movable part enclosed by the blue dotted line from the fifth row to the fourth one. In our notation, this movable part is denoted by $[2]^5_{Q_S}$. Similarly, $Q_{S''}$ is obtained from $Q_{S'}$ by moving the movable part $[1]^3_{Q_{S'}}$ to the fourth row.

Observe that $Q_{S''}$ cannot be obtained from Q_S by a single mutation, i.e. by moving only one movable part. Therefore, there are no mutations between Q_S and $Q_{S''}$.

Figure 8 displays the remaining two successor-closed subquivers obtained from Q_S by moving a movable part from the fifth row to another row. In particular, the quiver in Figure 8a is obtained by moving the segment $[2]^5_{Q_S}$ to the first row, while the quiver in Figure 8b is obtained by moving the segment $[2, 4]^5_{Q_S}$ to the second row.

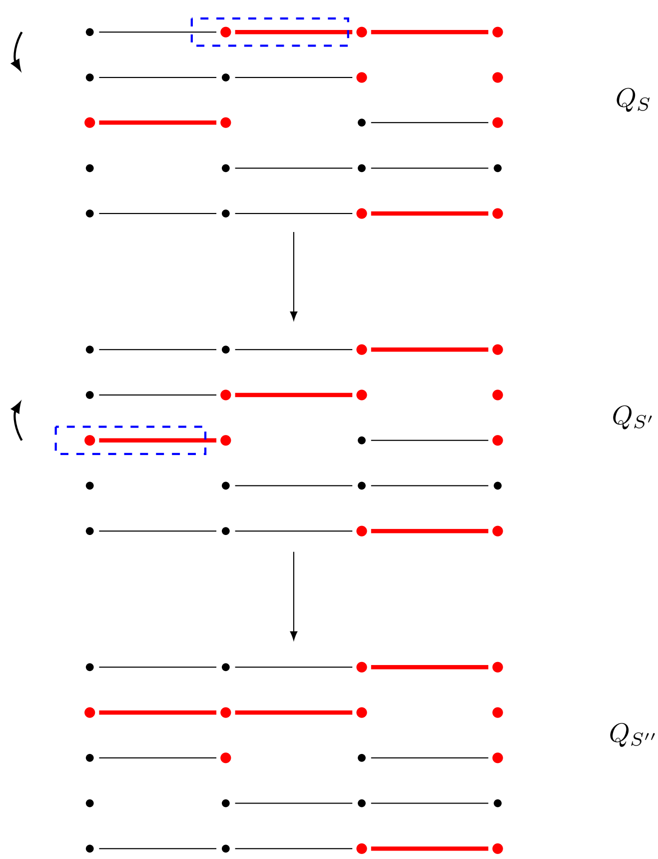


Figure 7. A sequences of mutations, from Q_S to $Q_{S''}$.

We denote by $\mathrm{Mut}(Q_S)$ the set of mutations of a successor closed subquiver $Q_S \in SC(M^{\mathbf{R}})$.

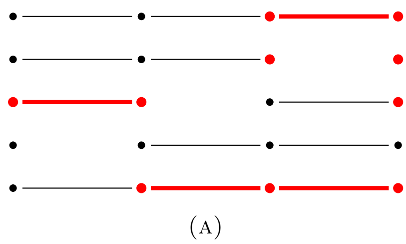


FIGURE 8. The only two other successor closed subquivers obtained from Q_S moving a single movable part from the fifth row.

REMARK 4.22. By definition of mutation and as a consequence of [13, Theorem 6.15], the cardinality of $\mathrm{Mut}(Q_S)$ equals the valence of p_S in the moment graph.

REMARK 4.23. Let $I = [a,b]^i_{Q_S}$ be a movable part (so, in particular, it is connected) and suppose that the linear segment $J = [a,b]^j_{Q(M^{\mathbf{R}})}$ is connected. Then I can be moved to J only if one of the following cases holds:

(A) $j \in S_{b+1}$; or
(B) $f_{b,b+1}(v^j_b) = 0$, i.e. there are no arrows between the b-th and $(b+1)$-th vertices in the j-th row of $Q(M^{\mathbf{R}})$.

Indeed, these are the only cases in which moving a movable part produces a successor-closed subquiver of $Q(M^{\mathbf{R}})$. These two cases are displayed in Figure 9a and Figure 9b, respectively. We write $[a,b]^i_{Q_S} \to j$ to denote that there is a mutation from $[a,b]^i_{Q_S}$ to $[a,b]^j_{Q(M^{\mathbf{R}})}$.

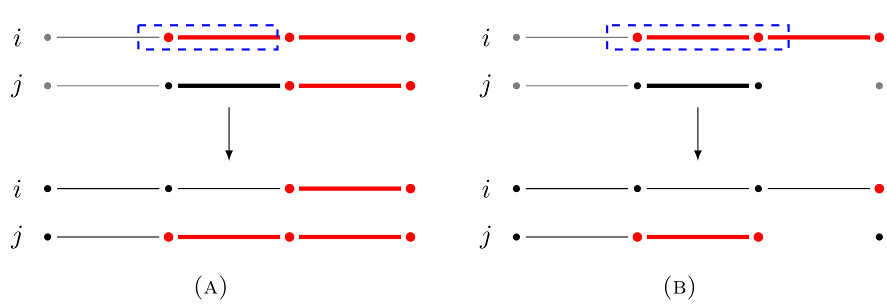


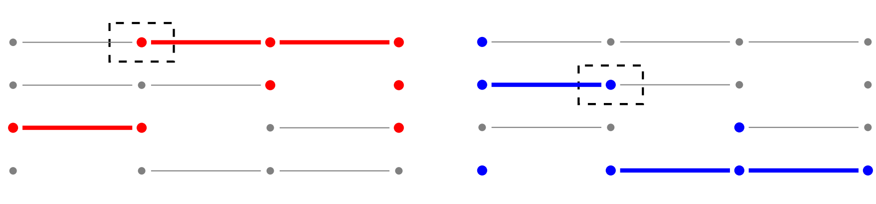
FIGURE 9. The two cases in which a mutation $[a,b]^i_{Q_S} \to j$ can be performed. Here the movable part $[a,b]^i_{Q_S}$ is enclosed by the blue dotted line, while the segment $[a,b]^j_{Q(M^{\mathbf{R}})}$ involved in the mutation is displayed in thick black.

To simplify the notation, we will denote by $[a,b]^i_S$ the movable part $[a,b]^i_{Q_S}$ in Q_S. Furthermore, by $\mathrm{Mut}(p_S)$ we denote the set of possible mutations of the successor closed subquiver associated to the point p_S. The following proposition establishes a relation between the combinatorial data of mutations of Q_S and the dimension of the tangent space of $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$ at p_S. As the proof is rather technical, the reader may find it helpful to consult Example 4.25, where the argument is illustrated through a concrete case.

PROPOSITION 4.24. *Let p_S be a fixed point, then* $|\mathrm{Mut}(p_S)| = \dim T_{p_S}\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$.

Proof. Because of Theorem 4.16 and by Formula (3.2), it is sufficient to prove that a mutation $[a,b]^i_S \to j$ occur if and only if there exist an indecomposable summand $U^i_{a,b'}$ of M_S and an indecomposable summand $U^j_{a',b}$ of $M^{\mathbf{R}}/M_S$ such that $\mathrm{Hom}(U^i_{a,b'}, U^j_{a',b}) \neq 0$, and that the pair $(U^i_{a,b'}, U^j_{a',b})$ is uniquely determined by the mutation $[a,b]^i_S \to j$. It is clear by definition that if we have a mutation $[a,b]^i_S \to j$, then there exists a maximal connected segment $[a,b']^i_S$ of Q_S containing $[a,b]^i_S$. In particular, it corresponds to the indecomposable summand $U^i_{a,b'}$ of M_S. Moreover, by definition of mutation, the segment $[a,b]^j_{Q(M^{\mathbf{R}})}$ need to be connected. By Remark 4.23, there exists a maximal connected segment of the j-th row of $Q(M^{\mathbf{R}}) \setminus Q_S$ of the form $[a',b]^j$, with $a' \leqslant a$. The linear segment $[a',b]^j$ corresponds to an indecomposable summand $U^j_{a',b}$ of $M^{\mathbf{R}}/M_S$ and $\mathrm{Hom}(U^i_{a,b'}, U^j_{a',b}) \neq 0$. Observe that such a pair of indecomposables is uniquely determined by the mutation $[a,b]^i_S \to j$. On the other hand, if $\mathrm{Hom}(U^i_{a,b}, U^j_{a',b'}) \neq 0$, where $U^i_{a,b}$ is an indecomposable summand of M_S and $U^j_{a',b'}$ is an indecomposable summand of $M^{\mathbf{R}}/M_S$, then by Formula (3.2) we have $a' \leqslant a \leqslant b' \leqslant b$. Consequently $[a,b']^i_S$ is a a connected segment of the i-th row of Q_S such that $[a-1,b']^i_S$ is not a linear segment of Q_S. Moreover $[a',b']^i_S$ is a maximal connected segment of $Q(M^{\mathbf{R}}) \setminus Q_S$ and then the pair $(U^i_{a,b}, U^j_{a',b'})$ corresponds to an uniquely determined mutation $[a,b']^i_S \to j$. □

Example 4.25. Consider the quiver Q_S displayed in Figure 7. The indecomposable decomposition of M_S is given by $U^5_{2,4} \oplus U^4_{3,3} \oplus U^4_{4,4} \oplus U^3_{1,2} \oplus U^3_{4,4} \oplus U^1_{3,4}$, corresponding to the red linear segments in Figure 10a. Analogously, the indecomposable decomposition of $M^{\mathbf{R}}/M_S$ is given by $U^5_{1,1} \oplus U^4_{1,2} \oplus U^3_{3,3} \oplus U^2_{1,1} \oplus U^2_{2,4} \oplus U^1_{1,2}$, and is displayed in blue in Figure 10b. The mutation $[2]^5_S \to 4$ is obtained moving the movable part $[2]^5_S$ to the segment $[2]^4_{Q(M^{\mathbf{R}})}$, both enclosed by dotted lines in Figure 10a and Figure 10b, respectively. The movable part $[2]^5_S$ is contained in the segment $[2,4]^5_S$ and so it corresponds to the indecomposable $U^5_{2,4}$ of M_S. Analogously $[2]^4_{Q(M^{\mathbf{R}})}$ corresponds to the indecomposable $U^4_{1,2}$ of $M^{\mathbf{R}}/M_S$. So the mutation $[2]^5_S \to 4$ is associated to the pair of indecomposables $(U^5_{2,4}, U^4_{1,2})$



(a) The indecomposable components of M_S

(b) The indecomposable components of $M^{\mathbf{R}}/M_S$

Figure 10. How to associate a pair of indecomposable to a mutation.

Observe that if $|\mathrm{Mut}(p_S)| = \binom{n+1}{2}$, then p_S is smooth. As a consequence of Remark 4.22, it is possible to link the dimension of the tangent space at p_S to the valence of p_S in the moment graph.

Corollary 4.26. *The valence of the vertex corresponding to p_S in the moment graph is equal to the dimension of tangent space to $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$ at p_S.*

5. A combinatorial formula for the dimension of tangent spaces

The first step for our combinatorial characterization of smooth fixed points is to define a suitable set that encodes relevant information about the dimension of the tangent space at a fixed point p_S.

Definition 5.1. *Consider the family* $\mathbf{R} = (R_1, \dots, R_{n-1})$ *of subsets of* $\{1, \dots, n+1\}$. *We define the set of* i*-positions of* $\mathbf{R}$ *as the set*

$$\mathrm{pos}(i) = \{j \in \{1, \dots, n-1\} \mid i \in R_j\}.$$

We will also refer to $\mathrm{pos}(i)$ as the set of positions of the i-th row in $Q(M^{\mathbf{R}})$.

Definition 5.2. *We define* $\mathrm{Sing}_i(p_S)$ *as the set of pairs* $(j, h+1)$*, with* $h \in \mathrm{pos}(i)$ *and* $j \notin S_{h+1}$*, such that there exists* $k \leqslant h$, $k \in \mathrm{pos}(j)$ *with the following characteristics:*

(1) $i \in S_t$ *for all* t *such that* $k \leqslant t \leqslant h$
(2) *the segment spanned by the vertices* $\{v_i^k, \dots, v_i^h\}$ *is connected in* Q_S.

We say that an element of $\mathrm{Sing}_i(p_S)$ is a singular point for the i-th row of Q_S.

Definition 5.3. *The set of singularities of* Q_S *is the set* $\mathrm{Sing}(p_S) = \bigsqcup \mathrm{Sing}_i(p_S)$.

Example 5.4. We remark that each pair $(j, h+1) \in \mathrm{Sing}(p_S)$ can be represented as a vertex of $Q(M^{\mathbf{R}})$ (possibly with multiplicities). In Figure 11, the set $\mathrm{Sing}(p_S)$ has a unique contribution, coming from the singular points for the third row. We now explain how $\mathrm{Sing}(p_S)$ has been computed in this example. Observe first that $\mathrm{pos}(i) \neq \varnothing$ if and only if $i = 2, 3$.

If $i = 2$, we have $\mathrm{pos}(2) = \{1\}$. To describe $\mathrm{Sing}_2(p_S)$, we need to determine all pairs of the form $(j, 2)$ such that $j \notin S_2$ and both conditions (1) and (2) are satisfied. Since $2 \notin S_1$, condition (1) cannot be satisfied by any pair of the form $(j, 2)$, and hence $\mathrm{Sing}_2(p_S)$ is empty.

Similarly, since $\mathrm{pos}(3) = 2$, to describe $\mathrm{Sing}_3(p_S)$ we need to determine all pairs of the form $(j, 3)$ such that $j \notin S_3$ and conditions (1) and (2) hold. The only possible candidate is $j = 2$. As already noted, $\mathrm{pos}(2) = \{1\}$, and $3 \in S_1, S_2$, so condition (1) is satisfied. Furthermore, the segment spanned by $\{v_3^1, v_3^2\}$ in Q_S is connected, so condition (2) is also satisfied.

Therefore, the set $\mathrm{Sing}_3(p_S)$ consists of the single pair $(2, 3)$, which corresponds to the blue vertex in Figure 11.

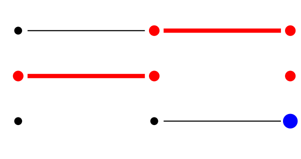


Figure 11. The set $\mathrm{Sing}_3(p_S)$, where $S = \{\{3\}, \{3, 4\}, \{1, 3, 4\}\}$

We prove now that, for *every* linear degeneration of the form in $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$, the following formula for the dimension of the tangent space at p_S holds:

$$\dim T_{p_S} \mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1}) = \frac{n(n+1)}{2} + |\mathrm{Sing}(p_S)|. \tag{5.1}$$

More precisely, in the following proposition we prove that $\mathrm{Sing}(p_S)$ measures the dimension of $\mathrm{Ext}^1(M_S, M^{\mathbf{R}}/M_S)$. The reader may find it helpful to consult Example 5.7 while reading the proof of this result.

Proposition 5.5.

$$|\mathrm{Sing}(p_S)| = \dim \mathrm{Ext}^1(M_S, M^{\mathbf{R}}/M_S)$$

Proof. First of all observe that $(j, h+1) \in \mathrm{Sing}_i(p_S)$ implies that there exist some uniquely determined indices $a = \max\{t \in \mathrm{pos}(j) | t \leqslant h\} + 1$ and $b > h$ such that $[a,b]^j_{Q(M^{\mathbf{R}})}$ is a connected segment in $Q(M^{\mathbf{R}}) \smallsetminus Q_S$. This segment is associated to an indecomposable subrepresentation $U^j_{a,b}$ of $M^{\mathbf{R}}/M_S$. Consider the two integers $\bar{k}$ and k_i such that

$$\bar{k} = \max\{t \in \mathrm{pos}(i) \cup \{1\} | t \leqslant h\} \qquad k_i = \min\{t \leqslant h | i \in S_s \ \forall s, t \leqslant s \leqslant h\}.$$

The segment $[\max\{\bar{k}, k_i\}, h]^i_S$ is then a connected component of the i-th row of Q_S that corresponds to an indecomposable summand $U^i_{\max\{\bar{k},k_i\},h}$ of M_S. Moreover, the condition (2) in Definition 5.2, implies that $\max\{\bar{k}, k_i\} \leqslant a \leqslant h < b$. Consequently, for each element in $(j, h+1) \in \mathrm{Sing}_i(p_S)$ we constructed pair $(U^i_{\max\{\bar{k},k_i\},h}, U^j_{a,b})$ of indecomposable representations in M_S and M/M_S respectively, such that $\mathrm{Ext}^1(U^i_{\max\{\bar{k},k_i\},h}, U^j_{a,b}) \neq 0$. Observe now that, once we fixed the index i, the pair $(U^i_{\max\{\bar{k},k_i\},h}, U^j_{a,b})$ is uniquely determined by $(j, h+1) \in \mathrm{Sing}_i(p_S)$. Then we need only to prove that each pair $(U_{k,h}, U_{a,b})$ of indecomposable summands of M_S and $M^{\mathbf{R}}/M_S$ respectively, such that $\mathrm{Ext}^1(U_{k,h}, U_{a,b}) \neq 0$, is associated to a pair $(j, h+1) \in \mathrm{Sing}(p_S)$. By condition expressed in Formula (3.3), we have $k+1 \leqslant a \leqslant h+1 \leqslant b$. Without loss of generality we can suppose that $U_{k,h}$ and $U_{a,b}$ corresponds to connected components of the i-th row of Q_S and of j-th row of $Q(M^{\mathbf{R}}) \smallsetminus Q_S$ respectively. This implies that the pair of indices $(j, h+1)$ satisfies the conditions in Definition 5.2 and then $(j, h+1) \in \mathrm{Sing}_i(p_S)$. □

Formula (5.1) now follows from Formula (3.1).

Example 5.6. We provide now an example of singularity sets for two different rows of the same Q_S that are not disjoint. Consider the flat degeneration $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^7)$ defined by the sequence of subsets $\mathbf{R} = \{\{5\}, \{6\}, \varnothing, \{3,6\}, \varnothing\}$ of $\{1, \ldots, 7\}$ and consider the fixed point p_S associated to the subquiver in red in Figure 12. It is possible to check directly that $\mathrm{Sing}_i(p_S)$ is non empty only if $i = 3, 6$. The singular sets for $i = 3, 6$ are then displayed in Figure 13. As an immediate consequence of Formula 5.1, the dimension of tangent space at p_S is equal to 24.

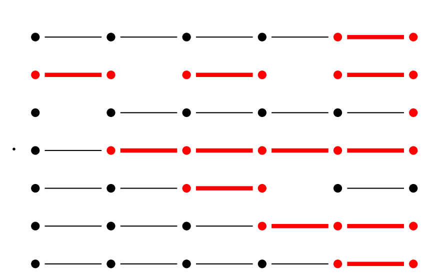
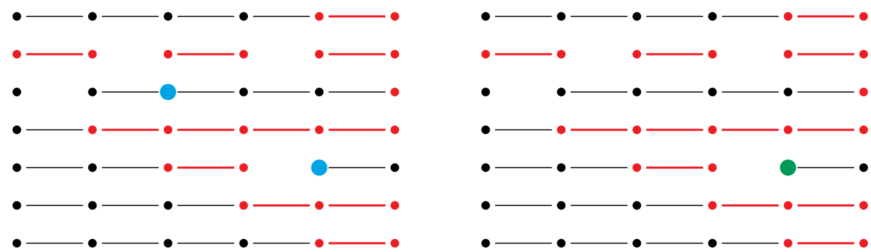


Figure 12. The fixed point p_S.

Example 5.7. In this example, we explicitly describe how to associate to each element of $\mathrm{Sing}(p_S)$ a pair of indecomposable summands of M_S and $M^{\mathbf{R}}$ with nontrivial Ext^1, in the case of the quiver Q_S displayed in Figure 12. Example 5.6 shows that

$$\mathrm{Sing}(p_S) = \mathrm{Sing}_3(p_S) \sqcup \mathrm{Sing}_6(p_S) = \{(3,5)\} \sqcup \{(3,5), (5,3)\}.$$



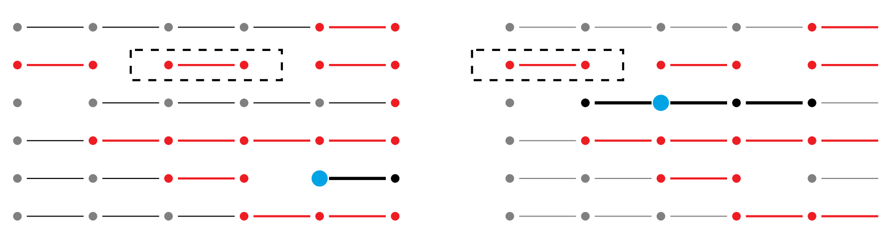
(A) The set $\mathrm{Sing}_6(p_S)$. (B) The set $\mathrm{Sing}_3(p_S)$

FIGURE 13. The singularity sets of point p_S.

We first describe how to associate pairs of indecomposables to the elements of $\mathrm{Sing}_6(p_S)$. The pairs of indices $\{(3,5),(5,3)\}$ can be visualized as vertices of the linear segments corresponding to the indecomposable summands $U^3_{5,6}$ and $U^5_{2,4}$ of $M^{\mathbf{R}}/M_S$. These indecomposable summands are shown in thick black in Figures 14a and 14b, respectively.

We now pair the indecomposable $U^3_{5,6}$ with an indecomposable U such that $\mathrm{Ext}^1(U,U^3_{5,6}) \neq 0$. Since we are analyzing $\mathrm{Sing}_6(p_S)$, the indecomposable U should correspond to a connected linear segment in the sixth row of Q_S. A direct inspection shows that the only possible choice is $U^6_{3,4}$, enclosed by the dotted line in Figure 14a. Analogously, $U^5_{2,4}$ is paired with the indecomposable component $U^6_{1,2}$.

The element $(3,5)$ also appears in $\mathrm{Sing}_3(p_S)$. In this case, since it contributes to $\mathrm{Sing}_3(p_S)$, the indecomposable summand $U^3_{5,6}$ should be paired with an indecomposable summand associated with a connected linear segment in the third row of Q_S. A direct check shows that such a component is $U^3_{3,4}$, as displayed in Figure 15.



(A) The pair $(U^6_{3,4}, U^3_{5,6})$ (B) The pair $(U^6_{1,2}, U^5_{2,4})$

FIGURE 14. The set $\mathrm{Sing}_6(p_S)$ and the associated indecomposable components.

6. Criteria for smoothness at fixed points

In this section we provide a list of equivalent smoothness criteria for a fixed point p_S. More closely, we identify a property of the sequence S that generalizes the smoothness condition proved in [7] for Feigin degenerations. Moreover to each p_S is associated a graph encoding information about dimension of tangent space at p_S. Our results are proved in Section 7

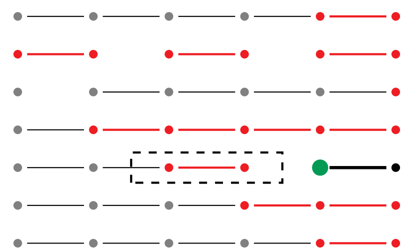


Figure 15. The set $\mathrm{Sing}_3(p_S)$ and the associated indecomposable components.

6.1. The generalized CI-F-R condition. We recall that Feigin degeneration is the $\mathbf{R}$ degeneration of $\mathcal{F}l(\mathbb{C}^{n+1})$ such that for every i we have $R_i = \{i+1\}$. A fixed point p_S for the torus action described in Section 4 is associated to an admissible sequence $S = (S_1, \ldots, S_n)$ such that $S_i \subset S_{i+1} \cup \{i+1\}$. In [7] the fixed points in the smooth locus are classified and enumerated, providing the following combinatorial smoothness criterion.

Definition 6.1 (Cerulli Irelli-Feigin-Reineke Condition). *We say that an admissible sequence S for the Feigin degeneration has the Cerulli Irelli-Feigin-Reineke Condition (for short CI-F-R Condition) if for all h, k such that $1 \leqslant h < k \leqslant n$ we have:*

$$k \in S_h \Rightarrow h+1 \in S_k. \tag{6.1}$$

Criterion 6.2 (Smoothness Criterion - Feigin Degeneration, [7, Theorem 4.2]). *A fixed point p_S is smooth if and only if S has the CI-F-R condition.*

Our first smoothness criterion is the natural generalization of this one.

Definition 6.3 (Generalized CI-F-R Condition). *Let S be an $\mathbf{R}$-admissible sequence of subsets of $\{1, \ldots, n+1\}$. We say that S has the Generalized CI-F-R Condition if, for every $i \in \{1, \ldots, n+1\}$,*

$$i \in S_k,\ k \in \mathrm{pos}(j) \Rightarrow j \in S_{h+1},\ \forall h \in \mathrm{pos}(i),\ h \geqslant k. \tag{6.2}$$

Observe that in the Feigin degeneration we have $\mathrm{pos}(i) = \{i-1\}$ for all i. Consequently an admissible sequence S has Generalized CI-F-R Condition if for all j, h such that $j \leqslant h$ we have $h \in S_j \Rightarrow j+1 \in S_h$, recovering the CI-F-R condition of Definition 6.1.

Proposition 6.4. *An $\mathbf{R}$-admissible sequence S has the Generalized CI-F-R Condition if and only if the set $\mathrm{Sing}(p_S)$ is empty.*

Proof. It is an immediate consequence of the definition that if S has the Generalized CI-F-R Condition then $\mathrm{Sing}(p_S) = \varnothing$. On the other hand, $\mathrm{Sing}(p_S) = \varnothing$ is equivalent to requiring that $\mathrm{Sing}_i(p_S) = \varnothing$ for all i. Suppose now that there exist i, j and $k \in \mathrm{pos}(j)$, $h \in \mathrm{pos}(i)$ such that $k \leqslant h$ and $i \in S_k, j \notin S_{h+1}$. For a fixed pair i, j, consider h minimal with this property. First of all, observe that j must be different from i, otherwise $(i, h+1)$ is in $\mathrm{Sing}_i(p_S)$. Note that in particular this implies that there exists a minimal index k_i such that $i \in S_t$ for all $t \geqslant k_i$. So, we are in the case $i \neq j$ and set $\widetilde{k} = \max\{t \in \mathrm{pos}(j) \,|\, t \leqslant h\}$. The segment $[\widetilde{k}+1, h+1]^j_{Q(M^{\mathbf{R}})}$ is connected by definition of $\widetilde{k}$ and consequently $j \notin S_t$ for all $t \in \{\widetilde{k}+1, \ldots, h+1\}$. By minimality of h this implies that the segment $[\widetilde{k}, h]^i_S$ is connected too. Moreover, because $k \leqslant \widetilde{k}$, the fact that $i \in S_t$ for all $t \geqslant k_i$ implies that $i \in S_t$, $\forall t \in \{\widetilde{k}, \ldots, h\}$ and $(j, h+1)$ satisfies the conditions of Definition 5.2, i.e. $\mathrm{Sing}_i(p_S) \neq \varnothing$, that is absurd. ∎

As a consequence of Formula 5.1, if $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$ is a flat degeneration, one obtains the following smoothness criterion that generalize Criterion 6.2:

Corollary 6.5. *Let $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$ be a flat degeneration. A fixed point p_S is smooth if and only if S satisfies the Generalized CI-F-R Condition.*

Remark 6.6. Suppose that S has the Generalized CI-F-R Condition, then $i \in S_h$ implies $i \in S_k$ for all k greater or equal than h, or equivalently that $S_i \subset S_{i+1}$ for every i. The converse is not true, in fact the admissible sequence $S = \{\{3\}, \{3,4\}, \{1,3,4\}\}$ defining the successor closed subquiver in Figure 11 satisfies the condition $S_i \subset S_{i+1}$ for every i, but as a consequence of Formula (5.1) and of Example 5.4, the associated fixed point p_S is not smooth.

Example 6.7. The next example shows that Corollary 6.5 does not hold if the degeneration is not flat. The algebraic variety $\mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$ can be identified with the degeneration $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^3)$ such that $\mathbf{R} = (\{1,2,3\})$. In particular, it is not a flat degeneration. In Figure 16 the coefficient quiver of $\mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$ is displayed. Moreover, the second diagram shows the successor closed subquiver associated to the point p_S defined by the admissible sequence $S = (\{1\}, \{1,2\})$. The set of singularities of p_S is non empty, in fact $\mathrm{Sing}(p_S) = \mathrm{Sing}_1(p_S) = \{(3,2)\}$, and then S does not have the Generalized CI-F-R Condition. However, $\mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$ is a smooth algebraic variety and so every fixed point is smooth. Nevertheless, we remark that if we compute the dimension of tangent space at p_S using Equation (5.1), our formula produces the correct result.

Figure 16. The coefficient quiver of $\mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$ and the fixed point associated to $S = \{\{1\}, \{1,2\}\}$.

6.2. The mutation graph. We introduce now a graph that encodes some information about the mutations that occur between the coefficient quiver of a fixed point and its adjacent vertices in the moment graph. We prove that the topology of this graph is linked with the property of a fixed point p_S of being smooth.

We say that we have a mutation between the i-th row and the j-th row of Q_S if there exists a fixed point S' such that $Q_{S'}$ is obtained from Q_S moving a unique movable part from its i-th row to the j-th row. For a fixed subquiver Q_S, it is possible to have many mutations from row i to row j. We will denote the set of these mutations by $\mathrm{Mut}_S(i,j)$.

Definition 6.8 (Oriented Mutation Graphs). *Consider a fixed point $p_S \in \mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$. The oriented mutation graph of p_S is the oriented multigraph $\widetilde{G}_S = (V(\widetilde{G}_S), E(\widetilde{G}_S))$ described by the following data:*

1. *The set of vertices $V(\widetilde{G}_S)$ is the set $\{1, \dots, n+1\}$,*
2. *There is an oriented edge from i to j for every mutation in $\mathrm{Mut}_S(i,j)$.*

We are going to extensively use the undirected version G_S of the graph $\widetilde{G}_S$, i.e. the same graph where we forgot the edges orientation. We will refer to G_S as the *Mutation Graph* of S.

Example 6.9. In Figure 17 a fixed point for the Feigin Degeneration when $n = 3$ is displayed. The associated oriented mutation graph and its unoriented version are

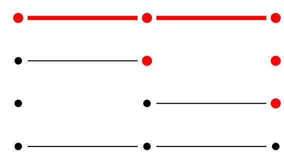


Figure 17. The fixed point p_S defined by $S = (\{4\}, \{3, 4\}, \{2, 3, 4\})$

displayed in Figure 18.

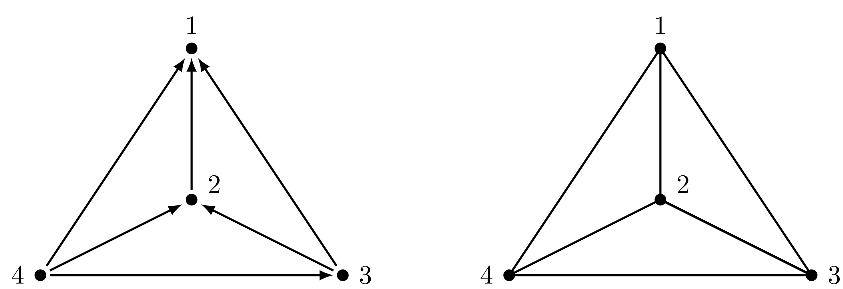


Figure 18. The oriented and unoriented mutation graphs of p_S in Figure 17

Example 6.10. In this example we exhibit a fixed point for a flat degeneration that has a non simple mutation graph. In Figure 19 , the successor closed subquiver associated to the admissible sequence $S = \{\{3\}, \{3, 4\}, \{1, 3, 4\}\}$ is displayed; it defines a T-fixed point p_S for the Feigin degeneration of $\mathcal{F}l(\mathbb{C}^4)$. It is immediate to check that the set $\mathrm{Mut}_S(3, 2)$ has 2 elements. The associated oriented mutation graph is the multigraph displayed on the right side of Figure 19.

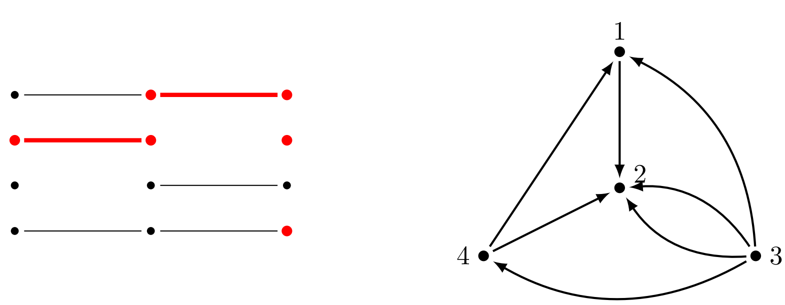


Figure 19. A fixed point p_S for Feigin degeneration with non simple oriented mutation graph.

A consequence of our work is that certain combinatorial properties of G_S and $\widetilde{G}_S$ are equivalent to the fact that p_S is smooth. Nevertheless, an explicit link between the structure of G_S and $\widetilde{G}_S$ and geometric properties of p_S is far from clear. As an example, it results that all smooth fixed points for a fixed degeneration $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$ have the same oriented mutation graphs up to vertices relabeling (see Theorem 6.12).

Moreover, in the case of the complete flag variety $\mathcal{F}l(\mathbb{C}^{n+1})$, where the cell decomposition essentially coincides with the Bruhat decomposition, we observed that starting from the mutation graph of a fixed point p_S it is possible to recover the inversion set of the permutation $w \in S_{n+1}$ associated to p_S and then the dimension of the corresponding cell. These phenomena are quite interesting and we aim to investigate deeper links between topology of mutation graphs and cell decomposition in some future works.

6.3. Smoothness criteria. Using the tools introduced in Section 6.1 and 6.2 it is possible to provide a list of smoothness criteria for a T-fixed point p_S, summarized in Theorem 6.12.

Definition 6.11. *A* Tournament Graph T *over* n *vertices is the assignment of a direction of the edges of the complete graph over* n *vertices. We say that* T *is* transitive *if the edge directions induce a total ordering on the vertices.*

Theorem 6.12 (Smoothness Criteria). *Let* p_S *be a* T *fixed point for a flat degeneration* $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$ *associated to an* $\mathbf{R}$*-admissible sequence* S*. The following conditions are equivalent:*

(1) *The point* p_S *is smooth;*
(2) *The set* $\mathrm{Sing}(p_S)$ *is empty;*
(3) *The* $\mathbf{R}$*-admissible sequence* S *has the Generalized CI-F-R Condition.*
(4) *The Mutation graph* G_S *is the complete graph over* $n+1$ *vertices.*
(5) *The Oriented Mutation graph* $\widetilde{G}_S$ *is a transitive tournament* $n+1$ *vertices;*

The equivalence of conditions (1) and (2) is a direct consequence of Formula (5.1). The fact that (2) $\iff$ (3) follows by Proposition 6.4. Moreover the fact that (5) $\Rightarrow$ (4) $\Rightarrow$ (1) is a consequence of flatness, of Proposition 4.24 and of definition of transitive tournaments. In Section 7.2 we give a direct proof of the fact that (1) $\Rightarrow$ (5) using techniques coming from representation theory of quivers of type A.

Example 6.13. We provide here an example of a smooth fixed point in a non flat degeneration with non a simple mutation graph. We consider the fixed point p_S of Example 6.7. We already proved that p_S is smooth. Moreover it is easy to check that $|\mathrm{Mut}(1,3)| = 2$ and so its mutation graph is not simple. In Figure 20 the successor closed subquiver associated to p_S and its oriented mutation graph are displayed. We finally remark that, up to vertices relabeling, this graph is the same for every T-fixed point in $\mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$.

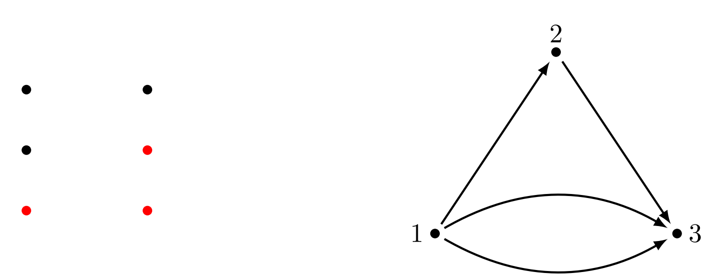


Figure 20. A smooth point p_S with non simple mutation graph.

6.4. Smooth locus in PBW degenerations. Using our characterization of smooth T-fixed points it is possible to describe the smooth locus in certain linear degenerations, endowed by the action of a suitable algebraic group.

Definition 6.14 (PBW Degeneration cf. [5, Section 1]). *An* $\mathbf{f}$*-degeneration is a PBW degeneration if kernels of the* f_i *are at most one dimensional and linearly independent.*

In particular any PBW degeneration is flat irreducible and has a $GL_{n+1}(\mathbb{C})^n$-orbit representative of the form $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$ such that $|R_i| \leqslant 1$ for every i and $R_i \cap R_j = \varnothing$ for every pair of indices $i \neq j$. These special degenerations are studied in [5, Section 5], where it is proved that they can be realized as Schubert varieties into certain parabolic flag varieties.

More precisely, denote by d_0 the number of indecomposable summands appearing in the representation $M^{\mathbf{R}}$ and set

$$\lambda_0 = 0, \qquad \lambda_k = \sum_{i=1}^{k} |R_i|, \qquad \ell_k = k + \lambda_{k-1},$$

Fix as a Borel subgroup B for $SL_{d_0}(\mathbb{C})$ the subgroup of upper triangular matrices and let Δ be the set of simple roots associated to B. We denote by ω_i the i-th fundamental weight for $SL_{d_0}(\mathbb{C})$ with respect to Δ; set $\omega = \omega_{\ell_1} + \cdots + \omega_{\ell_n}$ and consider the partial flag variety $SL_{d_0}(\mathbb{C})/P_\omega$, where P_ω is the parabolic subgroup stabilizing the weight ω.

If I is the subset of Δ corresponding to P_ω, then the degeneration $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$ can be identified with the parabolic Schubert variety X_w in $SL_{d_0}(\mathbb{C})/P_\omega$, where w is a minimal coset representative for W/W_I satisfying the conditions of [5, Section 5.2].

Under this identification, the action of the torus T of diagonal matrices in $SL_{d_0}(\mathbb{C})$ coincides with the torus action defined in Section 4. Since the Borel subgroup B acts on $SL_{d_0}(\mathbb{C})/P_\omega$ by left multiplication and Schubert cells coincide with B orbits of T-fixed points, the smooth locus in $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$ can be described as the union of B orbits of smooth fixed points.

Our result provides then a new combinatorial description of smooth locus in a special case of parabolic Schubert varieties. We point out that a different characterization of the smooth locus can be obtained as a consequence of some more general classical results about smooth and singular loci in Schubert varieties (see [2] for a complete reference).

In fact it holds in general that, if P_I is the parabolic subgroup of an algebraic group G associated to the subset of simple roots I and B is a Borel subgroup contained in P_I, then the projection map

$$\beta : G/B \to G/P_I$$

is a smooth B-equivariant fibre bundle; consequently if X_w is a parabolic Schubert variety in G/P_I, with $w \in W/W_I$, then the smooth locus of X_w is the image under β of the smooth locus of $X_{\widetilde{w}}$, where $\widetilde{w}$ is the maximal length element in the coset wW_I (cf. [9, Section 2]).

7. Proof of our smoothness criteria

In this section we complete the proof of equivalences in the statement of Theorem 6.12. Our proofs work by induction and use extensively the combinatorics of coefficient quivers and of their successor closed subquivers. In Section 7.1 we develop some techniques that allow us to restrict to suitable subquivers and then proceed by induction.

7.1. Restriction of coefficient quiver. Let p_S be a T fixed point in a linear degeneration $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$ associated to an $\mathbf{R}$-admissible sequence S. Moreover, let $M^{\mathbf{R}}$ be the quiver representation of the equioriented quiver A_n associated to the degeneration $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$.

Definition 7.1. *We say that* Q_S has an empty row *if there exist* $j \in \{1, \dots, n+1\}$ *such that* $j \notin S_i$ *for all* i.

Observe that this is coherent with the fact that, if we see Q_S as a subquiver of $Q(M^{\mathbf{R}})$, no connected components of Q_S belongs to the j-th row of $Q(M^{\mathbf{R}})$. Our induction is based on the possibility, if Q_S has an empty row, of restrict mutations of Q_S to mutations of a certain subquiver of $Q(M^{\mathbf{R}})$, that corresponds to a fixed point for a torus action on suitable quiver Grassmannian $\mathrm{Gr}_d(M^{\mathbf{R}'})$, where $M^{\mathbf{R}'}$ is a representation of the equioriented quiver of type A_{n-1}. We now describe our inductive reasoning in a more accurate way.

Suppose that the j-th row of the quiver Q_S is empty and consider the family of sets $\mathbf{R}' = (R'_1, \dots R'_{n-2})$ such that

$$R'_i = \begin{cases} R_i \setminus \{j\} & \text{if } j \in R_i \\ R_i & \text{otherwise} \end{cases}$$

Up to identifying $\mathbb{C}^n$ as the subspace of $\mathbb{C}^{n+1}$ spanned by the vectors $\{v^i_h\}_{i \neq j}$ for each h, the family $\mathbf{R}'$ defines a linear degeneration $\mathcal{F}l^{\mathbf{R}'}(\mathbb{C}^n)$ of the flag variety $\mathcal{F}l(\mathbb{C}^n)$. Moreover the coefficient quiver $Q(M^{\mathbf{R}'})$ associated $\mathcal{F}l^{\mathbf{R}'}(\mathbb{C}^n)$ is the subquiver of $Q(M^{\mathbf{R}})$ obtained deleting the j-th row and the column corresponding to the vectors $\{v^i_n\}$. Since the j-th row of Q_S is empty, starting from the $\mathbf{R}$-admissible sequence S it is possible to define the $\mathbf{R}'$-admissible sequence S' setting $S'_i = S_i$ for all $i \leqslant n-1$.

Remark 7.2. Let d_j be the number of indecomposable components of $M^{\mathbf{R}}$ that correspond to the connected components of the j-th row of $Q(M^{\mathbf{R}})$. It is possible to define an action of an algebraic torus $T' = (\mathbb{C}^*)^{d_0 - d_j}$ over $\mathrm{Gr}_{\mathbf{e}}(M^{\mathbf{R}'})$ restricting the T action on $\mathrm{Gr}_{\mathbf{e}}(M^{\mathbf{R}})$. The T'-action endows $\mathcal{F}l^{\mathbf{R}'}(\mathbb{C}^n)$ with the structure of GKM variety and the $\mathbf{R}'$-admissible sequence $S' = (S'_1, \dots, S'_{n-1})$ then corresponds to a T'-fixed point $p_{S'}$.

Remark 7.3. By Theorem 2.2, if the linear degeneration $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$ is flat, the degeneration $\mathcal{F}l^{\mathbf{R}'}(\mathbb{C}^n)$ is also flat.

It is natural to ask how the mutations of Q_S are linked to the ones of $Q_{S'}$. The following lemma explains this relation, crucial to prove our results about mutation graphs.

Lemma 7.4. *A mutation from the h-th row of $Q_{S'}$ to the k-th one corresponds to a unique mutation from h-th row to the k-th one of Q_S.*

Proof. First of all, we remark that because of our construction, we can identify the rows of $Q_{S'}$ with the corresponding rows of Q_S. In particular, supposing that $Q_{S'}$ is obtained from Q_S deleting the j-th row and the last column, if we have a mutation $[a,b]^h_{S'} \to k$ then k must be different from j. Observe now that a mutation of the form $[a,b]^h_{S'} \to k$ with $b < n-1$ corresponds to a unique mutation of Q_S because by definition of R', if $b < n$ the segment $[a,b]^h_{Q(M^{\mathbf{R}'})}$ is connected if and only if $[a,b]^h_{Q(M^{\mathbf{R}})}$ is connected and $[a,b]^h_{S'}$ is a movable part if and only if $[a,b]^h_S$ is a movable part. Moreover, by Remark 4.23, a mutation $[a,n-1]^h_{S'} \to k$ lift to a mutation $[a,n-1]^h_S \to k$

if and only if $\pi_{R_{n-1}}(k) = 0$ or $k \in S_n$. The latter condition is always satisfied because $|S_n| = n$ and the Lemma follows. □

A consequence of the previous Lemma is that $\widetilde{G}_{S'}$ embeds into $\widetilde{G}_S$ by the restriction process, i.e. there exists a subgraph of $\widetilde{G}_S$, spanned by vertices $\{1, \ldots, n+1\} \setminus \{j\}$ that is isomorphic to $\widetilde{G}_{S'}$. As a consequence, if $\widetilde{G}_S$ is a complete graph, then $\widetilde{G}_{S'}$ need to be again a complete graph, because $|\mathrm{Mut}(p_{S'})|$ is always greater or equal than $\frac{n(n-1)}{2}$.

Remark 7.5. A closer analysis of the mutations of Q_S and $Q_{S'}$ implies that $\widetilde{G}_{S'}$ is a full subgraph of $\widetilde{G}_S$. We omit the proof of this fact since it is not necessary to prove our results.

Example 7.6. On the left of Figure 21 are displayed a fixed point p_S for the T-action on the flat degeneration $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^5)$ with $\mathbf{R} = (\{3,4\}, \{3\}, \{2,5\})$. On the right there is the successor closed subquiver corresponding to the fixed point $p_{S'}$ obtained from p_S by restriction process. In particular, as observed in the Remark 7.2 it is a fixed point for the T' action on $\mathcal{F}l^{\mathbf{R}'}(\mathbb{C}^5)$ where $\mathbf{R}' = (\{3,4\}, \{3\})$. In Figure 22 the oriented mutation graphs of p_S and of $p_{S'}$ are displayed. In particular the blue full subgraph of $\widetilde{G}_S$ corresponds to the immersion of the oriented mutation graph $\widetilde{G}_{S'}$ induced by the restriction process.

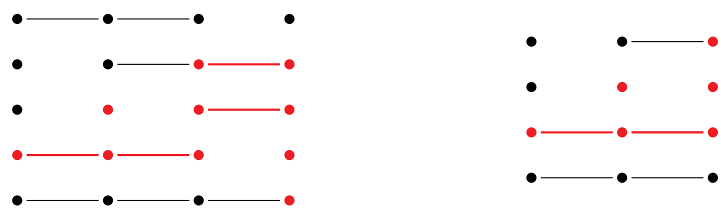


Figure 21. A fixed point p_S (on the left) and its restriction $p_{S'}$ (on the right).

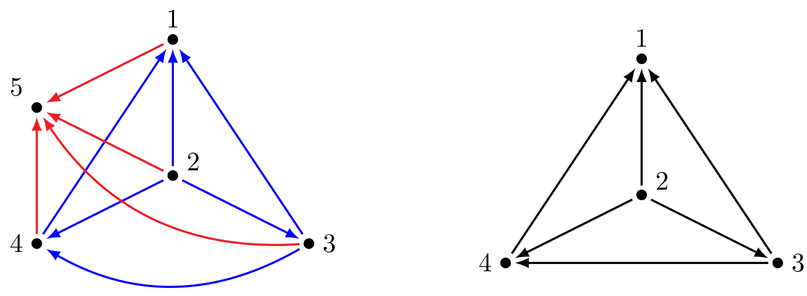


Figure 22. The oriented mutation graphs of p_S and of $p_{S'}$

7.2. Smoothness and mutation graph. In this section we prove that if a point p_S is smooth, then the oriented mutation graph $\widetilde{G}_S$ is a transitive tournament. We deduce preliminary information about the orientation of $\widetilde{G}_S$ as a consequences of Lemmata 7.7 and 7.8 and we use the tools developed in Section 7.1 to show that the condition (1) in Theorem 6.12 implies that the condition (5) holds.

Firstly we remark that the mutation graph G_S is the complete graph over $n+1$ vertices if and only if for every pair of indices $1 \leqslant i, j \leqslant n+1$

$$\mathrm{Mut}_S(i,j) + \mathrm{Mut}_S(j,i) = 1$$

To prove that conditions (1) and (5) in Theorem 6.12 are equivalent, we need two preliminary Lemmata:

Lemma 7.7. *If p_S is smooth, then Q_S has an empty row.*

Proof. Suppose that Q_S does not have empty rows. By Remark 4.16, it is enough to prove that under this assumption $\mathrm{Ext}^1(M_S, M^{\mathbf{R}}/M_S) \neq 0$. By definition of admissible sequence and because Q_S does not have an empty row, there exists $i \in \{1,\dots,n+1\} \smallsetminus S_n$ and h such that $i \in S_h$ and $i \notin S_j$ for all $j \geqslant h$. Set $k_i = \min\{k \in \{1,\dots,n\} \,|\, i \in S_j,\ \forall j\ k \leqslant j \leqslant h\}$ and $m_i = \max\{m \in \mathrm{pos}(i) \,|\, m < h\}$, the segment $[\max\{k_i, m_i\}, h]^i_S$ is a connected component of Q_S ad it corresponds to the indecomposable summand $U^i_{\max\{k_i,m_i\},h}$ of M_S. Moreover, if $P = \min\{p \in \mathrm{pos}(i) \cup \{n\} \,|\, h < p\}$, we have that $[h+1, P]^i_S$ is a connected component of $Q(M^{\mathbf{R}}) \smallsetminus Q_S$ and the representation $U_{h+1,P}$ is an indecomposable summand of $M^{\mathbf{R}}/M_S$. By Formula (3.3) we have $\mathrm{Ext}^1(U_{\max\{k_i,m_i\},h}, U_{h+1,P}) \neq 0$ and p_S cannot be smooth. ☐

Lemma 7.8. *Let p_S be a fixed point for the degeneration $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^{n+1})$. If the j-th row of Q_S is empty, then $|\mathrm{Mut}_S(i,j)| \geqslant 1$ for all $i \in \{1,\dots,n+1\}$, $i \neq j$.*

Proof. First of all observe that such a j is unique, because $|S_n| = n$. For $i \neq j$ set $m_i = \max\{\mathrm{pos}(i) \cup \{0\}\} + 1$ and denote by k_i the minimum of the set $\{h | i \in S_h, m_i \leqslant h\}$ (observe that k_i exists because j is the unique index not in S_n). Set now $m(i,j) = \min\{k \in \mathrm{pos}(j) \cup \{n+1\} \mid k \geqslant k_i\}$. By definition $[k_i, m(i,j)]^i_S$ is a movable part and $[k_i, m(i,j)]^j_{Q(M^{\mathbf{R}})}$ is a connected segment with ending point equal to $n+1$ or to an element in $\mathrm{pos}(j)$. This implies, by Remark 4.23, that we have a mutation $[k_i, m(i,j)]^i_S \to j$ and the lemma is proved. ☐

Remark 7.9. Previous Lemmata highlight also some properties of the oriented mutation graph. In fact Lemma 7.8 implies that if the j-th row of Q_S is empty, then the vertex j is a sink for $\widetilde{G}_S$.

Moreover, in our proof we need some classical results about transitive tournaments. We refer to [11] for a complete survey on the Tournament Graphs and related properties.

Definition 7.10. *Let v be a vertex of a directed graph G, the indegree $\mathrm{id}_G(v)$ (rep. outdegree $\mathrm{od}_G(v)$) of v is the number of edges $e \in E(G)$ such that v is the target (resp. the source) of e.*

The property of being transitive is linked to acyclicity and to the sets of outdegrees and indegrees, respectively.

Theorem 7.11 ([11, Corollary 5a]). *Let T be a tournament graph over $n+1$ vertices. The following are equivalent:*

- *T is a transitive tournament,*
- *T is acyclic,*
- *T does not contain a cycle of length 3,*
- *The set of outdegrees of T is $\{0,\dots,n\}$.*
- *The set of indegrees of T is $\{0,\dots,n\}$.*

In particular a full subgraph of a transitive tournament is again a transitive tournament.

Proposition 7.12. *If the fixed point p_S for $\mathcal{F}l^{\mathbf{R}}(\mathbb{C}^n)$ is smooth then $\widetilde{G}_S$ is a transitive tournament over n vertices.*

Proof. We proceed by induction on n. The base case $n = 1$ is completely trivial. For the general case, firstly we remark that by Lemma 7.7, there exists j such that $j \notin S_h$ for all $h \leqslant n$. Using the restriction process described in Section 7.1, we can consider an $\mathbf{R}'$-admissible sequence S' for the degeneration associated to the quiver $Q(M^{\mathbf{R}'})$ obtained by $Q(M^{\mathbf{R}})$ deleting the j-th row and the last column. We can easily obtain the that $p_{S'}$ is smooth: as a consequence of Lemma 7.4 we have

$$|\mathrm{Mut}(p_{S'})| + \sum_{i\neq j} |\mathrm{Mut}_S(i,j)| \leqslant \sum_{h,k\neq j} |\mathrm{Mut}_S(h,k)| + \sum_{i\neq j} |\mathrm{Mut}_S(i,j)| = \frac{n(n+1)}{2},$$

moreover, Lemma 7.8 implies that $\sum_{i\neq j} |\mathrm{Mut}_S(i,j)| \geqslant n$ and consequently

$$\frac{n(n-1)}{2} \leqslant |\mathrm{Mut}(p_{S'})| \leqslant \frac{n(n+1)}{2} - n.$$

So, by inductive reasoning, we can suppose $\widetilde{G}_{S'}$ to be a transitive tournament over n vertices. The previous computation, combined with the Lemma 7.8, implies that $|\mathrm{Mut}_S(i,j)| = 1$ for every $i \neq j$. As an immediate consequence, the mutation graph G_S is a complete graph. Moreover structure of $\widetilde{G}_S$ is more clear: its full subgraph spanned by the vertices $V(\widetilde{G}_S) \smallsetminus \{j\}$ can be identified with $\widetilde{G}_{S'}$ and the vertex corresponding to j is connected to each other by a single edge. In particular, by Remark 7.9, the vertex j is a sink for $\widetilde{G}_S$ and consequently $\mathrm{id}_{\widetilde{G}_S}(j) = n$. Moreover, the fact that j is a sink implies that the indegree of a vertex $v \neq j$ in $\widetilde{G}_S$ is equal to $\mathrm{id}_{\widetilde{G}_{S'}}(v)$. Consequently the set of indegrees of $\widetilde{G}_S$ is $\{0, \ldots, n\}$ and by Theorem 7.11 we obtain that $\widetilde{G}_S$ is a transitive tournament over $n+1$ vertices. □

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References

[1] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński, *Elements of the representation theory of associative algebras. Vol. 1: Techniques of representation theory*, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006.

[2] Sara Billey and V. Lakshmibai, *Singular loci of Schubert varieties*, Progress in Mathematics, vol. 182, Birkhäuser Boston, Inc., Boston, MA, 2000.

[3] Philippe Caldero and Markus Reineke, *On the quiver Grassmannian in the acyclic case*, J. Pure Appl. Algebra **212** (2008), no. 11, 2369–2380.

[4] Giovanni Cerulli Irelli, *Quiver Grassmannians associated with string modules*, J. Algebraic Combin. **33** (2011), no. 2, 259–276.

[5] Giovanni Cerulli Irelli, Xin Fang, Evgeny Feigin, Ghislain Fourier, and Markus Reineke, *Linear degenerations of flag varieties*, Math. Z. **287** (2017), no. 1-2, 615–654.

[6] Giovanni Cerulli Irelli, Evgeny Feigin, and Markus Reineke, *Quiver Grassmannians and degenerate flag varieties*, Algebra Number Theory **6** (2012), no. 1, 165–194.

[7] Giovanni Cerulli Irelli, Evgeny Feigin, and Markus Reineke, *Degenerate flag varieties: moment graphs and Schröder numbers*, J. Algebraic Combin. **38** (2013), no. 1, 159–189.

[8] E. B. Feĭgin, *PBW degeneration: algebra, geometry and combinatorics*, Proceedings of the Samara University Algebra and Geometry Seminar (Russian), Itogi Nauki Tekh. Ser. Sovrem. Mat. Prilozh. Temat. Obz., vol. 136, Vseross. Inst. Nauchn. i Tekhn. Inform. (VINITI), Moscow, 2017, pp. 3–30.

[9] V. Gasharov and V. Reiner, *Cohomology of smooth Schubert varieties in partial flag manifolds*, J. London Math. Soc. (2) **66** (2002), no. 3, 550–562.

[10] Mark Goresky, Robert Kottwitz, and Robert MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. **131** (1998), no. 1, 25–83.

[11] Frank Harary and Leo Moser, *The theory of round robin tournaments*, Amer. Math. Monthly **73** (1966), 231–246.

[12] Alexander Kirillov, Jr., *Quiver representations and quiver varieties*, Graduate Studies in Mathematics, vol. 174, American Mathematical Society, Providence, RI, 2016.

[13] Martina Lanini and Alexander Pütz, *GKM-theory for torus actions on cyclic quiver Grassmannians*, Algebra Number Theory **17** (2023), no. 12, 2055–2096.

[14] Martina Lanini and Alexander Pütz, *Permutation actions on quiver Grassmannians for the equioriented cycle via GKM-theory*, J. Algebraic Combin. **57** (2023), no. 3, 915–956.

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