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Splicing positroid varieties

Eugene Gorsky & Tonie Scroggin

Abstract We construct an explicit isomorphism between an open subset in the open positroid variety $\Pi^\circ_{k,n}$ in the Grassmannian $\mathrm{Gr}(k,n)$ and the product of two open positroid varieties $\Pi^\circ_{k,n-a+1}\times\Pi^\circ_{k,a+k-1}$. In the respective cluster structures, this isomorphism is given by freezing a certain subset of cluster variables and applying a cluster quasi-equivalence.

1. INTRODUCTION

In this paper, we study certain maps between the open positroid varieties. Recall that the Grassmannian $\mathrm{Gr}(k,n)$ can be realized as the space of $k\times n$ matrices of rank k, modulo row operations. We will denote the columns of such a matrix V by $(v_1,\ldots,v_n)$, and extend them periodically by $v_{i+n}=v_i$. The **open positroid variety** $\Pi^\circ_{k,n}$ is then defined [14] by the condition that the k cyclically consecutive vectors $v_i,v_{i+1},\ldots,v_{i+k-1}$ are linearly independent for all i. Equivalently, the minors $\Delta_{i,i+1,\ldots,i+k-1}(V)$ do not vanish. In the last decades, open positroid varieties (and more general positroid strata in $\mathrm{Gr}(k,n)$) have attracted a lot of attention in combinatorics, algebra, geometry and physics. We focus on two results which motivated this paper.

First, by the work of Scott [7, 15] open positroid varieties carry a **cluster structure** which we review in Section 2.3. In particular, there is a certain collection of minors of V providing cluster coordinates on $\Pi^\circ_{k,n}$. Many more cluster coordinates are obtained from these by mutations.

Second, by the work of Galashin and Lam [10] the torus-equivariant homology of $\Pi^\circ_{k,n}$ is closely related to the **Khovanov-Rozansky homology** HHH of the torus link $T(k,n-k)$. In fact, $\Pi^\circ_{k,n}$ (up to a certain algebraic torus) is isomorphic to a braid variety in the sense of [5, 4], and by [18] the torus-equivariant homology of any braid variety is related to the Khovanov-Rozansky homology of the corresponding link. Furthermore, by [3, 11] any braid variety admits a cluster structure.

The multiplication of braids $T(k,s)\cdot T(k,t)\to T(k,s+t)$ induces a natural map in Khovanov-Rozansky homology

$$\mathrm{HHH}(T(k,s))\otimes\mathrm{HHH}(T(k,t))\to\mathrm{HHH}(T(k,s+t)).$$

This suggests that there might be a map of open positroid varieties:

$$\Pi^\circ_{k,k+s}\times\Pi^\circ_{k,k+t}\to\Pi^\circ_{k,k+s+t}. \tag{1}$$

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In this paper, we construct such a map with particularly nice properties and describe its image.

Given a $k \times n$ matrix $V = (v_1, \dots, v_n)$ as above and $1 \leqslant a \leqslant n-k$, we consider a submatrix $V_1 = (v_a, \dots, v_n)$. This submatrix belongs to $\Pi^\circ_{k,n-a+1}$ if and only if certain additional minors of V do not vanish, which defines an open subset $U_a \subset \Pi^\circ_{k,n}$. On this open subset, we define the second matrix

$$V_2 = (v_1, \dots, v_a, u_1, \dots, u_{k-2}, v_n)$$

where u_i are certain vectors (8) spanning the one-dimensional intersection (see Lemma 3.7):

$$\langle v_a, v_{a+1}, \dots, v_{a+i}\rangle \cap \langle v_{n-k+i+1}, \dots, v_{n-1}, v_n\rangle = \langle u_i\rangle.$$

Diagrammatically, we can visualize the rule for computing u_i in Figure 1. The diagram represents a specific braid word for the half-twist braid, thought of as a subword of a longer torus braid (see Figure 6). The regions in the diagram are labeled by certain subspaces in $\mathbb{C}^k$ such that every vertical cross-section forms a complete flag and two neighboring flags have a prescribed relative position. For example, in Figure 1 we get complete flags

$$0 \subset \langle v_3\rangle \subset \langle v_3, v_4\rangle \subset \langle v_3, v_4, v_5\rangle \subset \langle v_3, v_4, v_5, v_6\rangle \subset \mathbb{C}^5$$

on the left and

$$0 \subset \langle v_{10}\rangle \subset \langle v_9, v_{10}\rangle \subset \langle v_8, v_9, v_{10}\rangle \subset \langle v_7, v_8, v_9, v_{10}\rangle \subset \mathbb{C}^5$$

on the right which determine all intermediate subspaces. See Section 5 for more details. Curiously, this figure resembles the process of mRNA splicing [1, Figure 2].

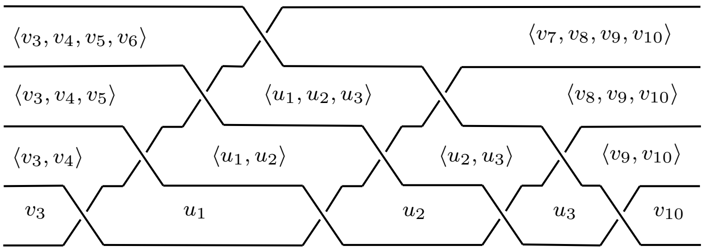

Figure 1. Braid diagram and flags for $k = 5, n = 10$ and $a = 3$. Here $\langle u_1\rangle = \langle v_3, v_4\rangle \cap \langle v_7, v_8, v_9, v_{10}\rangle$, $\langle u_2\rangle = \langle v_3, v_4, v_5\rangle \cap \langle v_8, v_9, v_{10}\rangle$ and $\langle u_3\rangle = \langle v_3, v_4, v_5, v_6\rangle \cap \langle v_9, v_{10}\rangle$.

We can now state our main result.

Theorem 1.1. *Choose an integer a such that $1 \leqslant a \leqslant n-k$.*

a) The map $\Phi_a : V \mapsto (V_1, V_2)$ is well defined and yields an isomorphism

$$U_a \simeq \Pi^\circ_{k,n-a+1} \times \Pi^\circ_{k,a+k-1}.$$

b) The subset U_a admits a cluster structure obtained by freezing certain explicit cluster variables in the rectangle seed for $\Pi^\circ_{k,n}$.

c) The map Φ_a is a cluster quasi-isomorphism (in the sense of [8]*) between U_a and $\Pi^\circ_{k,n-a+1} \times \Pi^\circ_{k,a+k-1}$.*

Remark 1.2. For $a = 1$ we have $V_1 = V \in \Pi^\circ_{k,n}$ and $V_2 \in \Pi^\circ_{k,k} = \mathrm{pt}$, so Φ_1 is the identity map. Most of the time, we will assume $2 \leqslant a \leqslant n-k$ so that Φ_a is nontrivial.

Note that the inverse map $\Phi_a^{-1} : \Pi^\circ_{k,n-a+1} \times \Pi^\circ_{k,a+k-1} \to \Pi^\circ_{k,n}$ has image U_a and fulfills our expectation (1) with $s = n - a + 1 - k$ and $t = a - 1$. We give an interpretation of Φ_a in terms of braid varieties in Section 5. In a future work, we plan to use the explicit description of the map Φ_a to study a more explicit relation between the homology of $\Pi^\circ_{k,n}$ and link homology. For example, in the case $k = 2$ there are no vectors u_i, and the map Φ_a has the form

$$(v_1, \dots, v_n) \mapsto (v_a, \dots, v_n), (v_1, \dots, v_a, v_n).$$

Such a map and its generalizations were studied in detail by the second author in [16], where the corresponding maps on de Rham cohomology were described explicitly.

Finally, we would like to notice a striking similarity between our map and the so-called BCFW recursion in theory of scattering amplitudes. In [6] this recursion was reinterpreted as a rational map (see [6, Definition 4.2]) $\Psi : \Pi_{4,n} \to \Pi_{4,N_L} \times \Pi_{4,N_R}$ which is closely related but slightly different from our Φ_a:

$$\Psi(v_1, \dots, v_n) \mapsto (v_1, \dots, v_a, \widetilde{u_1}, v_n), (v_{a+1}, \dots, v_{n-2}, \widetilde{u_2}, \widetilde{u_3}).$$

Here $\widetilde{u_i}$ are defined similarly to our u_i, but are distributed between both matrices V_1, V_2. As a result, the effect of Ψ on the cluster structure is significantly more complicated, see [6, Theorem 4.7]. It would be interesting to find a knot-theoretic interpretation of Ψ and generalizations of Ψ for $k \neq 4$.

The paper is organized as follows. In Section 2 we recall some background on open positroids and cluster structures. In Section 3 we describe the construction and the main properties of the map Φ_a, and prove Theorem 1.1(a). In Section 3 we consider the cluster structures on all varieties in question and prove Theorem 1.1(b,c). Finally, in Section 5 we relate our constructions to braid varieties.

After this paper first appeared on the arxiv, the authors collaborated with Soyeon Kim and José Simental on [13, 12]. In [13], we introduce a class of positroid varieties called **skew shaped positroids**, and generalized the map Φ_a and Theorem 1.1 for these. In [12], we study even more general splicing maps for braid varieties. See [13, 12] for more details.

2. BACKGROUND

2.1. GRASSMANNIANS AND OPEN POSITROIDS.

The Grassmannian $\mathrm{Gr}(k, n)$ is the space of all k-dimensional subspaces of $\mathbb{C}^n$, presented as the row span of a $k \times n$ matrix V of maximal rank.

Let $v_1, \dots, v_n$ be the columns of V where v_i are k-dimensional vectors. Given an ordered subset $I \in \binom{[n]}{k}$, the *Plücker coordinate* $\Delta_I(V)$ is the minor of $k \times k$ submatrix of V in column set I. We will sometimes consider the exterior algebra $\wedge^\bullet \mathbb{C}^k$, and identify $\Delta_I(V)$ with $v_{i_1} \wedge \cdots \wedge v_{i_k} \in \wedge^k(\mathbb{C}^k) \simeq \mathbb{C}$ for $I = \{i_1, \dots, i_k\}$.

The row operations have the effect of changing V to AV for an invertible $k \times k$ matrix A. This implies $v_i \mapsto Av_i$ and $\Delta_I \mapsto \det(A)\Delta_I$ for all I. In particular, Δ_I can be considered as projective coordinates on $\mathrm{Gr}(k, n)$, or as affine coordinates on the affine cone $\widehat{\mathrm{Gr}}(k, n)$. For a subvariety $X \subset \mathrm{Gr}(k, n)$, we define $\widehat{X} \subset \widehat{\mathrm{Gr}}(k, n)$ as the affine cone over X.

Knutson-Lam-Speyer [14] constructed the stratification

$$\mathrm{Gr}(k, n) = \bigsqcup_{f \in \mathbf{B}_{k,n}} \Pi^\circ_f$$

where Π°_f are open positroid varieties indexed by a finite set $\mathbf{B}_{k,n}$ of bounded affine permutations, see [10, Section 4.1] for more information. This positroid stratification contains a unique open stratum, the top dimensional positroid variety, defined such that cyclically consecutive Plücker coordinates are non-vanishing, i.e.

$$\Pi^\circ_{k,n} := \{V \in \mathrm{Gr}(k,n) : \Delta_{1,2,\ldots,k}(V), \Delta_{2,3,\ldots,k+1}(V), \ldots, \Delta_{n,1,2,\ldots,k-1}(V) \neq 0\}\,.$$

As above, we can also consider the open subset $\widehat{\Pi}^\circ_{k,n} \subset \widehat{\mathrm{Gr}}(k,n)$ defined by the same inequalities. Note that

$$\dim \Pi^\circ_{k,n} = \dim \mathrm{Gr}(k,n) = k(n-k),\ \dim \widehat{\Pi}^\circ_{k,n} = \dim \widehat{\mathrm{Gr}}(k,n) = k(n-k)+1.$$

2.2. CLUSTER ALGEBRAS. Cluster algebras are an interesting class of commutative rings which are defined by a seed s consisting of a quiver, or exchange matrix, and cluster variables, which are a finite collection of algebraically independent elements of the algebra. To simplify the exposition, we are focused on skew-symmetric exchange matrices.

A quiver Q is a directed graph with d vertices and no length 1 or 2 cycles. It corresponds to a skew-symmetric integer matrix ε_{ij} called the exchange matrix defined by

$$(\varepsilon_{ij}) = \begin{cases} a & \text{if there are } a \text{ arrows from vertex } i \text{ to vertex } j; \\ -a & \text{if there are } a \text{ arrows from vertex } j \text{ to vertex } i; \\ 0 & \text{otherwise} \end{cases}$$

The vertices of Q can be either mutable or frozen. Given a mutable vertex k, one can define the new quiver $\mu_k(Q)$ using the following steps:

1. If there is a path of the vertices $i \to k \to j$, then we add an arrow from i to j.
2. Any arrows incident to k change orientation.
3. Remove a maximal disjoint collection of 2-cycles produced in Steps (1) and (2).

The cluster variables $A_1, \ldots, A_d$ corresponding to the initial seed are a collection of algebraically independent variables, labeled by the vertices of Q. We assume that the variables corresponding to frozen vertices are invertible. For each seed s and each mutable variable A_k, there is another chart $\mu_k(s)$ with the quiver $\mu_k(Q)$ and cluster coordinates $A_1, \ldots, A'_k, \ldots, A_d$ where A'_k is defined by

$$A'_k = \frac{\left(\prod_{i:\varepsilon_{ki}>0} A_i^{\varepsilon_{ki}} + \prod_{i:\varepsilon_{ki}<0} A_i^{-\varepsilon_{ki}}\right)}{A_k} \tag{2}$$

The cluster algebra $\mathcal{A}_Q$ is a subalgebra of the field $\mathbb{C}(A_1, \ldots, A_d)$ of rational functions in the initial seed, generated by all cluster variables in all seeds. The rank of a cluster algebra is the number of mutable vertices of Q.

The cluster variety $X = X_Q$ is an affine algebraic variety defined as $X = \mathrm{Spec}\mathcal{A}_Q$. It has the algebra of regular functions $\mathbb{C}[X] \simeq \mathcal{A}_Q$ and field of rational functions $\mathbb{C}(X)$. Geometrically, a seed s of $\mathcal{A}_Q$ correspond to an open toric chart $U \simeq (\mathbb{C}^*)^d \subset X$ parametrized by the cluster coordinates $A_1, \ldots, A_d$ in s, which are invertible on U and extend to regular functions on X. The frozen variables correspond to global invertible functions on X. For more details on cluster algebras and cluster varieties (and the more general case of skew-symmetrizable exchange matrices), see [7, 19].

Additionally, for each mutable variable A_k we define the exchange ratio $\widehat{y_k}$ as the ratio of incoming and outgoing cluster variables :

$$\widehat{y_k} = \frac{\prod_{i:\varepsilon_{ki}>0} A_i^{\varepsilon_{ki}}}{\prod_{i:\varepsilon_{ki}<0} A_i^{-\varepsilon_{ki}}}.$$

We will need the notion of exchange ratios in the definition of quasi-equivalences.

Definition 2.1 ([8, 9]). *Let Σ and Σ_0 be seeds of rank r in $\mathbb{C}(X)$. Let $Q, A_i, \hat{y}_i$ denote the quiver, cluster variables and exchange ratios in Σ and use primes to denote these quantities in Σ_0. We assume that $A_{r+1}, \dots, A_d$ are frozen. Then Σ and Σ_0 are quasi-equivalent, denoted $\Sigma \sim \Sigma_0$, if the following hold:*

- *The groups $\mathbf{P}, \mathbf{P}_0 \subset \mathbb{C}[X]$ of Laurent monomials in frozen variables coincide. That is, each frozen variable A'_i is a Laurent monomial in $\{A_{r+1}, \dots, A_d\}$ and vice versa.*
- *Corresponding mutable variables coincide up to multiplication by an element of $\mathbf{P}$: for $i \in [r]$, there is a Laurent monomial $M_i \in \mathbf{P}$ such that $A_i = M_i A'_i \in \mathbb{C}(X)$.*
- *The exchange ratios (3) coincide: $\hat{y}_i = \hat{y}'_i$ for $i \in [r]$.*

Quasi-equivalence is an equivalence relation on seeds. Seeds Σ, Σ_0 are related by a quasi-cluster transformation if there exists a finite sequence $\underline{\mu}$ of mutations such that $\underline{\mu}(\Sigma) \sim \Sigma_0$.

By the main result of [8], it is sufficient to check the conditions of quasi-equivalence for one cluster in Σ and Σ_0, and this automatically implies quasi-equivalence between all corresponding clusters in Σ and Σ_0.

2.3. Cluster structures on open positroids.

In 2003, Scott [15] established that the coordinate ring of $\widehat{\mathrm{Gr}}(k,n)$ has a cluster structure using Postnikov arrangements. In this paper, will we use a different construction using rectangles seed $\Sigma_{k,n}$ that generate the cluster structure for the Plücker ring $R_{k,n}$ isomorphic to $\mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]$ as detailed in [7, Section 6.7]. The cluster structure in the Plücker ring $R_{k,n}$ is generated from the mutations on the rectangular seed $\Sigma_{k,n}$. Since (unlike [15]) we always assume that the frozen variables are invertible, we in fact consider a cluster structure on $\mathbb{C}[\widehat{\Pi}^\circ_{k,n}]$.

We first construct the quiver $Q_{k,n}$ where vertices are labeled by rectangles r contained in the $k \times (n-k)$ rectangle R along with the empty rectangle $\varnothing$. The frozen vertices are defined as rectangles of size $k \times j$ for $1 \leqslant j \leqslant n-k$, size $i \times (n-k)$ for $1 \leqslant i \leqslant k$, and $\varnothing$. The arrows connect from the $i \times j$ rectangle to the $i \times (j+1)$ rectangle, the $(i+1) \times j$ rectangle, and the $(i-1) \times (j-1)$ rectangle with the conditions that the rectangle has nonzero dimension, fits inside of R and does not connect two frozens. There is also an arrow from the $\varnothing$ rectangle to the 1×1 rectangle, see Figure 2.

Each rectangle r contained in the $k \times (n-k)$ rectangle R corresponds to a k-element subset of $[n]$ representing a Plücker coordinate. This correspondence is determined by positioning r in R such that the upper left corner coincides with the upper left corner of R. There exists a path from the upper right corner to the lower left corners of R which traces out the smaller rectangle r, with steps from 1 to n, where the map from r to $I(r)$ is given by the vertical steps of the path, see Figure 3. Define

$$\tilde{x}^{k,n} = \{\Delta_{I(r)} : r \text{ rectangle contained in } k \times (n-k) \text{ rectangle}\}$$

We may now define the rectangles seed $\Sigma_{k,n} = (\tilde{x}^{k,n}, \varepsilon_{ij}(Q_{k,n}))$.

We can summarize (and slightly rephrase) the above constructions as follows. We define ordered subsets

$$I(a,i) = \{a, a+1, \dots, a+i-1, n-k+i+1, \dots, n\}, \tag{3}$$

where $a = n-k-j+1$.

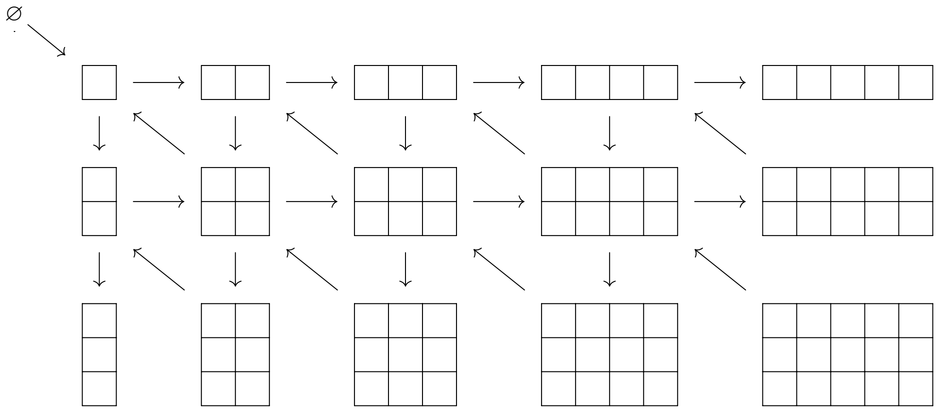

Figure 2. The quiver $Q_{3,8}$. Vertices are labeled by rectangles contained in a 3×5 rectangle. The grid is arranged such that the rectangles width increases from left to right and the heights increase from top to bottom.

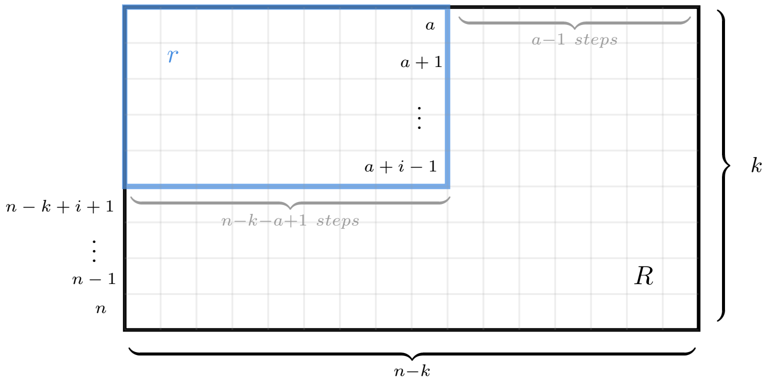

Figure 3. The Plücker coordinate $\Delta_{I(r)}$ corresponding to a rectangle r is given by the vertical steps in the path from the upper right corner to the lower left corner of the rectangle R of size $k\times(n-k)$ that cuts out the rectangle r positioned in the upper left corner of R.

Theorem 2.2 ([15]). *The cluster variables in the initial seed are given by the minors $\Delta_{I(a,i)}$ for $1\leqslant a\leqslant n-k$ and $1\leqslant i\leqslant k$, and an additional frozen variable $\Delta_{n-k+1,\ldots,n}$. Furthermore:*

1) The variables $\Delta_{I(a,i)}$ are frozen for $a=1$ and $i=k$, and mutable otherwise.

2) The quiver $Q_{k,n}$ consists of the following arrows:

$$\begin{tikzcd}
\Delta_{I(a,i)} \arrow[r] \arrow[d] & \Delta_{I(a-1,i)} \arrow[d] \\
\Delta_{I(a,i+1)} \arrow[r] & \Delta_{I(a-1,i+1)} \arrow[ul]
\end{tikzcd} \tag{4}$$

3) There is an additional arrow $\Delta_{n-k+1,\ldots,n}\to\Delta_{I(n-k,1)}$.

$$\begin{tikzcd}
\boxed{\Delta_{678}} \arrow[dr] & & & & & \\
& \Delta_{578} \arrow[r] \arrow[d] & \Delta_{478} \arrow[r] \arrow[d] & \Delta_{378} \arrow[r] \arrow[d] & \Delta_{278} \arrow[r] \arrow[d] & \boxed{\Delta_{178}} \\
& \Delta_{568} \arrow[r] \arrow[d] & \Delta_{458} \arrow[r] \arrow[d] \arrow[ul] & \Delta_{348} \arrow[r] \arrow[d] \arrow[ul] & \Delta_{238} \arrow[r] \arrow[d] \arrow[ul] & \boxed{\Delta_{128}} \arrow[ul] \\
& \boxed{\Delta_{567}} & \boxed{\Delta_{456}} \arrow[ul] & \boxed{\Delta_{345}} \arrow[ul] & \boxed{\Delta_{234}} \arrow[ul] & \boxed{\Delta_{123}} \arrow[ul]
\end{tikzcd}$$

FIGURE 4. Cluster variables in $\widehat{\Pi}^\circ_{3,8}$ corresponding to the rectangles seed.

See Figure 4 for the $\widehat{\Pi}^\circ_{3,8}$ example and Figure 5 for the general case.

Below we will also use the cluster structure on $\Pi^\circ_{k,n}$. The corresponding quiver can be obtained from $Q_{k,n}$ by deleting the vertex $\varnothing$, and the cluster variables in the initial seed are given by the ratios $\Delta_{I(a,i)}/\Delta_{n-k+1,\ldots,n}$. More precisely, we have the following:

PROPOSITION 2.3. *We have a quasi-equivalence of cluster varieties*

$$\widehat{\Pi}^\circ_{k,n} \simeq \mathbb{C}^* \times \Pi^\circ_{k,n} \tag{5}$$

and the corresponding quasi-equivalence of cluster algebras

$$\mathbb{C}[\widehat{\Pi}^\circ_{k,n}] \simeq \mathbb{C}[\Delta^{\pm}_{n-k+1,\ldots,n}] \otimes \mathbb{C}[\Pi^\circ_{k,n}].$$

Proof. This result is well known, but we provide a proof for the sake of completeness.

Define the map $f : \widehat{\Pi}^\circ_{k,n} \to \mathbb{C}^* \times \Pi^\circ_{k,n}$ by sending each Plücker coordinate $\Delta_{I(a,i)}$ to $\widetilde{\Delta}_{I(a,i)} := \Delta_{I(a,i)}/\Delta_{n-k+1,\ldots,n}$. We note that this map is well defined since $\Delta_{n-k+1,\ldots,n} = \Delta_{I(n-k+1,k)}$ is nonzero by definition of $\widehat{\Pi}^\circ_{k,n}$. We show that this map defines a quasi-equivalence by verifying that it preserves exchange ratios in the cases illustrated in Figure 5.

(a) Left corner: In $\mathbb{C}[\widehat{\Pi}^\circ_{k,n}]$, the mutable variable $\Delta_{I(n-k,1)}$ in the left corner has a total of two incoming arrows and two outgoing arrows. However, under the map f the cluster variable $\Delta_{I(n-k+1,k)}$ is mapped to 1, and therefore, the arrow from $\widetilde{\Delta}_{I(n-k+1,k)}$ to $\widetilde{\Delta}_{I(n-k,1)}$ vanishes. We now have that the mutable variable $\widetilde{\Delta}_{I(n-k,1)}$ in $\mathbb{C}[\Pi^\circ_{k,n}]$ has one incoming arrow and two outgoing arrows. However, we see that the exchange ratios under the map f are equivalent:

$$\widehat{y}_{\widetilde{\Delta}_{I(n-k,1)}} = \frac{\frac{\Delta_{I(n-k-1,2)}}{\Delta_{I(n-k+1,k)}}}{\frac{\Delta_{I(n-k-1,1)}}{\Delta_{I(n-k+1,k)}}\frac{\Delta_{I(n-k,2)}}{\Delta_{I(n-k+1,k)}}} = \frac{\Delta_{I(n-k-1,2)}\Delta_{I(n-k+1,k)}}{\Delta_{I(n-k-1,1)}\Delta_{I(n-k,2)}} = \widehat{y}_{\Delta_{I(n-k,1)}}.$$

(b) Boundary: Either $2 \leqslant a \leqslant n-k-1$ and $i=1$, or $a = n-k$ and $2 \leqslant i \leqslant k-1$. In $\mathbb{C}[\widehat{\Pi}^\circ_{k,n}]$, the mutable variable $\Delta_{I(a,i)}$ has two incoming arrows and two outgoing arrows. Under the map f, the mutable variable $\widetilde{\Delta}_{I(a,i)}$ in $\mathbb{C}[\Pi^\circ_{k,n}]$ still has two incoming and outgoing arrows where each of the corresponding variables have a factor of $(\Delta_{I(n-k+1,k)})^{-1}$ which cancels in the computation of the exchange ratio $y_{\widetilde{\Delta}_{I(a,i)}}$.

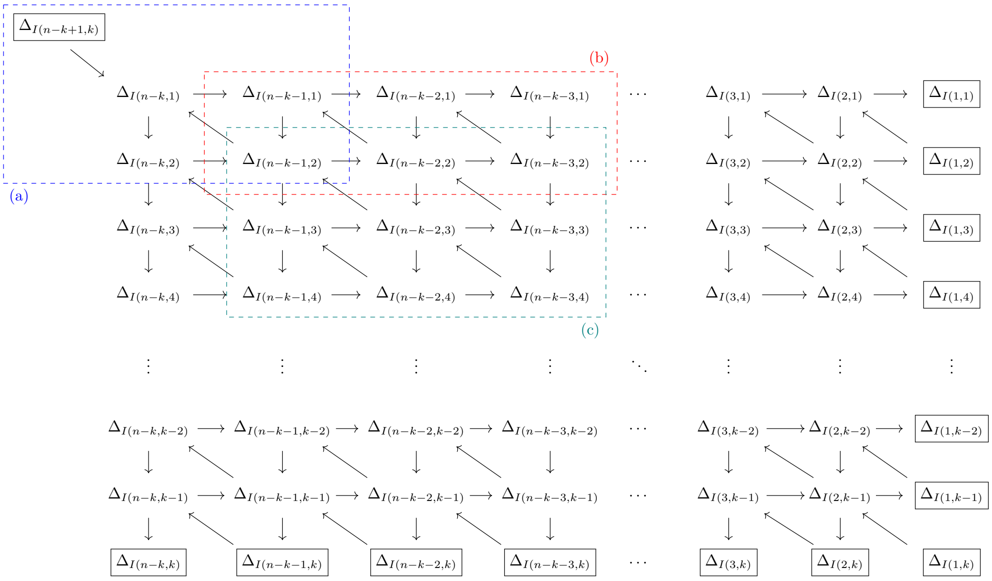

FIGURE 5. Cluster variables in $\widehat{\Pi}^\circ_{k,n}$ in the rectangle seed.

(c) Interior: Similarly to the boundary case, the mutable variable $\Delta_{I(a,i)}$ in $\mathbb{C}[\widehat{\Pi}^\circ_{k,n}]$ and $\widetilde{\Delta}_{I(a,i)}$ in $\mathbb{C}[\Pi^\circ_{k,n}]$ both have three incoming arrows and three outgoing arrows. Therefore, the factor $(\Delta_{I(n-k+1,k)})^{-1}$ cancels out in the computation of the exchange ratio $y_{\widetilde{\Delta}_{I(a,i)}}$. □

Thanks to Proposition 2.3, below we will freely translate various results and computations between the cluster structures on $\widehat{\Pi}^\circ_{k,n}$ and on $\Pi^\circ_{k,n}$. In particular, we will always compute the exchange ratios in $\widehat{\Pi}^\circ_{k,n}$, since they coincide with the ones in $\Pi^\circ_{k,n}$.

3. CONSTRUCTION

Let $\Pi^\circ_{k,n}$ be the open positroid variety in the Grassmannian $\mathrm{Gr}(k,n)$.

DEFINITION 3.1. *Given $2 \leqslant a \leqslant n-k$, we define an open subset $U_a \subset \Pi^\circ_{k,n}$ by the inequalities*

$$U_a = \{\Delta_{I(a,i)} \neq 0,\ 1 \leqslant i \leqslant k-1\}. \tag{6}$$

For $0 \leqslant s \leqslant i-1$ and $0 \leqslant t \leqslant k-i-1$ we define ordered subsets

$$I'(a,s,i) = \{a,\ldots,a+s-1,a+i,a+s+1,\ldots,a+i-1,n-k+i+1,\ldots,n\},$$

$$I'_{\text{sort}}(a,s,i) = \{a,\ldots,a+s-1,a+s+1,\ldots,a+i-1,a+i,n-k+i+1,\ldots,n\},$$

and

$$I''(a,t,i) = \{a,\ldots,a+i-1,n-k+i+1,\ldots,n-t-1,a+i,n-t+1,\ldots,n\},$$

$$I''_{\text{sort}}(a,t,i) = \{a,\ldots,a+i-1,a+i,n-k+i+1,\ldots,n-t-1,n-t+1,\ldots,n\},$$

Note that $I'(a,s,i)$ is obtained from $I(a,i)$ by replacing $a+s$ by $a+i$ (without changing the order), while $I''(a,t,i)$ is obtained from $I(a,i)$ by replacing $n-t$ by $a+i$. Also, $I'(a,s,i)$ and $I'_{\text{sort}}(a,s,i)$ are related by an $(i-s)$-cycle while $I''(a,t,i)$ and $I''_{\text{sort}}(a,t,i)$ are related by a $(k-i-1-t)$-cycle.

Lemma 3.2. *Given a matrix $V \in U_a$, for all $1 \leqslant i \leqslant k-2$ we have an identity*

$$\begin{aligned} v_{a+i} &= \sum_{s=0}^{i-1} \frac{\Delta_{I'(a,s,i)}}{\Delta_{I(a,i)}} v_{a+s} + \sum_{t=0}^{k-i-1} \frac{\Delta_{I''(a,t,i)}}{\Delta_{I(a,i)}} v_{n-t} \\ &= \sum_{s=0}^{i-1} (-1)^{i-s-1} \frac{\Delta_{I'_{\mathrm{sort}}(a,s,i)}}{\Delta_{I(a,i)}} v_{a+s} + \sum_{t=0}^{k-i-1} (-1)^{k-i-t} \frac{\Delta_{I''_{\mathrm{sort}}(a,t,i)}}{\Delta_{I(a,i)}} v_{n-t} \end{aligned}$$

Proof. Since $\Delta_{I(a,i)}(V) \neq 0$, the vectors $v_a, \ldots, v_{a+i-1}, v_{n-k+i+1}, \ldots, v_n$ span $\mathbb{C}^k$. We can uniquely write v_{a+i} as linear combination of these:

$$v_{a+i} = x_0 v_a + \ldots + x_{i-1} v_{a+i-1} + y_{k-i-1} v_{n-k+i+1} + \ldots + y_0 v_n .$$

Now the coefficients x_s, y_t are determined by Cramer's Rule:

$$x_s = \frac{\Delta_{I'(a,s,i)}}{\Delta_{I(a,i)}} = (-1)^{i-s-1} \frac{\Delta_{I'_{\mathrm{sort}}(a,s,i)}}{\Delta_{I(a,i)}}, \ y_t = \frac{\Delta_{I''(a,t,i)}}{\Delta_{I(a,i)}} = (-1)^{k-i-t} \frac{\Delta_{I''_{\mathrm{sort}}(a,t,i)}}{\Delta_{I(a,i)}}. \ \square$$

Example 3.3. For $a = 3$ the open subset $U_3 \subset \Pi^\circ_{3,8}$ is defined by $\Delta_{348}, \Delta_{378} \neq 0$, indicating that the vectors v_3, v_7, v_8 span $\mathbb{C}^3$ and the vectors v_3, v_4, v_8 span $\mathbb{C}^3$. Using Cramer's rule we may express the vector v_4 as

$$v_4 = \frac{\Delta_{478}}{\Delta_{378}} v_3 + \frac{\Delta_{348}}{\Delta_{378}} v_7 + \frac{\Delta_{374}}{\Delta_{378}} v_8 = \frac{\Delta_{478}}{\Delta_{378}} v_3 + \frac{\Delta_{348}}{\Delta_{378}} v_7 - \frac{\Delta_{347}}{\Delta_{378}} v_8$$

Example 3.4. The open subset $U_3 \subset \Pi^\circ_{5,10}$ is defined by

$$\Delta_{3,7,8,9,10}, \Delta_{3,4,8,9,10}, \Delta_{3,4,5,9,10}, \Delta_{3,4,5,6,10} \neq 0.$$

From this collection of nonvanishing Plücker coordinates we have

$$\mathbb{C}^5 = \langle v_3, v_7, v_8, v_9, v_{10} \rangle = \langle v_3, v_4, v_8, v_9, v_{10} \rangle = \langle v_3, v_4, v_5, v_9, v_{10} \rangle.$$

By Cramer's Rule, we can expand v_4, v_5, v_6 in the respective bases:

$$\begin{aligned} v_4 &= \frac{\Delta_{4,7,8,9,10}}{\Delta_{3,7,8,9,10}} v_3 + \frac{\Delta_{3,4,8,9,10}}{\Delta_{3,7,8,9,10}} v_7 - \frac{\Delta_{3,4,7,9,10}}{\Delta_{3,7,8,9,10}} v_8 + \frac{\Delta_{3,4,7,8,10}}{\Delta_{3,7,8,9,10}} v_9 - \frac{\Delta_{3,4,7,8,9}}{\Delta_{3,7,8,9,10}} v_{10} \\ v_5 &= -\frac{\Delta_{4,5,8,9,10}}{\Delta_{3,4,8,9,10}} v_3 + \frac{\Delta_{3,5,8,9,10}}{\Delta_{3,4,8,9,10}} v_4 + \frac{\Delta_{3,4,5,9,10}}{\Delta_{3,4,8,9,10}} v_8 - \frac{\Delta_{3,4,5,8,10}}{\Delta_{3,4,8,9,10}} v_9 + \frac{\Delta_{3,4,5,8,9}}{\Delta_{3,4,8,9,10}} v_{10} \\ v_6 &= \frac{\Delta_{4,5,6,9,10}}{\Delta_{3,4,5,9,10}} v_3 - \frac{\Delta_{3,5,6,9,10}}{\Delta_{3,4,5,9,10}} v_4 + \frac{\Delta_{3,4,6,9,10}}{\Delta_{3,4,5,9,10}} v_5 + \frac{\Delta_{3,4,5,6,10}}{\Delta_{3,4,5,9,10}} v_9 - \frac{\Delta_{3,4,5,6,9}}{\Delta_{3,4,5,9,10}} v_{10} \end{aligned}$$

Definition 3.5. *Given a matrix $V = (v_1, \ldots, v_n)$ in U_a, we define two matrices V_1, V_2 as follows:*

$$V_1 = (v_a, \ldots, v_n), \ V_2 = (v_1, \ldots, v_a, u_1 \ldots, u_{k-2}, v_n) \tag{7}$$

where

$$u_i = v_{a+i} - \sum_{s=0}^{i-1} \frac{\Delta_{I'(a,s,i)}}{\Delta_{I(a,i)}} v_{a+s} = \sum_{t=0}^{k-i-1} \frac{\Delta_{I''(a,t,i)}}{\Delta_{I(a,i)}} v_{n-t}. \tag{8}$$

The second equation in (8) follows from Lemma 3.2.

Example 3.6. Continuing Example 3.3, we decompose the matrix

$$V = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{pmatrix}$$

into

$$V_1 = \begin{pmatrix} v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{pmatrix}, \quad V_2 = \begin{pmatrix} v_1 & v_2 & v_3 & u & v_8 \end{pmatrix}$$

where

$$u = v_4 - \frac{\Delta_{478}}{\Delta_{378}} v_3 = \frac{\Delta_{348}}{\Delta_{378}} v_7 - \frac{\Delta_{347}}{\Delta_{378}} v_8 .$$

Lemma 3.7. *Assume that $V \in U_a$. Then for $1 \leqslant i \leqslant k-2$ the intersection of two subspaces*

$$\langle v_a, v_{a+1}, \dots, v_{a+i}\rangle \cap \langle v_{n-k+i+1}, v_{n-k+i+2}, \dots, v_n\rangle.$$

is one-dimensional and spanned by the vector u_i.

Proof. By (8) and Lemma 3.2 the vector u_i is indeed contained in this intersection. Since $\Delta_{I(a,i+1)} \neq 0$, the vectors $v_a, \dots, v_{a+i}$ are linearly independent and hence $u_i \neq 0$. Since $\Delta_{I(a,i)} \neq 0$, the vectors $v_{n-k+i+1}, \dots, v_n$ are linearly independent as well and altogether the two subspaces span $\mathbb{C}^k$. Now

$$\dim\langle v_a, v_{a+1}, \dots, v_{a+i}\rangle \cap \langle v_{n-k+i+1}, v_{n-k+i+2}, \dots, v_n\rangle = (i+1)+(k-i)-k = 1. \ \square$$

Lemma 3.8. *a) We have*

$$v_a \wedge u_1 \wedge \dots \wedge u_i = v_a \wedge v_{a+1} \wedge \dots \wedge v_{a+i}.$$

b) We have

$$u_i \wedge \dots \wedge u_{k-2} \wedge v_n = \frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,i)}} v_{n-k+i+1} \wedge \dots \wedge v_{n-1} \wedge v_n.$$

Proof. a) By (8) we have

$$u_i \in v_{a+i} + \langle v_a, \dots, v_{a+i-1}\rangle,$$

so

$$v_a \wedge u_1 \wedge \dots \wedge u_i = v_a \wedge (v_{a+1}+\dots) \wedge \dots (v_{a+i}+\dots) = v_a \wedge v_{a+1} \wedge \dots \wedge v_{a+i}.$$

b) Similarly, by the second equation in (8) we have

$$u_i \in \frac{\Delta_{I''(a,k-i-1,i)}}{\Delta_{I(a,i)}} v_{n-k+i+1} + \langle v_{n-k+i+2}, \dots, v_n\rangle.$$

Note that $I''(a,k-i-1,i)$ is obtained from $I(a,i) = \{a, \dots, a-i-1, n-k+i+1, \dots, n\}$ by replacing $n-k+i+1$ with $a+i$, so in fact $I''(a,k-i-1,i) = I(a,i+1)$. Now

$$u_i \wedge \dots \wedge u_{k-2} \wedge v_n = \left(\frac{\Delta_{I(a,i+1)}}{\Delta_{I(a,i)}} v_{n-k+i+1} + \dots\right) \wedge \dots \wedge \left(\frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,k-2)}} v_{n-1} + \dots\right) \wedge v_n =$$

$$\frac{\Delta_{I(a,i+1)}}{\Delta_{I(a,i)}} \cdot \frac{\Delta_{I(a,i+2)}}{\Delta_{I(a,i+1)}} \cdots \frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,k-2)}} v_{n-k+i+1} \wedge \dots \wedge v_n.$$

The factors in the coefficient cancel pairwise except for $\Delta_{I(a,k-1)}/\Delta_{I(a,i)}$. $\square$

Lemma 3.9. *If $V \in U_a$ then $V_1 \in \Pi^\circ_{k,n-a+1}$ and $V_2 \in \Pi^\circ_{k,a+k-1}$.*

Proof. The first statement is clear by the definition of U_a. To prove the second one, we need to compute the following minors:

1) $\Delta_{b,\dots,b+k-1}(V_2)$, $b+k-1 \leqslant a$. This minor does not change, so $\Delta_{b,\dots,b+k-1}(V_2) = \Delta_{b,\dots,b+k-1}(V) \neq 0$.

2) $\Delta_{b,\dots,b+k-1}(V_2)$, $b < a < b+k-1$. Let $i = b+k-1-a$, then by Lemma 3.8(a) we get

$$\Delta_{b,\dots,b+k-1}(V_2) = v_b \wedge \dots \wedge v_a \wedge u_1 \wedge \dots \wedge u_i =$$
$$v_b \wedge \dots \wedge v_a \wedge v_{a+1} \wedge \dots \wedge v_{a+i} = \Delta_{b,\dots,b+k-1}(V) \neq 0.$$

3) $\Delta_{a,\dots,a+k-1}(V_2) = v_a \wedge u_1 \wedge \dots \wedge u_{k-2} \wedge v_n = v_a \wedge \dots \wedge v_{a+k-2} \wedge v_n \neq 0$ by definition of U_a.

4) Finally, we need to consider the minor $u_i \wedge \dots \wedge u_{k-2} \wedge v_n \wedge v_1 \cdots v_i$ which by Lemma 3.8(b) equals

$$\frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,i)}} v_{n-k+i+1} \wedge \dots \wedge v_n \wedge v_1 \cdots v_i = \frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,i)}} \Delta_{n-k+i+1,\dots,n,1,\dots,i} \neq 0. \ \square$$

Theorem 3.10. *The map $\Phi_a : V \mapsto (V_1, V_2)$ defined by* (7) *is an isomorphism between $U_a \subset \Pi^\circ_{k,n}$ and the product $\Pi^\circ_{k,n-a+1} \times \Pi^\circ_{k,a+k-1}$.*

Remark 3.11. We have $\dim U_a = \dim \Pi^\circ_{k,n} = k(n-k)$ while

$$\dim \Pi^\circ_{k,n-a+1} + \dim \Pi^\circ_{k,a+k-1} = k(n-a+1-k) + k(a-1) = k(n-k).$$

Proof. By Lemma 3.9 the map $\Phi_a : U_a \to \Pi^\circ_{k,n-a+1} \times \Pi^\circ_{k,a+k-1}$ is well defined. We need to construct the inverse map, reconstructing V from V_1 and V_2. Since V_1 and V_2 are both defined up to row operations, we need to choose appropriate representatives in their equivalence classes and make sure that they glue correctly to V.

For V_1, choose a representative in the equivalence class arbitrarily and label the column vectors by $(v_a, \ldots, v_n)$. Since $V_1 \in \Pi^\circ_{k,n-a+1}$, we have $\Delta_{I(a,i)} \neq 0$. By Lemma 3.2, we can define the vectors $u_1, \ldots, u_{k-2}$ by (8). Applying row operations to V_1 is equivalent to the multiplication by an invertible $(k \times k)$ matrix A on the left. It transforms v_i to Av_i, multiplies all the minors of V_1 by $\det A$, and transforms u_i to (9)

$$u_i \to Av_{a+i} - \sum_{j=0}^{i-1} \frac{\Delta_{I'(a,j,i)} \det(A)}{\Delta_{I(a,i)} \det(A)} (Av_{a+j}) = A\left[v_{a+i} - \sum_{j=0}^{i-1} \frac{\Delta_{I'(a,j,i)}}{\Delta_{I(a,i)}} v_{a+j}\right] = Au_i.$$

By Lemma 3.8(a) we get $v_a \wedge u_1 \cdots u_{k-2} \wedge v_n = v_a \wedge v_{a+1} \cdots v_{a+k-2} \wedge v_n$. This is nonzero since $V_1 \in \Pi^\circ_{k,n-a+1}$, so the vectors $v_a, u_1, \ldots, u_{k-2}, v_n$ form a basis of $\mathbb{C}^k$. Therefore we can uniquely find a representative for V_2 of the form $V_2 = (v_1, \ldots, v_{a-1}, v_a, u_1, \ldots, u_{k-2}, v_n)$. Indeed, if $V_2' = (v_1', \ldots, v_{a+k-1}')$ is some other representative then

$$V_2 = (v_a, u_1, \ldots, u_{k-2}, v_n)(v_a', \ldots, v_{a+k-1}')^{-1} V_2'.$$

By (9), row operations $V_1 \mapsto AV_1$ also change $V_2 \to AV_2$. Now we can define $V = (v_1, \ldots, v_{a-1}, v_a, \ldots, v_n)$ where the vectors $v_1, \ldots, v_{a-1}$ are the first $(a-1)$ columns of V_2 and $(v_a, \ldots, v_n) = V_1$. By the above, this is well defined up to row operations.

Similarly to the proof of Lemma 3.9, one can check that $V \in \Pi^\circ_{k,n}$, and $V_1 \in \Pi^\circ_{k,n-a+1}$ immediately implies that $V \in U_a$. This completes the proof. ☐

4. Cluster algebra interpretation

We would like to compare the quivers and cluster coordinates (4) for the matrices V, V_1 and V_2, which we denote by Q_V, Q_{V_1} and Q_{V_2}. As in Proposition 2.3, we prefer to work with the affine cones $\widehat{U_a}$, $\widehat{\Pi}^\circ_{k,n-a+1}$ and $\widehat{\Pi}^\circ_{k,a+k-1}$. By construction, the empty rectangle in both Q_V and Q_{V_1} corresponds to $\Delta_{n-k+1,\ldots,n}(V)$. On the other hand, by Lemma 3.8(a) the empty rectangle in Q_{V_2} corresponds to the minor

$$\begin{aligned}\Delta_{I(a,k)}(V_2) = v_a \wedge u_1 \wedge \cdots u_{k-2} \wedge v_n &= v_a \wedge v_{a+1} \wedge \cdots v_{a+k-2} \wedge v_n \\ &= \Delta_{I(a,k-1)}(V) = \Delta_{I(1,k-1)}(V_1)\end{aligned}$$

which is connected to $\Delta_{a-1,1}(V_2)$.

Clearly, the open subset $U_a \subset \Pi^\circ_{k,n}$ is defined by freezing the cluster variables $\Delta_{I(a,i)}(V)$ in Q_V, which are identified with $\Delta_{I(1,i)}(V_1)$. We need to analyze the behavior of all other minors in Q_V under Φ_a.

Lemma 4.1. *a) If $b \geqslant a$ then $\Delta_{I(b,i)}(V) = \Delta_{I(b-a+1,i)}(V_1)$.*

b) If $b < a$ then

$$\frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,i)}} \Delta_{I(b,i)}(V) = \Delta_{I(b,i)}(V_2).$$

Proof. Part (a) is clear from (7). For part (b), we first assume $b+i-1 \geqslant a$ and write

$$\Delta_{I(b,i)}(V_2) = v_b \wedge \cdots v_a \wedge (u_1 \wedge \cdots \wedge u_{i-(a-b+1)}) \wedge (u_i \wedge \cdots \wedge u_{k-2} \wedge v_n).$$

By Lemma 3.8 we get

$$v_a \wedge u_1 \wedge \cdots \wedge u_{i-(a-b+1)} = v_a \wedge v_{a+1} \wedge \cdots \wedge v_{b+i-1}$$

and

$$u_i \wedge \cdots \wedge u_{k-2} \wedge v_n = \frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,i)}} v_{n-k+i+1} \wedge \cdots \wedge v_{n-1} \wedge v_n,$$

so

$$\begin{aligned}\Delta_{I(b,i)}(V_2) &= \frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,i)}}(v_b \wedge \cdots v_{b+i-1}) \wedge (v_{n-k+i+1} \wedge \cdots \wedge v_{n-1} \wedge v_n)\\ &= \frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,i)}} \Delta_{I(b,i)}(V).\end{aligned}$$

Similarly, if $b+i-1 < a$ then

$$\Delta_{I(b,i)}(V_2) = v_b \wedge \cdots v_{b+i-1} \wedge (u_i \wedge \cdots \wedge u_{k-2} \wedge v_n) =$$

$$\frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,i)}}(v_b \wedge \cdots v_{b+i-1}) \wedge (v_{n-k+i+1} \wedge \cdots \wedge v_{n-1} \wedge v_n) = \frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,i)}} \Delta_{I(b,i)}(V). \quad \square$$

Example 4.2. For $a = 3,\ k = 3, n = 8$ we get $\Delta_{378}, \Delta_{348} \neq 0$, as in Example 3.6. We have $V_1 = (v_3, v_4, v_5, v_6, v_7, v_8)$ and $V_2 = (v_1, v_2, v_3, u, v_8)$. The quiver Q_V after freezing Δ_{378} and Δ_{348} has the form:

$$\begin{tikzcd}
\boxed{\Delta_{678}} \arrow[dr] & & & & & \\
& \Delta_{578} \arrow[r] \arrow[d] & \Delta_{478} \arrow[r] \arrow[d] & \boxed{\Delta_{378}} \arrow[r] \arrow[d] & \Delta_{278} \arrow[r] \arrow[d] & \boxed{\Delta_{178}} \\
& \Delta_{568} \arrow[r] \arrow[d] & \Delta_{458} \arrow[r] \arrow[d] \arrow[ul] & \boxed{\Delta_{348}} \arrow[r] \arrow[d] \arrow[ul] & \Delta_{238} \arrow[r] \arrow[d] \arrow[ul] & \boxed{\Delta_{128}} \arrow[ul] \\
& \boxed{\Delta_{567}} & \boxed{\Delta_{456}} \arrow[ul] & \boxed{\Delta_{345}} \arrow[ul] & \boxed{\Delta_{234}} \arrow[ul] & \boxed{\Delta_{123}} \arrow[ul]
\end{tikzcd}$$

while the quivers Q_{V_1} and Q_{V_2} have the form

$$\begin{tikzcd}
\boxed{\Delta_{678}} \arrow[dr] & & & & & \\
& \Delta_{578} \arrow[r] \arrow[d] & \Delta_{478} \arrow[r] \arrow[d] & \boxed{\Delta_{378}} & \Delta_{2u8} \arrow[r] \arrow[d] & \boxed{\Delta_{1u8}} \\
& \Delta_{568} \arrow[r] \arrow[d] & \Delta_{458} \arrow[r] \arrow[d] \arrow[ul] & \boxed{\Delta_{348}} \arrow[ul] \arrow[ur] & \Delta_{238} \arrow[r] \arrow[d] & \boxed{\Delta_{128}} \arrow[ul] \\
& \boxed{\Delta_{567}} & \boxed{\Delta_{456}} \arrow[ul] & \boxed{\Delta_{345}} \arrow[ul] & \boxed{\Delta_{23u}} & \boxed{\Delta_{123}} \arrow[ul]
\end{tikzcd}$$

Note that we identified $\Delta_{3u8} = \Delta_{348}$. We claim that the two cluster structures are related by a quasi-equivalence. Indeed,

$$\Delta_{3u8} = \Delta_{348},\ \Delta_{23u} = \Delta_{234},\ \Delta_{2u8} = \alpha\Delta_{278},\ \Delta_{1u8} = \alpha\Delta_{178} \text{ where } \alpha = \frac{\Delta_{348}}{\Delta_{378}},$$

and all other cluster variables are unchanged. Therefore all cluster variables are the same up to monomials in frozen. We need to check the exchange ratios:

$$y_{2u8}(V_2) = \frac{\Delta_{348}\Delta_{128}}{\Delta_{1u8}\Delta_{238}} = \alpha^{-1}\frac{\Delta_{128}\Delta_{348}}{\Delta_{178}\Delta_{238}} = \frac{\Delta_{128}\Delta_{378}}{\Delta_{178}\Delta_{238}} = y_{278}(V).$$

while

$$y_{238}(V_2) = \frac{\Delta_{2u8}\Delta_{123}}{\Delta_{128}\Delta_{23u}} = \alpha\frac{\Delta_{278}\Delta_{123}}{\Delta_{128}\Delta_{234}} = \frac{\Delta_{278}\Delta_{123}\Delta_{348}}{\Delta_{128}\Delta_{234}\Delta_{378}} = y_{238}(V).$$

Since the exchange ratios agree, we indeed get a quasi-equivalence.

We are ready to state and prove our main result.

Theorem 4.3. *The map $\Phi_a : V \mapsto (V_1, V_2)$ defined by (7) is a cluster quasi-isomorphism between $\widehat{U_a} \subset \widehat{\Pi}^\circ_{k,n}$ and the product $\widehat{\Pi}^\circ_{k,n-a+1} \times \widehat{\Pi}^\circ_{k,a+k-1}$ with identified frozen variables $\Delta_{I(a,k)}(V_2) = \Delta_{I(1,k-1)}(V_1)$.*

As a consequence, Φ_a yields a cluster quasi-isomorphism between U_a and the product $\Pi^\circ_{k,n-a+1} \times \Pi^\circ_{k,a+k-1}$.

Proof. The second statement follows from the first by Proposition 2.3, so we focus on $\widehat{U_a}$. By Lemma 4.1(a) all Scott minors $\Delta_{I(b,i)}(V_1)$ are the same as the minors in the left half of Q_V.

We need to analyze the right half of Q_V. By Lemma 4.1(b) all minors in the right half are multiplied by some monomials in $\Delta_{I(a,i)}$ which are frozen on U_a. It remains to compute the exchange ratios. We have the following cases:

(a) Interior: $b < a$, $i > 1$. The piece of the quiver Q_V around $\Delta_{I(b,i)}$ has the form

$$\begin{tikzcd}
\Delta_{I(b+1,i-1)} \arrow[r] \arrow[d] & \Delta_{I(b,i-1)} \arrow[r] \arrow[d] & \Delta_{I(b-1,i-1)} \arrow[d] \\
\Delta_{I(b+1,i)} \arrow[r] \arrow[d] & \Delta_{I(b,i)} \arrow[r] \arrow[d] \arrow[ul] & \Delta_{I(b-1,i)} \arrow[d] \arrow[ul] \\
\Delta_{I(b+1,i+1)} \arrow[r] & \Delta_{I(b,i+1)} \arrow[r] \arrow[ul] & \Delta_{I(b-1,i+1)} \arrow[ul]
\end{tikzcd}$$

The exchange ratios are equal to

$$y_{I(b,i)} = \frac{\Delta_{I(b,i-1)}\Delta_{I(b+1,i)}\Delta_{I(b-1,i+1)}}{\Delta_{I(b-1,i)}\Delta_{I(b,i+1)}\Delta_{I(b+1,i-1)}}$$

so by Lemma 4.1 we get

$$\frac{y_{I(b,i)}(V)}{y_{I(b,i)}(V_2)} = \frac{\Delta_{I(a,i-1)}\Delta_{I(a,i)}\Delta_{I(a,i+1)}}{\Delta_{I(a,i)}\Delta_{I(a,i+1)}\Delta_{I(a,i-1)}} = 1.$$

and $y_{I(b,i)}(V) = y_{I(b,i)}(V_2)$. Note that $\Delta_{I(a,k-1)}$ cancels out.

(b) Top boundary $i = 1$:

$$\begin{tikzcd}
\Delta_{I(b+1,1)} \arrow[r] \arrow[d] & \Delta_{I(b,1)} \arrow[r] \arrow[d] & \Delta_{I(b-1,1)} \arrow[d] \\
\Delta_{I(b+1,2)} \arrow[r] & \Delta_{I(b,2)} \arrow[r] \arrow[ul] & \Delta_{I(b-1,2)} \arrow[ul]
\end{tikzcd}$$

The exchange ratios are equal to

$$y_{I(b,1)} = \frac{\Delta_{I(b+1,1)}\Delta_{I(b-1,2)}}{\Delta_{I(b-1,1)}\Delta_{I(b,2)}}$$

so by Lemma 4.1 we get

$$\frac{y_{I(b,1)}(V)}{y_{I(b,1)}(V_2)} = \frac{\Delta_{I(a,1)}\Delta_{I(a,2)}}{\Delta_{I(a,1)}\Delta_{I(a,2)}} = 1.$$

and $y_{I(b,i)}(V) = y_{I(b,i)}(V_2)$. Note that $\Delta_{I(a,k-1)}$ cancels again.

(c) Left boundary, $b = a - 1$:

$$\begin{tikzcd}
\Delta_{I(a-1,i-1)} \arrow[r] \arrow[d] & \Delta_{I(a-2,i-1)} \arrow[d] \\
\Delta_{I(a-1,i)} \arrow[r] \arrow[d] & \Delta_{I(a-2,i)} \arrow[d] \arrow[ul] \\
\Delta_{I(a-1,i+1)} \arrow[r] & \Delta_{I(a-2,i+1)} \arrow[ul]
\end{tikzcd}$$

The exchange ratios are equal to

$$y_{I(a-1,i)}(V_2) = \frac{\Delta_{I(a-1,i-1)}(V_2)\Delta_{I(a-2,i+1)}(V_2)}{\Delta_{I(a-2,i)}(V_2)\Delta_{I(a-1,i+1)}(V_2)}$$

so by Lemma 4.1 we get

$$y_{I(a-1,i)}(V_2) = \left[\frac{\Delta_{I(a,i-1)}(V)\Delta_{I(a,i+1)}(V)}{\Delta_{I(a,i)}(V)\Delta_{I(a,i+1)}(V)}\right]^{-1} \cdot \frac{\Delta_{I(a-1,i-1)}(V)\Delta_{I(a-2,i+1)}(V)}{\Delta_{I(a-2,i)}(V)\Delta_{I(a-1,i+1)}(V)} =$$

$$\frac{\Delta_{I(a,i)}(V)\Delta_{I(a-1,i-1)}(V)\Delta_{I(a-2,i+1)}(V)}{\Delta_{I(a,i-1)}(V)\Delta_{I(a-2,i)}(V)\Delta_{I(a-1,i+1)}(V)} = y_{I(a-1,i)}(V).$$

(d) Corner, $b = a - 1, i = 1$:

$$\begin{tikzcd}
 & \Delta_{I(a-1,1)} \arrow[r] \arrow[d] & \Delta_{I(a-2,1)} \arrow[d] \\
\boxed{\Delta_{I(a,k-1)}} \arrow[ur] & \Delta_{I(a-1,2)} \arrow[r] & \Delta_{I(a-2,2)} \arrow[ul]
\end{tikzcd}$$

Here we identify $\Delta_{I(a,k-1)}(V_2)$ with $\Delta_{I(a,k-1)}(V) = \Delta_{I(1,k-1)}(V_1)$ as above. The exchange ratio is equal to

$$y_{I(a-1,1)}(V_2) = \frac{\Delta_{I(a,k-1)}(V)\Delta_{I(a-2,2)}(V_2)}{\Delta_{I(a-2,1)}(V_2)\Delta_{I(a-1,2)}(V_2)}$$

so by Lemma 4.1 we get

$$\begin{aligned}\Delta_{I(a,k-1)}(V) \cdot \frac{\Delta_{I(a-2,2)}(V)\Delta_{I(a,k-1)}(V)}{\Delta_{I(a,2)}(V)}& \\ \cdot \frac{\Delta_{I(a,1)}(V)}{\Delta_{I(a-2,1)}(V)\Delta_{I(a,k-1)}(V)} &\cdot \frac{\Delta_{I(a,2)}(V)}{\Delta_{I(a-1,2)}(V)\Delta_{I(a,k-1)}(V)} \\ = \frac{\Delta_{I(a-2,2)}(V)\Delta_{I(a,1)}(V)}{\Delta_{I(a-2,1)}(V)\Delta_{I(a-1,2)}(V)} &= y_{I(a-1,1)}(V). \qquad \square\end{aligned}$$

5. Relation to braid varieties

In this section we describe the map Φ in terms of braid varieties. We refer to [5, 4, 3, 11] for more information and context and braid varieties, and only use some basic definitions. We will work on the variety of complete flags

$$\mathrm{Fl}_k = \{0 = \mathcal{F}_0 \subset \mathcal{F}_1 \cdots \subset \mathcal{F}_k = \mathbb{C}^k\},\ \dim \mathcal{F}_i = i.$$

Definition 5.1. *We say that two flags $\mathcal{F}$ and $\mathcal{F}'$ are in position s_i if $\mathcal{F}_j = \mathcal{F}'_j$ for $j \neq i$ and $\mathcal{F}_i \neq \mathcal{F}'_i$. We will denote this by $\mathcal{F} \xrightarrow{s_i} \mathcal{F}'$.*

We say that $\mathcal{F}$ and $\mathcal{F}'$ are in position w_0 if $\mathcal{F}_i \oplus \mathcal{F}'_{n-i} = \mathbb{C}^k$, in other words $\mathcal{F}_i$ is transversal to $\mathcal{F}'_{n-i}$ for all i. Here w_0 is the longest element of the permutation group S_k.

Definition 5.2. *Given a braid $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}$, we define the braid variety as the space of sequences of flags*

$$\mathcal{F}^{(0)} \xrightarrow{s_{i_1}} \mathcal{F}^{(1)} \dots \mathcal{F}^{(\ell-1)} \xrightarrow{s_{i_\ell}} \mathcal{F}_{(\ell)}$$

such that $\mathcal{F}^{(0)}$ is the standard flag and $\mathcal{F}^{(\ell)}$ is the antistandard flag in $\mathbb{C}^k$:

$$\mathcal{F}^{(0)}_i = \langle e_1, \dots, e_i\rangle,\ \mathcal{F}^{(\ell)}_i = \langle e_{k-i+1}, \dots, e_k\rangle.$$

We will visualize the flags $\mathcal{F}^{(j)}$ by labeling the regions in the braid diagram for β by vector spaces such that each vertical cross-section provides a complete flag, see Figure 6. We recall an explicit construction [2, Section 4] relating $\Pi^\circ_{k,n}$ to braid variety $X(\beta_{k,n})$ where

$$\beta_{k,n} = (\sigma_1 \dots \sigma_{k-1})^{n-k}(\sigma_1 \dots \sigma_{k-1}) \dots (\sigma_2\sigma_1)\sigma_1 = (\sigma_1 \dots \sigma_{k-1})^{n-k} w_0$$

(see also [17]). Here $T(k, n-k) = (\sigma_1 \dots \sigma_{k-1})^{n-k}$ is the $(k, n-k)$ torus braid and $\sigma_1(\sigma_2\sigma_1)\cdots(\sigma_{k-1}\cdots\sigma_1)$ is the specific braid word for the half-twist braid.

Given a matrix $V = (v_1, \dots, v_n)$, we can fill in the bottom row of the braid diagram for $\beta_{k,n}$ by the vectors $v_1, \dots, v_n$. This uniquely determines the subspaces for all other regions as spans $\langle v_i, \dots, v_j\rangle$ for appropriate i, j, see Figures 6 and 7. The conditions $\Delta_{I(a,k)}(V) \neq 0$ are equivalent to the relative position conditions for each crossing of β. The conditions $\Delta_{I(1,i)}(V) \neq 0$ are equivalent to the fact that two flags

$$\mathcal{F}^{(0)} = \{0 \subset \langle v_1\rangle \subset \langle v_1, v_2\rangle \subset \dots \langle v_1, \dots, v_k\rangle\}$$

and

$$\mathcal{F}^{(N)} = \{0 \subset \langle v_n\rangle \subset \langle v_{n-1}, v_n\rangle \subset \dots \langle v_{n-k+1}, \dots, v_n\rangle\}$$

are in position w_0. Therefore there is a unique matrix M such that $M\mathcal{F}^{(0)}$ is the standard flag and $M\mathcal{F}^{(N)}$ is the antistandard flag.

Finally, the flags constructed as above determine the vectors v_i only up to scalars. This can be fixed either by rescaling v_i, or by considering framed flags as in [2]. As a result, we obtain the following.

THEOREM 5.3 ([2, 17]). *Let* $\Pi^{\circ,1}_{k,n}$ *be the subset of* $\widehat{\Pi}^{\circ}_{k,n}$ *defined by*

$$\Delta_{b,b+1,\ldots,b+k-1} = \Delta_{I(b,k)} = 1, \quad \text{for } 1 \leqslant b \leqslant n-k+1.$$

Then $X(\beta_{k,n}) \simeq \Pi^{\circ,1}_{k,n}$ *and* $\widehat{\Pi}^{\circ}_{k,n} \cong \Pi^{\circ,1}_{k,n} \times (\mathbb{C}^*)^{n-k+1}$.

Following Proposition 2.3, we can also regard $\Pi^{\circ,1}_{k,n}$ as the subset of $\Pi^{\circ}_{k,n}$ defined by

$$\Delta_{I(b,k)}/\Delta_{I(n-k+1,k)} = 1, \quad \text{for } 1 \leqslant b \leqslant n-k.$$

In particular,

$$\begin{aligned} \dim \Pi^{\circ,1}_{k,n} &= \dim \widehat{\Pi}^{\circ}_{k,n} - (n-k+1) \\ &= k(n-k)+1-(n-k+1) = (k-1)(n-k) = \ell(\beta_{k,n}) - \ell(w_0). \end{aligned} \tag{10}$$

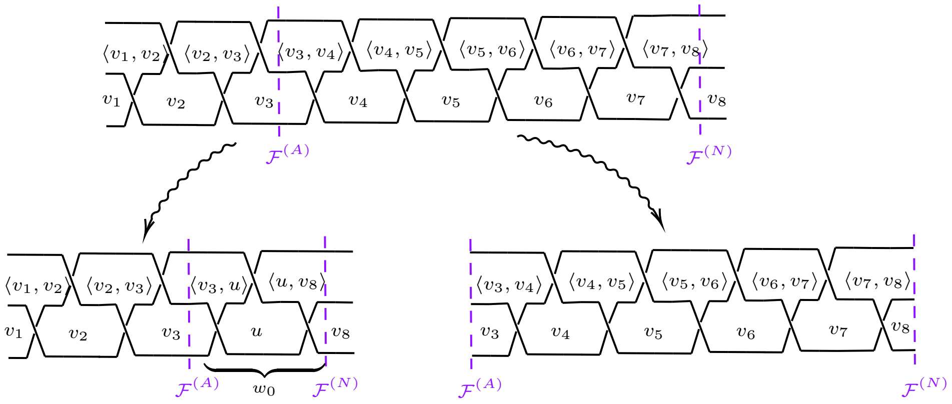

FIGURE 6. Freezing $\Delta_{348}, \Delta_{378}$ in the braid associated to $X(\beta_{3,8}) \simeq \Pi^{\circ,1}_{3,8}$.

Let $\beta = (\sigma_1 \ldots \sigma_{k-1})^{n-k}(\sigma_1 \ldots \sigma_{k-1}) \ldots (\sigma_1\sigma_2)\sigma_1$ be the braid where $X(\beta) \simeq \Pi^{\circ,1}_{k,n}$. Then the process of freezing $\Delta_{I(a)}$ corresponds to severing the braid β at flags $\mathcal{F}^{(A)}$ and $\mathcal{F}^{(N)}$ where $A = (a-1)(k-1)+1$ and $N = (n-k)(k-1)+\binom{k}{2}+1$. Upon severing the braid at the given flags we disassemble the braid into two separate braids decorated by the flags

$$\mathcal{F}^{(A)} \xrightarrow{s_1} \mathcal{F}^{(A+1)} \ldots \mathcal{F}^{(N-1)} \xrightarrow{s_1} \mathcal{F}^{(N)} \tag{11}$$

and

$$\mathcal{F}^{(0)} \xrightarrow{s_1} \mathcal{F}^{(1)} \ldots \mathcal{F}^{(A-1)} \xrightarrow{s_{k-1}} \mathcal{F}^{(A)} \xrightarrow{s_1} \widetilde{\mathcal{F}}^{(A+1)} \xrightarrow{\cdots} \widetilde{\mathcal{F}}^{A+\binom{k}{2}-1} \xrightarrow{s_1} \mathcal{F}^{(N)}. \tag{12}$$

The first braid is decorated by the flags between $\mathcal{F}^{(A)}$ and $\mathcal{F}^{(N)}$ and is associated to $X(\beta_1)$ where $\beta_1 = (\sigma_1 \cdots \sigma_{k-1})^{n-k-a+1} w_0$. Note that the conditions defining the open subset U_a guarantee that the flags $\mathcal{F}^{(A)}$ and $\mathcal{F}^{(N)}$ are in position w_0, so as above there is a unique matrix M such that $M\mathcal{F}^{(A)}$ is the standard flag and $M\mathcal{F}^{(N)}$ is the antistandard flag.

For the second braid, we "splice" together the flags $\mathcal{F}^{(A)}$ and $\mathcal{F}^{(N)}$ by adding the sequences of flags $\widetilde{\mathcal{F}}$ associated with the half twist on k strands. See Figure 6 for an example of the decomposition of the braid β into its two separate components, and Figure 7 for a depiction of the local splicing effect on the flags. Stitching the flags

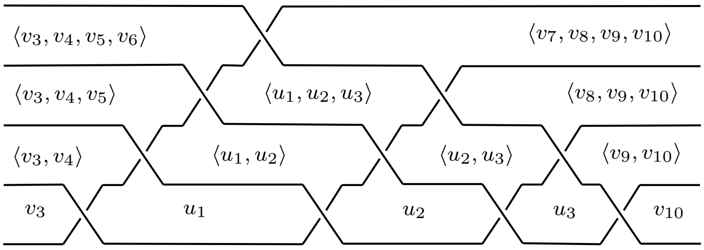

FIGURE 7. Braid diagram and flags for Example 3.4. Here $\langle u_1\rangle = \langle v_3, v_4\rangle \cap \langle v_7, v_8, v_9, v_{10}\rangle$, $\langle u_2\rangle = \langle v_3, v_4, v_5\rangle \cap \langle v_8, v_9, v_{10}\rangle$ and $\langle u_3\rangle = \langle v_3, v_4, v_5, v_6\rangle \cap \langle v_9, v_{10}\rangle$.

$\mathcal{F}^{(A)}$ and $\mathcal{F}^{(N)}$ together with the half twist fills the bottom row of the braid with $k-2$ vectors $u_1, \ldots, u_{k-2}$, and the intermediate flags $\widetilde{F}^{(A+j)}$ are uniquely determined by $\mathcal{F}^{(A)}$ and $\mathcal{F}^{(N)}$. Through this process the resulting braid is $\beta_2 = (\sigma_1 \cdots \sigma_{k-1})^{a-1} w_0$.

Finally, we can compare the cluster structures on braid varieties. The cluster structure on $\Pi^{\circ,1}_{k,n}$ is obtained from (4) by removing the frozen variables $\Delta_{I(b,k)}$ from Q_V.

THEOREM 5.4. *The map* $\overline{\Phi}_a : V \mapsto (V_1, \overline{V_2})$,

$$V_1 = (v_a, \ldots, v_n), \quad \overline{V_2} = \left(v_1, \ldots, v_a, u_1, \ldots, u_{k-2}, \frac{v_n}{\Delta_{I(a,k-1)}}\right)$$

defines a quasi-cluster isomorphism between $U^1_a = U_a \cap \Pi^{\circ,1}_{k,n}$ *and* $\Pi^{\circ,1}_{k,n-a+1} \times \Pi^{\circ,1}_{k,a+k-1}$.

Note that we do not need to change the matrix V_1 since all of its consecutive minors are still equal to 1. Also note that by (10) we get $\dim U^1_a = \dim \Pi^{\circ,1}_{k,n} = (k-1)(n-k)$ while

$$\dim \Pi^{\circ,1}_{k,n-a+1} + \Pi^{\circ,1}_{k,a+k-1} = (k-1)(n-a+1-k) + (k-1)(a-1) = (k-1)(n-k).$$

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Eugene Gorsky, Department of Mathematics, University of California at Davis, One Shields Avenue, Davis CA 95616
E-mail : `egorskiy@math.ucdavis.edu`

Tonie Scroggin, Department of Mathematics, University of California at Santa Barbara, Santa Barbara CA 93106
E-mail : `tmscroggin@ucsb.edu`