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Uniqueness for 2-intersecting families of permutations and perfect matchings

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ABSTRACT We give a characterization of the largest 2-intersecting families of permutations of $\{1, 2, \dots, n\}$ and of perfect matchings of the complete graph K_{2n} for all $n \geq 2$.

1. INTRODUCTION

Erdős–Ko–Rado theory [12] studies intersecting families of objects. One of the high points of the theory is the Ahlswede–Khachatrian theorem [1, 2].

THEOREM 1.1. *Let \mathcal{F} be a subset of $\binom{[n]}{k}$ (the collection of all subsets of $[n] := \{1, \dots, n\}$ of size k) which is t -intersecting: every $A, B \in \mathcal{F}$ satisfy $|A \cap B| \geq t$. Then*

$$|\mathcal{F}| \leq \max_{r \leq (n-t)/2} \left| \left\{ A \subseteq \binom{[n]}{k} : |A \cap [t+2r]| \geq t+r \right\} \right|.$$

Furthermore, if \mathcal{F} achieves this bound, then

$$\mathcal{F} = \left\{ A \subseteq \binom{[n]}{k} : |A \cap S| \geq t+r \right\}$$

for some $r \leq (n-t)/2$ and some $S \subseteq [n]$ of size $t+2r$.

The theorem consists of two statements: an *upper bound* on the size of t -intersecting families, and a characterization of the extremal families. The latter part is known as *uniqueness*.

The Ahlswede–Khachatrian theorem is about intersecting families of sets. In this paper, we will be interested in intersecting families of permutations and perfect matchings.

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PERMUTATIONS. Let S_n be the group of all permutations of $[n]$. Two permutations $\alpha, \beta \in S_n$ are t -intersecting if there exist t distinct indices $i_1, \dots, i_t \in [n]$ such that $\alpha(i_1) = \beta(i_1), \dots, \alpha(i_t) = \beta(i_t)$. Ellis, Friedgut and Pilpel [11] conjectured that the Ahlswede–Khachatrian theorem extends to intersecting families of permutations.

CONJECTURE 1.2. *Let \mathcal{F} be a subset of S_n which is t -intersecting. Then*

$$|\mathcal{F}| \leq \max_{r \leq (n-t)/2} |\{\alpha \in S_n : \alpha(i) = i \text{ for at least } t+r \text{ many } i \in [t+2r]\}|.$$

Furthermore, if \mathcal{F} achieves this bound, then

$$\mathcal{F} = \{\alpha \in S_n : \alpha(i_s) = j_s \text{ for at least } t+r \text{ many } s \in [t+2r]\}$$

for some $r \leq (n-t)/2$ and some distinct $i_1, \dots, i_{t+2r} \in [n]$ and distinct $j_1, \dots, j_{t+2r} \in [n]$.

When $t \leq 3$, the maximum is always attained at $r = 0$, in which case the upper bound is $(n-t)!$, and the conjectured extremal families are the t -cosets.

Deza and Frankl [16] proved the upper bound part of Conjecture 1.2 in the special case $t = 1$ by noticing that the n cyclic rotations of any fixed permutation are pairwise non-1-intersecting. Proving uniqueness in this case proved a lot harder, but eventually many proofs were found [5, 20, 17, 11].

The case $t > 1$ is significantly harder. For every $t > 1$, Ellis, Friedgut and Pilpel [11] proved that the upper bound in Conjecture 1.2 holds for large enough n , and Ellis [9] proved that the uniqueness part of Conjecture 1.2 holds for large enough n . Ellis, Friedgut and Pilpel used a spectral technique based on the so-called Hoffman bound (closely related to the Lovász theta function and to the linear programming bound in coding theory), which has found many other applications in Erdős–Ko–Rado theory [18].

Recently, Meagher and Razafimahatratra [26], refining the techniques of Ellis, Friedgut and Pilpel, proved that 2-intersecting subsets of S_n contain at most $(n-2)!$ permutations, thus verifying the upper bound part of Conjecture 1.2 for $t = 2$ and all n . However, they were unable to characterize the extremal families in this case. In this paper, we show that all such families are 2-cosets, thus verifying the uniqueness part of Conjecture 1.2 for $t = 2$ and all n .

THEOREM 1.3. *Let $n \geq 2$. If \mathcal{F} is a 2-intersecting subset of S_n of size $(n-2)!$, then there exist $i_1 \neq i_2$ and $j_1 \neq j_2$ such that*

$$\mathcal{F} = \{\alpha \in S_n : \alpha(i_1) = j_1 \text{ and } \alpha(i_2) = j_2\}.$$

PERFECT MATCHINGS. So far we have considered intersecting families of permutations. A related area of study is intersecting families of perfect matchings of the complete graph K_{2n} . This can be seen as the non-bipartite analog of the symmetric group, which is the set of perfect matchings in the complete bipartite graph $K_{n,n}$.

A simple argument shows that a 1-intersecting family of perfect matchings contains at most $(2n-3)!! = (2n-3)(2n-5)\cdots(1)$ perfect matchings. More difficult arguments [19, 21, 25, 8] show that this is attained uniquely by 1-cosets.

Lindzey [22, 23] extended the arguments of Ellis, Friedgut and Pilpel [11, 9] to the setting of perfect matchings, proving that for all t , the maximum size t -intersecting families are t -cosets, for large enough n .

Recently, Fallat, Meagher and Shirazi showed that the maximum size of a 2-intersecting family is at most $(2n-5)!!$ for all n . Extending our arguments in the setting of the symmetric group, we prove an analog of Theorem 1.3 in the setting of the perfect matching scheme.

THEOREM 1.4. *Let $n \geq 2$. If \mathcal{F} is a 2-intersecting subset of \mathcal{M}_{2n} of size $(2n-5)!!$, then there exist distinct i_1, j_1, i_2, j_2 such that \mathcal{F} consists of all perfect matchings containing the edges $\{i_1, j_1\}, \{i_2, j_2\}$.*

ON THE PROOF. The spectral technique used by Ellis, Friedgut and Pilpel shows that for every $t \geq 1$, if n is large enough and $\mathcal{F} \subseteq S_n$ is a t -intersecting family of size $(n-t)!$, then the characteristic function of \mathcal{F} , denoted as $1_{\mathcal{F}}$, has degree at most t . This means that $1_{\mathcal{F}}$ can be expressed as a polynomial of degree at most t in the Boolean variables x_{ij} , where $x_{ij} = 1$ if the input permutation maps i to j . Equivalently, x_{ij} is the (i, j) 'th entry of the permutation matrix representing the input permutation.

Similarly, Meagher and Razafimahatratra show that if $n \geq 5$ and $\mathcal{F} \subseteq S_n$ is a 2-intersecting family of size $(n-2)!$, then the characteristic function of \mathcal{F} has degree at most 2. This is the starting point of this work.

Ellis, Friedgut and Pilpel used polyhedral techniques (chiefly, the Birkhoff-von Neumann theorem on bistochastic matrices) to show that Boolean degree 1 functions are *dictators*, that is, they either depend only on $\alpha(i)$ for some $i \in [n]$ (where α is the input permutation), or they depend only on $\alpha^{-1}(j)$ for some $j \in [n]$. The only 1-intersecting dictators are 1-cosets, and this provides a proof of the uniqueness part of Conjecture 1.2 when $t = 1$.

Similar techniques do not work for larger degree [14]. Instead, we turn to the theory of *complexity measures of Boolean functions* [4], which was recently generalized to the symmetric group and to the perfect matching scheme by Dafni et al. [8]. Using these techniques, Dafni et al. give a simple proof of the uniqueness part of Conjecture 1.2, as well as of its analog for the perfect matching scheme, for large n .

The chief tool used by Dafni et al., which we also employ in our proof, is *certificate complexity*. Let $f: S_n \rightarrow \{0, 1\}$ be a Boolean function, and let $\alpha \in S_n$ be such that $f(\alpha) = b$. A *certificate* for α is a subset $\{i_1, \dots, i_m\} \subseteq [n]$ such that $f(\beta) = b$ whenever $\beta(i_1) = \alpha(i_1), \dots, \beta(i_m) = \alpha(i_m)$. The idea is that in order to verify that $f(\alpha) = b$, it suffices to check the value of $\alpha(i_1), \dots, \alpha(i_m)$. The *certificate complexity* of α is the minimum size of a certificate for α , and the certificate complexity of f is the maximum, over all $\alpha \in S_n$, of the certificate complexity of α . We denote the certificate complexity of f by $C(f)$.

Suppose that f is the characteristic function of a 2-intersecting subset of S_n . When $C(f) \leq n-2$, we give a structure theorem for f which suffices to bound the size of the family away from $(n-2)!$ unless $C(f) = 2$, in which case f is the characteristic function of a 2-coset. A simple argument (using another complexity measure, *sensitivity*) shows that if $n \geq 8$ and $f: S_n \rightarrow \{0, 1\}$ has degree at most 2 then $C(f) \leq n-2$, allowing us to apply the preceding argument. Finally, we handle the case $n \leq 7$ using exhaustive search, employing an algorithm for enumerating maximum cliques.

Similar techniques work in the case of the perfect matching scheme, with one complication: the proof of our structure theorem relies on the fact that Boolean degree 1 functions on the symmetric group have certificate complexity 1, but this fails for the perfect matching scheme. Using the maximality of maximum-size 2-intersecting families, we are able to overcome this hurdle.

FUTURE RESEARCH. Conjecture 1.2 is about families of permutations in which any two permutations agree on the images of t points. A related question concerns t -setwise-intersecting families of permutations, in which any two permutations α, β agree on the image of a set of size t : $\{\alpha(i_1), \dots, \alpha(i_t)\} = \{\beta(i_1), \dots, \beta(i_t)\}$ for a set of size t .

Ellis [10] showed that for every t , if n is large enough then the maximum size of a t -setwise-intersecting subset of S_n is $t!(n-t)!$, and this is achieved uniquely by

families of the form

$$\{\alpha \in S_n : \{\alpha(i_1), \dots, \alpha(i_t)\} = \{j_1, \dots, j_t\}\},$$

which we call *t-setwise-cosets*.

Meagher and Razafimahatratra [26] showed that when $t = 2$, the upper bound holds for all $n \geq 2$, and moreover, if $n \geq 4$ and f is the characteristic function of a 2-setwise-intersecting family of size $2(n - 2)!$, then f has degree at most 2 and additionally satisfies $f^{=(n-2,1,1)} = 0$ (see Section 2.1.1 for an explanation of this notation). In other words, $f^{=\lambda} \neq 0$ only for $\lambda = (n), (n - 1, 1), (n - 2, 2)$.

Behajaina, Maleki, Rasoamanana and Razafimahatratra [3] showed that when $t = 3$, the upper bound holds for all $n \geq 11$, and moreover, if f is the characteristic function of a 3-setwise-intersecting family of size $6(n - 3)!$, then $f^{=\lambda} \neq 0$ only for $\lambda = (n), (n - 1, 1), (n - 2, 2), (n - 3, 3)$.

In both cases, the authors were unable to classify the extremal families. Can we extend our techniques to show that the extremal families are *t-setwise-cosets*? The first step in this direction was already taken by the second author in her master’s thesis [7], who showed that this holds for large n . One of the difficulties in obtaining bounds for small n is taking advantage of the characterization of extremal f , which is stronger than a mere degree bound.

We mention in passing two more challenges. One is to extend the theory of setwise-intersecting families from permutations to perfect matchings. A *t-setwise intersecting* family of perfect matchings is one in which for any two perfect matchings m_1, m_2 there are two sets A, B of size t such that both m_1, m_2 match the vertices in A to vertices in B . Another possible generalization is to require the existence of a single set C of size $2t$ such that both m_1 and m_2 match vertices in C to vertices in C .

The other is a common generalization of *t-intersecting families* and *t-setwise-intersecting families*. Given a partition $\lambda = \lambda_1, \dots, \lambda_m$, a λ -intersecting family of permutations is one in which for any two permutations α, β there are disjoint sets A_1, \dots, A_m of sizes $\lambda_1, \dots, \lambda_m$ such that $\alpha(A_1) = \beta(A_1), \dots, \alpha(A_m) = \beta(A_m)$. A *t-intersecting family* corresponds to $\lambda = (1^t)$, and a *t-setwise-intersecting family* corresponds to $\lambda = (t)$.

STRUCTURE OF THE PAPER. We start with a few preliminaries in Section 1. Apart from introducing several key definitions and results, we also prove several results appearing in a more general form in [8], in order to keep the paper self-contained. Theorem 1.3 is then proved in Section 3, and Theorem 1.4 in Section 4.

2. PRELIMINARIES

2.1. SYMMETRIC GROUP. For integer $n \geq 1$, the *symmetric group* S_n is the collection of all permutations of $[n] = \{1, \dots, n\}$. We denote the identity permutation by id .

A *t-coset* of S_n is a set of the form

$$\{\alpha \in S_n : \alpha(i_1) = j_1, \dots, \alpha(i_t) = j_t\},$$

where $i_1, \dots, i_t \in [n]$ are distinct and $j_1, \dots, j_t \in [n]$ are distinct. A *t-coset* contains $(n - t)!$ permutations.

We use the term *coset* for a 1-coset. For $i, j \in [n]$, the coset $[i \mapsto j]$ consists of all permutations sending i to j :

$$[i \mapsto j] = \{\alpha \in S_n : \alpha(i) = j\}.$$

Two permutations $\alpha, \beta \in S_n$ are *t-intersecting* if there is a *t-coset* which contains both of them. Equivalently, the two permutations *t-intersect* if there are distinct indices $i_1, \dots, i_t \in [n]$ such that $\alpha(i_1) = \beta(i_1), \dots, \alpha(i_t) = \beta(i_t)$.

A subset $\mathcal{F} \subseteq S_n$ is *t-intersecting* if every pair of permutations in \mathcal{F} are *t*-intersecting.

2.1.1. *Degree.* We can represent permutations $\alpha \in S_n$ using n^2 variables x_{ij} whose semantics are: $x_{ij} = 1$ if $\alpha(i) = j$, and $x_{ij} = 0$ otherwise. The *degree* of a function $f: S_n \rightarrow \mathbb{R}$, denoted $\deg f$, is the minimal d such that f can be represented as a degree d polynomial over the variables x_{ij} . For example, the characteristic function of a *t*-coset has degree at most t , since it can be represented by the polynomial $x_{i_1 j_1} \cdots x_{i_t j_t}$.

An equivalent way to define degree is via the representation theory of the symmetric group. Let $\mathbb{R}[S_n]$ be the vector space of all real-valued functions over the symmetric group. Representation theory gives an orthogonal decomposition (with respect to the inner product $\langle f, g \rangle = \sum_{\alpha \in S_n} f(\alpha)g(\alpha)$)

$$\mathbb{R}[S_n] = \bigoplus_{\lambda \vdash n} V^\lambda,$$

where λ goes over all integer partitions of n , and V^λ are certain subspaces known as *isotypic components*, which we define explicitly in Section 2.1.3. Accordingly, every function $f: S_n \rightarrow \mathbb{R}$ has a unique decomposition

$$f = \sum_{\lambda \vdash n} f^{\lambda},$$

where $f^{\lambda} \in V^\lambda$. Ellis, Friedgut and Pilpel [11, Theorem 7] showed that $\deg f$ is the maximal d such that $f^{\lambda} \neq 0$ for some λ satisfying $\lambda_1 = n - d$. In particular, every function has degree at most $n - 1$.

We can now formally state the main result of Meagher and Razafimahatratra [26, Corollary 5.5].

THEOREM 2.1. *If $n \geq 5$ then any 2-intersecting family $\mathcal{F} \subseteq S_n$ contains at most $(n - 2)!$ permutations. Furthermore, if $f: S_n \rightarrow \{0, 1\}$ is the characteristic vector of a 2-intersecting family of size $(n - 2)!$, then $\deg f \leq 2$.*

We will also need a result of Ellis, Friedgut and Pilpel [11] classifying degree 1 functions.

THEOREM 2.2 ([11, Corollary 2]). *If $f: S_n \rightarrow \{0, 1\}$ has degree at most 1 then either*

$$f = \sum_{j \in J} x_{ij}$$

for some $i \in [n]$ and $J \subseteq [n]$, or

$$f = \sum_{i \in I} x_{ij}$$

for some $I \subseteq [n]$ and $j \in [n]$.

See also [8, Theorem 6.1] for an alternative (but very similar) proof.

2.1.2. *Certificate complexity.* Fix n . A *certificate* is a set $C = \{(i_1, j_1), \dots, (i_m, j_m)\} \subseteq [n] \times [n]$. A permutation $\alpha \in S_n$ satisfies the certificate C if $\alpha(i_1) = j_1, \dots, \alpha(i_m) = j_m$. The set of permutations satisfying a certificate C is either empty or a $|C|$ -coset.

We sometimes think of a permutation $\alpha \in S_n$ as the certificate $\{(1, \alpha(1)), \dots, (n, \alpha(n))\}$, which we call the *certificate representation* of α .

A *Boolean function* is a function $f: S_n \rightarrow \{0, 1\}$. Given $\alpha \in S_n$ such that $f(\alpha) = b$, a *certificate for α* (with respect to f) is a certificate C such that:

- (i) α satisfies C .
- (ii) If β satisfies C then $f(\beta) = b$.

Intuitively, in order to certify that $f(\alpha) = b$, it suffices to verify that α satisfies C .

The *certificate complexity* of α , denoted $C(f, \alpha)$, is the minimum size of a certificate for α . A certificate for α of this size is known as a *minimum certificate*. Since every permutation is determined by its values on the points $1, \dots, n - 1$, the certificate complexity of α is always at most $n - 1$.

The *certificate complexity* of f , denoted $C(f)$, is $\max_{\alpha \in S_n} C(f, \alpha)$. The certificate complexity of f is always at most $n - 1$.

Dafni et al. [8, Theorem 3.1] showed that certificate complexity is polynomially related to the degree: $C(f) = O((\deg f)^8)$ and $\deg f = O(C(f)^4)$.

In the special case of degree 1, Theorem 2.2 implies that $C(f) \leq 1$.

LEMMA 2.3. *If $f: S_n \rightarrow \{0, 1\}$ has degree at most 1 then $C(f) \leq 1$.*

Proof. According to Theorem 2.2, either $f = \sum_{j \in J} x_{ij}$ for some $i \in [n]$ and $J \subseteq [n]$, or $f = \sum_{i \in I} x_{ij}$ for some $I \subseteq [n]$ and $j \in [n]$. In the former case, every $\alpha \in S_n$ has the certificate $\{(i, \alpha(i))\}$, and in the latter case, every $\alpha \in S_n$ has the certificate $\{(\alpha^{-1}(j), j)\}$. \square

The following lemma, which essentially follows from [8, Lemma 7.3], shows that if \mathcal{F} is a 2-intersecting family then minimum certificates of any two permutations in \mathcal{F} must 2-intersect, unless $C(f) = n - 1$.

LEMMA 2.4. *Let $f: S_n \rightarrow \{0, 1\}$ be the characteristic function of a 2-intersecting family, and suppose that $C(f) \leq n - 2$. If $f(\alpha) = f(\beta) = 1$ and C_α, C_β are minimum certificates for α, β (respectively), then $|C_\alpha \cap C_\beta| \geq 2$.*

Proof. We will show that there exist $\alpha' \in S_n$ satisfying C_α and $\beta' \in S_n$ satisfying C_β such that $\alpha' \cap \beta' = C_\alpha \cap C_\beta$, where we identify α', β' with their certificate representations. Since $f(\alpha') = f(\beta') = 1$ and f is the characteristic function of a 2-intersecting family, we have $|\alpha' \cap \beta'| \geq 2$, and so $|C_\alpha \cap C_\beta| \geq 2$.

We start by constructing α' satisfying C_α such that $\alpha' \cap C_\beta = C_\alpha \cap C_\beta$. Let $C_\alpha = \{(i_1, j_1), \dots, (i_m, j_m)\}$, where $m \leq n - 2$, let i'_1, \dots, i'_{n-m} be the i -indices not mentioned in C_α , and let j'_1, \dots, j'_{n-m} be the j -indices not mentioned in C_α .

For each i'_s , there is at most one j'_t such that $(i'_s, j'_t) \in C_\beta$. We can therefore arrange the indices in such a way that if $(i'_s, j'_t) \in C_\beta$ then $s = t$. We can now define α' explicitly:

$$\alpha'(i_1) = j_1, \dots, \alpha'(i_m) = j_m, \alpha'(i'_1) = j'_2, \dots, \alpha'(i'_{n-m-1}) = j'_{n-m}, \alpha'(i'_{n-m}) = j'_1.$$

The same argument (replacing C_α, C_β by C_β, α') shows that we can find β' satisfying C_β such that $\alpha' \cap \beta' = \alpha' \cap C_\beta = C_\alpha \cap C_\beta$, completing the proof. \square

When $C(f) = n - 1$, the conclusion of Lemma 2.4 indeed fails. For example, consider the family $\mathcal{F} \subseteq S_5$ given by $\mathcal{F} = \{\text{id}, (1\ 2\ 3)\}$, and the minimum certificates $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$ and $\{(1, 2), (2, 3), (3, 1), (4, 4)\}$.

Using the concept of *sensitivity*, we can rule out the case $C(f) = n - 1$ for functions of degree at most 2. The proof is based on the technique used to prove [8, Lemma 3.6].

LEMMA 2.5. *If $n \geq 8$ and $f: S_n \rightarrow \{0, 1\}$ has degree at most 2, then $C(f) \leq n - 2$.*

Proof. The proof is by contradiction. Suppose that $n \geq 8$, that $f: S_n \rightarrow \{0, 1\}$ has degree at most 2, and that $C(f) = n - 1$. Without loss of generality, $C(f, \text{id}) = n - 1$ and $f(\text{id}) = 0$. Then $f((i\ j)) = 1$ for all $i \neq j \in [n]$ (here $(i\ j)$ is the transposition switching i and j), since otherwise $\{(k, k) : k \neq i, j\}$ would be a certificate for id of size $n - 2$.

We construct a function $g: \{0, 1\}^4 \rightarrow \{0, 1\}$ as follows:

$$g(y_1, y_2, y_3, y_4) = f((1\ 2)^{y_1} (3\ 4)^{y_2} (5\ 6)^{y_3} (7\ 8)^{y_4}).$$

Here $(i\ j)^0 = \text{id}$, $(i\ j)^1 = (i\ j)$, and the input to f is the product of four powers of this kind.

Since f has degree at most 2, it can be written as a polynomial of degree at most 2 in the variables x_{ij} . Given y_1, y_2, y_3, y_4 , the value of the variables x_{ij} is:

$$\begin{aligned} x_{11} = x_{22} &= 1 - y_1 & x_{12} = x_{21} &= y_1 \\ x_{33} = x_{44} &= 1 - y_2 & x_{34} = x_{43} &= y_2 \\ x_{55} = x_{66} &= 1 - y_3 & x_{56} = x_{65} &= y_3 \\ x_{77} = x_{88} &= 1 - y_4 & x_{78} = x_{87} &= y_4 \end{aligned}$$

All other variables are assigned zero. This shows that g can be expressed as a polynomial of degree at most 2 in the variables y_1, y_2, y_3, y_4 . We say that g has *degree at most 2*.

By construction, the function g satisfies the following constraints:

$$\begin{aligned} g(0, 0, 0, 0) &= f(\text{id}) = 0 \\ g(1, 0, 0, 0) &= f((1\ 2)) = 1 \\ g(0, 1, 0, 0) &= f((3\ 4)) = 1 \\ g(0, 0, 1, 0) &= f((5\ 6)) = 1 \\ g(0, 0, 0, 1) &= f((7\ 8)) = 1 \end{aligned}$$

We say that g has *sensitivity 4*.⁽¹⁾

There are only 2^{16} many functions from $\{0, 1\}^4$ to $\{0, 1\}$. For each function, we can check whether it has degree at most 2 by solving linear equations, since each function has a unique representation as a multilinear polynomial in y_1, y_2, y_3, y_4 (this is a basic fact in Boolean function analysis). Going over all such functions in SAGE [30], we find out that none of them has sensitivity 4, a contradiction. The relevant code appears in Appendix A.1. \square

For experts in Boolean function analysis, here is an alternative proof that g cannot have sensitivity 4. Since $\deg g \leq 2$, every influential variable has influence at least $1/2$ [29, Proposition 3.6]. If g has sensitivity 4, then it depends on 4 variables, and so its total influence is 2. Since g has degree 2, this can only happen if g is homogeneous of degree 2. But in that case, the sensitivity of g at every point is exactly 2 [15, Proposition 3.7].

The bound on the sensitivity cannot be improved: the Boolean function

$$g(y_1, y_2, y_3) = y_1 y_2 y_3 + (1 - y_1)(1 - y_2)(1 - y_3)$$

has degree 2 and satisfies

$$g(0, 0, 0) = 1, \quad g(1, 0, 0) = g(0, 1, 0) = g(0, 0, 1) = 0.$$

2.1.3. Degree reduction. Let $\mathcal{F} \subseteq S_n$. We denote the restriction of \mathcal{F} to the coset $[i \mapsto j]$ by $\mathcal{F}|_{[i \mapsto j]}$. We can think of $\mathcal{F}|_{[i \mapsto j]}$ as a subset of S_{n-1} . Formally speaking, we can think of S_n as the set of all injections from $[n]$ to $[n]$. The restriction of S_n to the coset $[i \mapsto j]$ is isomorphic to the set of all injections from $[n] \setminus \{i\}$ to $[n] \setminus \{j\}$, which is the same as S_{n-1} up to renumbering.

Let f be the characteristic function of \mathcal{F} . The definition of degree using polynomials shows that $\deg f|_{[i \mapsto j]} \leq \deg f$, where f is the characteristic function of $\mathcal{F}|_{[i \mapsto j]}$.

⁽¹⁾Formally, the sensitivity of a Boolean function $g: \{0, 1\}^n \rightarrow \{0, 1\}$ is defined as follows. The sensitivity of g at a point $y \in \{0, 1\}^n$ is the number of coordinates $i \in [n]$ such that $g(y^{\oplus i}) \neq g(y)$, where $y^{\oplus i}$ results from y by flipping the i 'th coordinate. The sensitivity of g is obtained by calculating its sensitivity at all points in $\{0, 1\}^n$ and taking the maximum.

Indeed, given a polynomial representing f , we can obtain a polynomial representing $f|_{[i \mapsto j]}$ by substituting $x_{ij} = 1$ and $x_{i'j'} = x_{i'j} = 0$ for any $i' \neq i$ and $j' \neq j$.

The following crucial lemma, which forms part of the proof of [8, Lemma 5.7], states that if \mathcal{F} is contained in $[i \mapsto j]$, then the degree of f strictly decreases when restricted to $[i \mapsto j]$.

LEMMA 2.6. *Suppose that $f: S_n \rightarrow \{0, 1\}$ is the characteristic function of a family which is a subset of the coset $[i \mapsto j]$. Then $\deg f|_{[i \mapsto j]} \leq \max(\deg f - 1, 0)$.*

The corresponding result for functions on the Boolean cube $\{0, 1\}^n$ is easy to prove. Suppose that $f: \{0, 1\}^n \rightarrow \{0, 1\}$ is such that $f(x) = 1$ implies $x_n = 1$. We can write

$$f(x_1, \dots, x_n) = (1 - x_n)f_0(x_1, \dots, x_{n-1}) + x_n f_1(x_1, \dots, x_{n-1}).$$

Substituting $x_n = 0$, we get $f_0 = 0$, and so $f = x_n f_1$. The restriction of f to the coset $\{x : x_n = 1\}$ is f_1 . If $f_1 \neq 0$ then clearly $\deg f = \deg f_1 + 1$, where $\deg f$ is the degree of the unique multilinear polynomial representing f .

The argument proving Lemma 2.6 is more subtle, and uses representation theory.

Proof of Lemma 2.6. Assume, for concreteness, that $i = j = n$. We can assume that $\deg f|_{[n \mapsto n]} > 0$, since otherwise there is nothing to prove.

Our starting point is a well-known representation-theoretic formula for $f^{=\lambda}$, where $\lambda \vdash n$ is arbitrary:

$$f^{=\lambda}(\sigma) = \frac{\dim \lambda}{n!} \sum_{\pi \in S_n} \chi_\lambda(\sigma^{-1}\pi) f(\pi).$$

In this formula, χ_λ is the *irreducible character* corresponding to V^λ , and $\dim \lambda = \chi_\lambda(\text{id})$.

Suppose $\sigma \in [n \mapsto n]$. Since f is supported on $[n \mapsto n]$, we can further write

$$f^{=\lambda}(\sigma) = \frac{\dim \lambda}{n!} \sum_{\pi \in [n \mapsto n]} \chi_\lambda(\sigma^{-1}\pi) f(\pi).$$

Observe that $\sigma^{-1}\pi \in [n \mapsto n]$ as well.

Identifying $[n \mapsto n]$ with a copy of S_{n-1} , we can apply the well-known branching rule to obtain

$$\chi_\lambda(\sigma^{-1}\pi) = \sum_{\mu \triangleleft \lambda} \chi_\mu(\sigma^{-1}\pi).$$

Here $\mu \vdash n - 1$ goes over all partitions whose Ferrers diagram results from that of λ by removing a single box, and we treat $\sigma^{-1}\pi$ as an element of S_{n-1} on the right-hand side.

Substituting this formula in the formula for $f^{=\lambda}$, we obtain

$$f^{=\lambda}(\sigma) = \sum_{\mu \triangleleft \lambda} \frac{\dim \lambda}{n!} \sum_{\pi \in [n \mapsto n]} \chi_\mu(\sigma^{-1}\pi) f(\pi).$$

Combining this with the representation-theoretic formula for $(f|_{[n \mapsto n]})^{=\mu}$, we finally conclude

$$(1) \quad (f^{=\lambda})|_{[n \mapsto n]} = \sum_{\mu \triangleleft \lambda} \frac{\dim \lambda}{n \dim \mu} (f|_{[n \mapsto n]})^{=\mu}.$$

Using this formula, we can finish the proof. Let $\deg f|_{[n \mapsto n]} = d$. According to the definition of degree, there exists a partition $\mu \vdash n - 1$ with $\mu_1 = n - 1 - d$ and $(f|_{[n \mapsto n]})^{=\mu} \neq 0$. Among all such partitions, choose one with a maximal number of parts, say k .

Let $\lambda \vdash n$ be obtained from μ by adding a new part $\lambda_{k+1} = 1$. Consider all partitions $\kappa \triangleleft \lambda$:

- If κ results from removing a box from the first row then $\kappa_1 = n - 1 - (d + 1)$, and so $(f|_{[n \rightarrow n]})^{=\kappa} = 0$ by the definition of degree.
- If κ results from removing a box from a row other than the first or the last then κ has $k + 1$ parts, and so $(f|_{[n \rightarrow n]})^{=\kappa} = 0$ by the choice of μ .
- Otherwise, κ results from removing the last row, and so $\kappa = \mu$.

Applying (1), this shows that

$$(f^{=\lambda})|_{[n \rightarrow n]} = \frac{\dim \lambda}{n \dim \mu} (f|_{[n \rightarrow n]})^{=\mu} \neq 0.$$

In particular, $f^{=\lambda} \neq 0$, and so $\deg f \geq d + 1$ by the definition of degree. □

2.2. PERFECT MATCHING SCHEME. For integer $n \geq 1$, the *perfect matching scheme* \mathcal{M}_{2n} is the collection of all perfect matchings of the complete graph K_{2n} . Since the symmetric group S_n can be viewed as the collection of all perfect matchings of the complete *bipartite* graph $K_{n,n}$, we can think of the perfect matching scheme as the non-bipartite analog of the symmetric group.

The perfect matching scheme \mathcal{M}_{2n} contains $(2n - 1)!!$ many different perfect matchings. Here

$$(2n - 1)!! = (2n - 1)(2n - 3)(2n - 5) \cdots (5)(3)(1) = \frac{(2n)!}{2^n n!}.$$

We sometimes think of a perfect matching in \mathcal{M}_{2n} as a set of n pairs of elements from $[2n]$, which together cover all of $[2n]$ (the *pair representation*). At other times, it will be useful to think of a perfect matching in \mathcal{M}_{2n} as a fixed-point-free involution on $[2n]$, that is, $m(i) \neq i$ is the element that i is matched to, and $m(m(i)) = i$.

A *t-coset* of \mathcal{M}_{2n} is a set of the form

$$\{m \in \mathcal{M}_{2n} : m(i_1) = j_1, \dots, m(i_t) = j_t\},$$

where $i_1, \dots, i_t, j_1, \dots, j_t \in [2n]$ are distinct.⁽²⁾ A *t-coset* contains $(2(n - t) - 1)!!$ perfect matchings.

We use the term *coset* for a 1-coset. For $i \neq j \in [2n]$, the coset $[i \leftrightarrow j]$ consists of all perfect matchings in which i, j are matched:

$$[i \leftrightarrow j] = \{m \in \mathcal{M}_{2n} : m(i) = j\}.$$

Note that $[i \leftrightarrow j] = [j \leftrightarrow i]$.

Two perfect matchings $m_1, m_2 \in \mathcal{M}_{2n}$ are *t-intersecting* if there is a *t-coset* which contains both of them. A subset $\mathcal{F} \subseteq \mathcal{M}_{2n}$ is *t-intersecting* if any two perfect matchings in \mathcal{F} are *t-intersecting*.

2.2.1. *Degree.* We can represent perfect matchings $m \in \mathcal{M}_{2n}$ using $\binom{2n}{2}$ variables x_{ij} (where the index is an unordered pair of elements) whose semantics are: $x_{ij} = 1$ if $m(i) = j$, and $x_{ij} = 0$ otherwise. The *degree* of a function $f: \mathcal{M}_{2n} \rightarrow \mathbb{R}$, denoted $\deg f$, is the minimal d such that f can be represented as a degree d polynomial over the variables x_{ij} . For example, the characteristic function of a *t-coset* has degree at most t .

An equivalent way to define degree is via the representation theory of the perfect matching scheme. Let $\mathbb{R}[\mathcal{M}_{2n}]$ be the vector space of all real-valued functions over the perfect matching scheme. Representation theory (for example, [6, Chapter 11]) gives an orthogonal decomposition (with respect to the inner product $\langle f, g \rangle = \sum_{m \in \mathcal{M}_{2n}} f(m)g(m)$)

$$\mathbb{R}[\mathcal{M}_{2n}] = \bigoplus_{\lambda \vdash n} V^{2\lambda},$$

⁽²⁾This is a slight abuse of terminology as \mathcal{M}_{2n} doesn't afford a natural group structure.

where λ goes over all integer partitions of n , 2λ is the partition of $2n$ obtained by doubling each part, and $V^{2\lambda}$ are the isotypic components, which we define explicitly in Section 2.2.3. Accordingly, every function $f: \mathcal{M}_{2n} \rightarrow \mathbb{R}$ has a unique decomposition

$$f = \sum_{\lambda \vdash n} f^{\lambda},$$

where $f^{\lambda} \in V^{2\lambda}$. Lindzey [23, Theorem 5.1.1] showed that $\deg f$ is the maximal d such that $f^{\lambda} \neq 0$ for some λ satisfying $\lambda_1 = n - d$. In particular, every function has degree at most $n - 1$.

We can now formally state the main result of Fallat, Meagher and Shirazi [13, Theorem 4.13].

THEOREM 2.7. *If $n \geq 3$ then any 2-intersecting family $\mathcal{F} \subseteq \mathcal{M}_{2n}$ contains at most $(2n - 5)!!$ perfect matchings. Furthermore, if $f: \mathcal{M}_{2n} \rightarrow \{0, 1\}$ is the characteristic vector of a 2-intersecting family of size $(2n - 5)!!$, then $\deg f \leq 2$.*

We will also need a result of Dafni et al. [8] classifying degree 1 functions.

THEOREM 2.8 ([8, Theorem 6.2]). *If $f: \mathcal{M}_{2n} \rightarrow \{0, 1\}$ has degree at most 1 then f is either a dictator:*

$$f = \sum_{j \in J} x_{ij},$$

for some $i \in [2n]$ and $J \subseteq [2n]$ not containing i ; or f is a triangle:

$$f = x_{ij} + x_{ik} + x_{jk},$$

for some distinct $i, j, k \in [2n]$; or f is an anti-triangle:

$$f = 1 - x_{ij} - x_{ik} - x_{jk},$$

for some distinct $i, j, k \in [2n]$.

2.2.2. Certificate complexity. Fix n . A certificate is a set $C = \{\{i_1, j_1\}, \dots, \{i_r, j_r\}\} \subseteq \binom{[2n]}{2}$, where $\binom{[2n]}{2}$ is the collection of all subsets of $[2n]$ of size 2. A perfect matching $m \in \mathcal{M}_{2n}$ satisfies the certificate C if $m(i_1) = j_1, \dots, m(i_r) = j_r$.

A Boolean function is a function $f: \mathcal{M}_{2n} \rightarrow \{0, 1\}$. Just as in Section 2.1.2, for each $m \in \mathcal{M}_{2n}$ we can define the following notions: a certificate for m (with respect to f); the certificate complexity of m , denoted $C(f, m)$; a minimum certificate; and the certificate complexity of f , denoted $C(f)$. As in the case of the symmetric group, $C(f) \leq n - 1$, since a perfect matching is determined by any $n - 1$ of its edges. Dafni et al. [8] showed that $C(f)$ is polynomially related to $\deg f$.

In the case of the symmetric group, degree 1 functions have certificate complexity 1. This is no longer the case for the perfect matching scheme, since triangles and anti-triangles have certificate complexity 2.

LEMMA 2.9. *If $f: \mathcal{M}_{2n} \rightarrow \{0, 1\}$ has degree at most 1 then $C(f) \leq 2$.*

Proof. According to Theorem 2.8, f is either a dictator, a triangle, or an anti-triangle. If $f = \sum_{j \in J} x_{ij}$ is a dictator, then every m has a certificate $\{\{i, m(i)\}\}$. If $f = x_{ij} + x_{ik} + x_{jk}$ is a triangle, then every m such that $f(m) = 1$ has one of the certificates $\{\{i, j\}\}, \{\{i, k\}\}, \{\{j, k\}\}$, and every m such that $f(m) = 0$ has the certificate $\{\{i, m(i)\}, \{j, m(j)\}\}$. If f is an anti-triangle then $1 - f$ is a triangle, so we can use the same certificates as in the case of a triangle. \square

The proof of Lemma 2.4 extends to the perfect matching scheme with minor changes.

LEMMA 2.10. *Let $f: \mathcal{M}_{2n} \rightarrow \{0, 1\}$ be the characteristic function of a 2-intersecting family, and suppose that $C(f) \leq n - 2$. If $f(m_1) = f(m_2) = 1$ and C_{m_1}, C_{m_2} are minimum certificates for m_1, m_2 (respectively), then $|C_{m_1} \cap C_{m_2}| \geq 2$.*

Proof. We will show how to construct a perfect matching m'_1 satisfying C_{m_1} such that $m'_1 \cap C_{m_2} = C_{m_1} \cap C_{m_2}$. The same argument can be used to construct a perfect matching m'_2 satisfying C_{m_2} such that $m'_1 \cap m'_2 = m'_1 \cap C_{m_2}$. It follows that $|C_{m_1} \cap C_{m_2}| = |m'_1 \cap m'_2| \geq 2$, since f is the characteristic function of a 2-intersecting family.

Let $C_{m_1} = \{\{i_1, j_1\}, \dots, \{i_r, j_r\}\}$, where $r \leq n - 2$, and let $k_1, \dots, k_{2(n-r)}$ be the indices not mentioned in C_{m_1} . Rearrange the indices so that if $\{k_s, k_t\} \in C_{m_2}$ then $\{s, t\} = \{2\ell - 1, 2\ell\}$ for some $\ell \in [n - r]$. The perfect matching m'_1 has the pair representation

$$\{i_1, j_1\}, \dots, \{i_r, j_r\}, \{k_1, k_{n-r+1}\}, \{k_2, k_{n-r+2}\}, \dots, \{k_{n-r}, k_{2(n-r)}\}.$$

Since $n - r \geq 2$, the new pairs do not intersect C_{m_2} , completing the proof. \square

Using sensitivity, we can rule out the case $C(f) = n - 1$ for functions of degree at most 2, just as we did for the symmetric group.

LEMMA 2.11. *If $n \geq 8$ and $f: \mathcal{M}_{2n} \rightarrow \{0, 1\}$ has degree at most 2, then $C(f) \leq n - 2$.*

Proof. The proof is by contradiction. Suppose that $n \geq 8$, that $f: \mathcal{M}_{2n} \rightarrow \{0, 1\}$ has degree at most 2, and that $C(f) = n - 1$, say $C(f, m) = n - 1$ for $m = \{\{1, 2\}, \{3, 4\}, \dots, \{2n - 1, 2n\}\}$. Since $C(f, m) > n - 2$, $m \setminus \{\{2i - 1, 2i\}, \{2j - 1, 2j\}\}$ is not a certificate for m for any $i \neq j \in [n]$, and so there is a perfect matching $m^{ij} \supset m \setminus \{\{2i - 1, 2i\}, \{2j - 1, 2j\}\}$ such that $f(m^{ij}) \neq f(m)$; the perfect matching m^{ij} is obtained from m by replacing $\{\{2i - 1, 2i\}, \{2j - 1, 2j\}\}$ with either $\{\{2i - 1, 2j\}, \{2j - 1, 2i\}\}$ or $\{\{2i - 1, 2j - 1\}, \{2i, 2j\}\}$. We think of ij as an operation replacing $\{\{2i - 1, 2i\}, \{2j - 1, 2j\}\}$ with the appropriate two pairs.

We construct a function $g: \{0, 1\}^4 \rightarrow \{0, 1\}$ as follows: $g(y_1, y_2, y_3, y_4)$ is the value of f on the perfect matching obtained from m by applying the subset of the operations ^{12, 34, 56, 78} specified by the input (for example, if $y_1 = 1$ then we apply ¹²). As in the proof of Lemma 2.5, this results in a function of degree at most 2 satisfying $g(0, 0, 0, 0) \neq g(1, 0, 0, 0) = g(0, 1, 0, 0) = g(0, 0, 1, 0) = g(0, 0, 0, 1)$, which is impossible. \square

2.2.3. *Degree reduction.* Let $\mathcal{F} \subseteq \mathcal{M}_{2n}$. We denote the restriction of \mathcal{F} to the coset $[i \leftrightarrow j]$ by $\mathcal{F}|_{[i \leftrightarrow j]}$, which we can think of as a subset of $\mathcal{M}_{2(n-1)}$.

Let f be the characteristic function of \mathcal{F} . The definition of degree using polynomials shows that $\deg f|_{[i \leftrightarrow j]} \leq \deg f$. The following analog of Lemma 2.6 shows that if \mathcal{F} is contained in $[i \leftrightarrow j]$, then the degree of f strictly decreases when restricted to $[i \leftrightarrow j]$.

LEMMA 2.12. *Suppose that $f: \mathcal{M}_{2n} \rightarrow \{0, 1\}$ is the characteristic function of a family which is a subset of the coset $[i \leftrightarrow j]$. Then $\deg f|_{[i \leftrightarrow j]} \leq \max(\deg f - 1, 0)$.*

The proof is along the lines of the proof of Lemma 2.6.

Proof. Assume, for concreteness, that $i = 2n - 1$ and $j = 2n$. We can assume that $\deg f|_{[2n-1 \leftrightarrow 2n]} > 0$, since otherwise there is nothing to prove.

Our starting point is a formula for f^{λ} coming from the theory of association schemes:

$$f^{\lambda}(m) = \frac{1}{(2n - 1)!!} \sum_{m' \in \mathcal{M}_{2n}} \omega_{\lambda}(m, m') f(m'),$$

where ω_{λ} is the zonal spherical function corresponding to $V^{2\lambda}$. The analogous expression in the proof of Lemma 2.6 is $\dim \lambda \cdot \chi_{\lambda}(\sigma^{-1}\pi)$.

If $m \in [2n - 1 \leftrightarrow 2n]$ then, since f is supported on $[2n - 1 \leftrightarrow 2n]$, we can further write

$$f^{=\lambda}(m) = \frac{1}{(2n - 1)!!} \sum_{m' \in [2n-1 \leftrightarrow 2n]} \omega_\lambda(m, m') f(m'),$$

At this point we invoke a “branching rule” for the perfect matching association scheme due to Stanley [31] (see also Macdonald’s monograph [24]):

$$\omega_\lambda(m, m') = \sum_{\mu < \lambda} c(\lambda, \mu) \omega_\mu(m, m'),$$

where the branching coefficients $c(\lambda, \mu)$ are strictly positive. In the case of the symmetric group, the branching coefficients are $\frac{\dim \mu}{\dim \lambda}$ which, via the hook-length formula, can be expressed in terms of hook-lengths. Stanley gives a similar formula in our case, involving upper and lower hook-lengths. Stanley’s formula is stated in terms of Jack polynomials; the perfect matching scheme corresponds to $\alpha = 2$ (and the symmetric group to $\alpha = 1$). For this connection, see Macdonald [24] or Muzychuk [27].

The rest of the proof is virtually identical to the proof in Lemma 2.6 (and indeed, using Jack polynomials, one could produce a combined proof). Substituting the branching rule in the formula for $f^{=\lambda}$ gives

$$f^{=\lambda}(m) = \sum_{\mu < \lambda} \frac{c(\lambda, \mu)}{(2n - 1)!!} \sum_{m' \in [2n-1 \leftrightarrow 2n]} \omega_\mu(m, m') f(m'),$$

implying an analog of (1):

$$f^{=\lambda}|_{[2n-1 \leftrightarrow 2n]} = \sum_{\mu < \lambda} \frac{c(\lambda, \mu)}{2n - 1} (f|_{[2n-1 \leftrightarrow 2n]})^{=\mu}.$$

Let $\deg f|_{[2n-1 \leftrightarrow 2n]} = d$. Let $\mu \vdash n - 1$ be a partition with $\mu_1 = n - 1 - d$ and $(f|_{[2n-1 \leftrightarrow 2n]})^{=\mu} \neq 0$ having a maximal number of parts k . If $\lambda \vdash n$ is obtained from μ by adding a new part $\lambda_{k+1} = 1$ then as in the proof of Lemma 2.6, we obtain

$$f^{=\lambda}|_{[2n-1 \leftrightarrow 2n]} = \frac{c(\lambda, \mu)}{2n - 1} (f|_{[2n-1 \leftrightarrow 2n]})^{=\mu} \neq 0.$$

In particular, $f^{=\lambda} \neq 0$, and so $\deg f \geq d + 1$. □

3. MAIN THEOREM: SYMMETRIC GROUP

3.1. STRUCTURE THEOREM. We start with a simple observation on 2-intersecting families, whose statement requires a few definitions.

Two cosets $[i_1 \mapsto j_1], [i_2 \mapsto j_2]$ are *compatible* if their intersection is non-empty. In other words, either the two cosets are equal, or $i_1 \neq i_2$ and $j_1 \neq j_2$. A family $\mathcal{F} \subseteq S_n$ is *m-covered*, for some integer $m \geq 1$, if there are m compatible cosets $[i_1 \mapsto j_1], \dots, [i_m \mapsto j_m]$ such that $\mathcal{F} \subseteq [i_1 \mapsto j_1] \cup \dots \cup [i_m \mapsto j_m]$.

LEMMA 3.1. *Let f be the characteristic function of a 2-intersecting family $\mathcal{F} \subseteq S_n$, and suppose that $C(f) \leq n - 2$.*

If $\alpha \in \mathcal{F}$ then \mathcal{F} is m-covered for $m = C(f, \alpha) - 1$.

Proof. Let $C_\alpha = \{(i_1, j_1), \dots, (i_{m+1}, j_{m+1})\}$ be a minimal certificate for α . We will show that $\mathcal{F} \subseteq [i_1 \mapsto j_1] \cup \dots \cup [i_m \mapsto j_m]$.

Suppose that $\beta \in \mathcal{F}$, and let C_β be a minimal certificate for β . According to Lemma 2.4, $|C_\alpha \cap C_\beta| \geq 2$. Therefore $(i_s, j_s) \in C_\beta$ for some $s \in [m]$, implying that $\beta \in [i_s \mapsto j_s] \subseteq [i_1 \mapsto j_1] \cup \dots \cup [i_m \mapsto j_m]$. □

The following structure lemma provides a kind of converse to Lemma 3.1.

LEMMA 3.2. Let $\mathcal{F} \subseteq S_n$ be an m -covered family, say $\mathcal{F} \subseteq [i_1 \mapsto j_1] \cup \dots \cup [i_m \mapsto j_m]$. Suppose that its characteristic function f has degree at most 2.

Every α has a certificate of the form $\{(i_1, \alpha(i_1)), \dots, (i_m, \alpha(i_m)), (i, j)\}$, and in particular, $C(f) \leq m + 1$.

Proof. We will prove by induction on m that if $\mathcal{F} \subseteq S_n$ is m -covered, say $\mathcal{F} \subseteq [i_1 \mapsto j_1] \cup \dots \cup [i_m \mapsto j_m]$, and its characteristic function f has degree at most 2, then for every distinct $k_1, \dots, k_m \in [n]$, the restriction of f to $[i_1 \mapsto k_1] \cap \dots \cap [i_m \mapsto k_m]$ has degree at most 1. By Lemma 2.3, this restriction has certificate complexity at most 1. For any $\alpha \in S_n$, choosing $k_1 = \alpha(i_1), \dots, k_m = \alpha(i_m)$, we deduce that α has a certificate of the required form.

We start with the base case $m = 1$. Let $\mathcal{F} \subseteq S_n$ be such that $\mathcal{F} \subseteq [i_1 \mapsto j_1]$, and suppose that its characteristic function f has degree at most 2. Lemma 2.6 directly implies that the restriction of f to $[i_1 \mapsto j_1]$ has degree at most 1, as required.

Suppose now that $m \geq 2$. Let $\mathcal{F} \subseteq S_n$ be such that $\mathcal{F} \subseteq [i_1 \mapsto j_1] \cup \dots \cup [i_m \mapsto j_m]$, and suppose that its characteristic function f has degree at most 2. Consider any distinct $k_1, \dots, k_m \in [n]$. We need to show that the restriction of f to $[i_1 \mapsto k_1] \cap \dots \cap [i_m \mapsto k_m]$ has degree at most 1.

Suppose first that $k_s \neq j_s$ for some $s \in [m]$, say $k_m \neq j_m$. The restriction of \mathcal{F} to $[i_m \mapsto k_m]$ is contained in $[i_1 \mapsto j_1] \cup \dots \cup [i_{m-1} \mapsto j_{m-1}]$, and its characteristic function has degree at most 2. Therefore the induction hypothesis implies that the restriction of f to $[i_1 \mapsto k_1] \cap \dots \cap [i_m \mapsto k_m]$ has degree at most 1.

It remains to show that the restriction of f to $[i_1 \mapsto j_1] \cap \dots \cap [i_m \mapsto j_m]$ has degree at most 1. To see this, consider the function

$$g := x_{i_1 j_1} \cdots x_{i_m j_m} f = f - \sum_{\substack{k_1, \dots, k_m \\ (k_1, \dots, k_m) \neq (j_1, \dots, j_m)}} x_{i_1 k_1} \cdots x_{i_m k_m} f.$$

We showed above that the restriction of f to $[i_1 \mapsto k_1] \cap \dots \cap [i_m \mapsto k_m]$ has degree at most 1 whenever $(k_1, \dots, k_m) \neq (j_1, \dots, j_m)$. Therefore every term in the sum on the right-hand side has degree at most $m + 1$, since it can be represented as $x_{i_1 k_1} \cdots x_{i_m k_m}$ times the linear polynomial representing the restriction of f to $[i_1 \mapsto k_1] \cap \dots \cap [i_m \mapsto k_m]$. Consequently, $\deg g \leq \max(\deg f, m + 1) = m + 1$.

By construction, g is the characteristic function of a family which is contained in the coset intersection $\mathcal{I} = [i_1 \mapsto j_1] \cap \dots \cap [i_m \mapsto j_m]$. Applying Lemma 2.6 repeatedly, we conclude that the restriction of g to \mathcal{I} has degree at most $\max(m + 1 - m, 0) = 1$. Since the restriction of f to \mathcal{I} is the same as the restriction of g to \mathcal{I} , this completes the proof of the induction step. \square

3.2. PROOF OF MAIN THEOREM. We are now ready to prove our main upper bound, from which Theorem 1.3 will quickly follow.

LEMMA 3.3. Let $\mathcal{F} \subseteq S_n$ be a 2-intersecting family whose characteristic function f has degree at most 2. Suppose that $C(f) \leq n - 2$.

The family \mathcal{F} is contained in the union of at most T many $C(f)$ -cosets, where

$$T = \frac{2^{\lfloor C(f)/2 \rfloor} (C(f) - 1)!}{2^{\lfloor C(f)/2 \rfloor}}.$$

Moreover, $C(f) \geq 2$.

Proof. If \mathcal{F} is empty then the lemma is vacuously true. Otherwise, choose an arbitrary $\beta \in \mathcal{F}$. Lemma 2.4, applied to $\alpha = \beta$, shows that $C(f, \beta) \geq 2$. Lemma 3.1 shows that \mathcal{F} is m -covered, where $m = C(f, \beta) - 1$. Without loss of generality, we can assume that $\mathcal{F} \subseteq [1 \mapsto 1] \cup \dots \cup [m \mapsto m]$.

Lemma 3.2 shows that $C(f) \leq m + 1$, and so $m = C(f) - 1$ and $C(f) = C(f, \beta) \geq 2$. The lemma also shows that every $\alpha \in \mathcal{F}$ has a certificate C_α of the form $\{(1, \alpha(1)), \dots, (m, \alpha(m)), (i, j)\}$. It also implies that \mathcal{F} is not $(m - 1)$ -covered (otherwise its certificate complexity would be at most $m < C(f)$). Therefore, if X is any set of at most $m - 1$ compatible cosets, then there is $\alpha_X \in \mathcal{F}$ which is not contained in the union of the cosets in X . Fix a certificate C_X for α_X of the form $\{(1, \alpha_X(1)), \dots, (m, \alpha_X(m)), (i_X, j_X)\}$.

We think of the certificates C_α, C_X as colored using $m + 1$ colors: for $k \in [m]$, a pair $(k, *)$ is colored k , and the remaining pair (i, j) is colored $m + 1$. We can determine the color of a pair (i', j') given only the pair: if $i' \in [m]$ then the color is i' , and otherwise the color is $m + 1$.

We will prove the lemma by showing that if $\alpha \in \mathcal{F}$ then C_α is one of at most T possible certificates. Since C_α corresponds to a $C(f)$ -coset containing α , the lemma would follow. In the proof, we will freely identify a pair (i, j) with the corresponding coset $[i \mapsto j]$, and from now on we will only ever mention cosets.

The proof is slightly different depending on the parity of $C(f)$. We start with the case in which $C(f) = 2r$ is even. Let $\alpha \in \mathcal{F}$. We define a sequence $X_0 \subset \dots \subset X_r \subseteq C_\alpha$, where X_s is a collection of $2s$ compatible cosets, as follows. The starting point is $X_0 = \emptyset$. Now let $s < r$, and suppose that X_s has been defined. Since $|X_s| = 2s \leq C(f) - 2 = m - 1$, there is a permutation $\alpha_{X_s} \in \mathcal{F}$ which lies outside of the union of the cosets in X_s . According to Lemma 2.4, the certificates C_α and C_{X_s} must have at least two cosets in common q_1, q_2 . The choice of α_{X_s} guarantees that $q_1, q_2 \notin X_s$, and we define $X_{s+1} = X_s \cup \{q_1, q_2\}$. By construction, $X_{s+1} \subseteq C_\alpha$, and so the cosets in X_{s+1} are indeed compatible.

Note that $|X_r| = 2r = |C_\alpha|$, and so $X_r = C_\alpha$. For $s < r$, given X_s , the set $X_{s+1} \setminus X_s$ consists of two cosets belonging to C_{X_s} . Moreover, since $X_{s+1} \subseteq C_\alpha$ and C_α contains one coset of each color, the colors of the two cosets in $X_{s+1} \setminus X_s$ are different from the colors of the cosets in X_s . Since C_{X_s} also contains one coset of each color, this means that there are $\binom{|C_{X_s}| - |X_s|}{2} = \binom{C(f) - 2s}{2}$ choices for $X_{s+1} \setminus X_s$. In total, the number of possible choices for C_α is

$$\prod_{s=0}^{r-1} \binom{C(f) - 2s}{2} = \frac{C(f)!}{2^r},$$

matching the formula for T in the statement of the lemma.

Now suppose that $C(f) = 2r + 1$ is odd. Let $\alpha \in \mathcal{F}$. We define a sequence $X_0 \subset \dots \subset X_r \subseteq C_\alpha$, where X_s is a collection of $2s + 1$ compatible cosets, as follows. We start by defining X_0 . According to Lemma 2.4, the certificates C_α and C_\emptyset must contain at least two cosets in common q_1, q_2 . At least one of these cosets is of the form $(k, \alpha(k))$ for some $k \in [m]$. We define $X_0 = \{(k, \alpha(k))\}$. Now let $s < r$, and suppose that X_s has been defined. Since $|X_s| = 2s + 1 \leq C(f) - 2 = m - 1$, we can define X_{s+1} as in the case of even $C(f)$.

Note that $|X_r| = 2r + 1 = |C_\alpha|$, and so $X_r = C_\alpha$. The certificate X_0 contains a single coset $(k, \ell) \in C_\emptyset$, where $k \in [m]$, and so there are $m = C(f) - 1$ choices for X_0 . For $s < r$, given X_s , there are $\binom{|C_{X_s}| - |X_s|}{2} = \binom{C(f) - 2s - 1}{2}$ choices for X_{s+1} (this is the same as in the case of even $C(f)$). In total, the number of possible choices for C_α is

$$(C(f) - 1) \prod_{s=0}^{r-1} \binom{C(f) - 2s - 1}{2} = (C(f) - 1) \frac{(C(f) - 1)!}{2^r},$$

matching the formula for T in the statement of the lemma. □

We can now prove Theorem 1.3.

Proof of Theorem 1.3. Let \mathcal{F} be a 2-intersecting subset of S_n of size $(n - 2)!$, where $n \geq 2$, and let f be its characteristic function. Our goal is to show that

$$\mathcal{F} = \{\alpha \in S_n : \alpha(i_1) = j_1 \text{ and } \alpha(i_2) = j_2\}$$

for some $i_1 \neq i_2 \in [n]$ and $j_1 \neq j_2 \in [n]$. In other words, we need to show that \mathcal{F} is a 2-coset.

Suppose first that $n \geq 8$. Since $n \geq 5$, Theorem 2.1 shows that $\deg f \leq 2$. Lemma 2.5 shows that $C(f) \leq n - 2$. Lemma 3.3 implies that $C(f) \geq 2$ and $|\mathcal{F}| \leq T(n - C(f))!$, where $T = 2^{\lfloor C(f)/2 \rfloor} (C(f) - 1)! / 2^{\lfloor C(f)/2 \rfloor}$. Therefore

$$\frac{(n - 2)!}{(n - C(f))!} \leq \frac{2^{\lfloor C(f)/2 \rfloor} (C(f) - 1)!}{2^{\lfloor C(f)/2 \rfloor}}.$$

The left-hand side is clearly increasing in n . Since $n \geq C(f) + 2$, it follows that

$$\frac{C(f)!}{2} \leq \frac{2^{\lfloor C(f)/2 \rfloor} (C(f) - 1)!}{2^{\lfloor C(f)/2 \rfloor}},$$

and so

$$C(f) \leq \frac{2^{\lfloor C(f)/2 \rfloor}}{2^{\lfloor C(f)/2 \rfloor - 1}}.$$

If $C(f) = 2r$ is even then this reads $2r \leq 2r/2^{r-1}$, and so $r = 1$. If $C(f) = 2r + 1$ is odd then this reads $2r + 1 \leq 2r/2^{r-1}$, which never holds. We conclude that $C(f) = 2$.

Let $\alpha \in \mathcal{F}$ be arbitrary. Since $C(f) = 2$, there is a certificate of the form $\{(i_1, j_1), (i_2, j_2)\}$ for α . The number of permutations satisfying this certificate is $(n - 2)!$, and we conclude that \mathcal{F} consists of all permutations satisfying the certificate, completing the proof in this case.

In order to complete the proof, we consider the case $n \leq 7$. It suffices to show that for $n \in \{2, 3, 4, 5, 6, 7\}$, any 2-intersecting family of size $(n - 2)!$ which contains the identity permutation is of the form

$$\{\alpha \in S_n : \alpha(i_1) = i_1, \alpha(i_2) = i_2\}$$

for some $i_1 \neq i_2$.

If $n = 2$ or $n = 3$ then $(n - 2)! = 1$, and the claim can be verified directly. In order to verify the remaining cases $n \in \{4, 5, 6, 7\}$, we formulate the task as a maximum clique problem. Let G_n be the graph whose vertex set consists of all permutations which 2-intersect the identity permutation, and in which two permutations are connected if they 2-intersect. A 2-intersecting family containing the identity permutation is the same as a clique in G_n . Using SAGE [30], which internally uses the software library Cliquer [28], we verify that a maximum clique in G_n contains $(n - 2)!$ many permutations, and furthermore, there are exactly $\binom{n}{2}$ many maximum cliques, matching the number of 2-cosets containing the identity permutation. The relevant code appears in Appendix A.2. This completes the proof. \square

4. MAIN THEOREM: PERFECT MATCHING SCHEME

The proof of Theorem 1.4 is similar to that of Theorem 1.3, but there is an additional complication: Lemma 2.9 only states that degree 1 Boolean functions have certificate complexity 2. This causes the proof of Lemma 3.2 to fail. In order to get around this, we show that the bad case never occurs for *(inclusion-)maximal* 2-intersecting families.

Fix $n \geq 1$, and work over \mathcal{M}_{2n} . We start by defining *extended certificates*. An extended certificate of size c is either a standard certificate of size c , or a pair $(C', \{i, j, k\})$, where C' is a certificate of size $c - 1$ and $i, j, k \in [2n]$ are distinct elements not appearing in C' . A perfect matching $m \in \mathcal{M}_{2n}$ satisfies a certificate $(C', \{i, j, k\})$ if it satisfies C' and doesn't contain any of the pairs $\{i, j\}, \{i, k\}, \{j, k\}$.

LEMMA 4.1. *Fix $n \geq 1$, work over \mathcal{M}_{2n} , and let $c \leq n - 2$. Let $C_1 = (C'_1, \{i, j, k\})$ be an extended certificate of size c , and let C_2 be an extended certificate of size c . If any perfect matching satisfying C_1 and any perfect matching satisfying C_2 are 2-intersecting, then the same holds when C_1 is replaced by C'_1 .*

Proof. Suppose that m_1 is a perfect matching satisfying C'_1 but not C_1 . Without loss of generality, $m_1(i) = j$. We need to show that if m_2 is a perfect matching satisfying C_2 then m_1 and m_2 are 2-intersecting.

Since m_1 satisfies C'_1 , its pair representation consists of C'_1 together with $n - (c - 1) \geq 3$ more pairs, one of which is $\{i, j\}$. Among the other pairs, at least one $\{a, b\}$ does not involve k . Consider the following two perfect matchings:

$$m'_1 = m_1 \setminus \{\{i, j\}, \{a, b\}\} \cup \{\{i, a\}, \{j, b\}\},$$

$$m''_1 = m_1 \setminus \{\{i, j\}, \{a, b\}\} \cup \{\{i, b\}, \{j, a\}\}.$$

Both of these perfect matchings avoid the edges $\{i, j\}, \{i, k\}, \{j, k\}$. If m_2 contains $\{i, a\}$ then it cannot contain $\{i, b\}$ or $\{j, a\}$, and similarly for $\{j, b\}, \{i, b\}, \{j, a\}$. Therefore m_2 cannot intersect both $\{\{i, a\}, \{j, b\}\}$ and $\{\{i, b\}, \{j, a\}\}$. Suppose that m_2 does not intersect $\{\{i, a\}, \{j, b\}\}$. Then $m'_1 \cap m_2 \subseteq m_1 \cap m_2$. On the other hand, m'_1 satisfies C_1 , and so $|m'_1 \cap m_2| \geq 2$. We conclude that $|m_1 \cap m_2| \geq 2$, as needed. \square

4.1. STRUCTURE THEOREM. Two cosets $[i_1 \leftrightarrow j_1], [i_2 \leftrightarrow j_2]$ are *compatible* if their intersection is non-empty. A family is *r-covered* if it is contained in the union of r compatible cosets.

LEMMA 4.2. *Let f be the characteristic function of a 2-intersecting family $\mathcal{F} \subseteq \mathcal{M}_{2n}$, and suppose that $C(f) \leq n - 2$.*

If $m \in \mathcal{F}$ then \mathcal{F} is r -covered for $r = C(f, m) - 1$.

Since the proof is identical to that of Lemma 3.1 (with Lemma 2.10 standing for Lemma 2.4), we omit it.

The following structure theorem is the analog of Lemma 3.2. There are three differences in the statement. First, we assume that \mathcal{F} is *maximal*: it is not properly contained in any larger 2-intersecting family. Second, we assume that $r \leq n - 2$. Third, we only provide certificates to perfect matchings which belong to \mathcal{F} .

LEMMA 4.3. *Let $\mathcal{F} \subseteq \mathcal{M}_{2n}$ be an r -covered family, say $\mathcal{F} \subseteq [i_1 \leftrightarrow j_1] \cup \dots \cup [i_r \leftrightarrow j_r]$, where $r \leq n - 3$. Suppose that \mathcal{F} is maximal, and that its characteristic function f has degree at most 2.*

Every $m \in \mathcal{F}$ has a certificate of the form $\{\{i_1, m(i_1)\}, \dots, \{i_r, m(i_r)\}, \{i, j\}\}$.

Note that the certificate could mention the same pair twice, for example if $m(i_1) = i_2$.

Proof. The first step is to prove that for every distinct k_1, \dots, k_r , the restriction of f to $[i_1 \leftrightarrow k_1] \cap \dots \cap [i_r \leftrightarrow k_r]$ has degree at most 1, assuming the coset intersection is non-empty. This part is identical to the proof of Lemma 3.2, so we do not repeat it here (the only difference is that we never consider k_1, \dots, k_r such that the coset intersection is empty).

Now suppose that $m \in \mathcal{F}$. Then the restriction of f to $I = [i_1 \leftrightarrow m(i_1)] \cap \dots \cap [i_r \leftrightarrow m(i_r)]$ (which is non-empty) has degree 1. According to Theorem 2.8, $f|_I$ is either a dictator, a triangle, or an anti-triangle. If $f|_I$ is a dictator or a triangle, then $m|_I$ has a certificate of size 1, and we are done. We would like to show that the remaining case is impossible.

Indeed, suppose that $f|_I$ is an anti-triangle. Then m has an extended certificate $(C', \{i, j, k\})$ of size $r + 1 \leq n - 2$. The same argument shows that every perfect

matching in \mathcal{F} has an extended certificate of size $r + 1 \leq n - 2$. Therefore Lemma 4.1 implies that if $m' \notin \mathcal{F}$ is a perfect matching satisfying C' , then $\mathcal{F} \cup \{m'\}$ is also 2-intersecting. Hence either \mathcal{F} is not maximal (if such $m' \notin \mathcal{F}$ exists), or in fact C' is already a certificate for m . \square

4.2. PROOF OF MAIN THEOREM. We can now prove the analog of Lemma 3.3, from which Theorem 1.4 will easily follow. The only difference in the statement is the additional assumption that \mathcal{F} is maximal.

LEMMA 4.4. *Let $\mathcal{F} \subseteq \mathcal{M}_{2n}$ be a maximal 2-intersecting family whose characteristic function f has degree at most 2, and suppose that $C(f) \leq n - 2$. Let $C_1(f)$ the maximum certificate complexity of any $m \in \mathcal{F}$.*

The family \mathcal{F} is contained in the union of at most T many $C_1(f)$ -cosets, where

$$T = \frac{2 \lfloor C_1(f)/2 \rfloor (C_1(f) - 1)!}{2^{\lfloor C_1(f)/2 \rfloor}}.$$

Moreover, $C_1(f) \geq 2$.

Proof. Let $\mu \in \mathcal{F}$ be a perfect matching satisfying $C(f, \mu) = C_1(f)$. Lemma 2.10, applied to $m_1 = m_2 = \mu$, shows that $C(f, \mu) \geq 2$, and so $C_1(f) \geq 2$. Lemma 4.2 shows that \mathcal{F} is r -covered, where $r = C(f, \mu) - 1$. Note that $r \leq C(f) - 1 \leq n - 3$. Without loss of generality, we can assume that $\mathcal{F} \subseteq [1 \leftrightarrow r + 1] \cup [2 \leftrightarrow r + 2] \cup \dots \cup [r \leftrightarrow 2r]$.

Lemma 4.3 shows that every $m \in \mathcal{F}$ has a certificate of the form $\{\{1, m(1)\}, \dots, \{r, m(r)\}, \{i, j\}\}$, which we call the *standard certificate*. The lemma also implies that \mathcal{F} is not $(r - 1)$ -covered: otherwise, every perfect matching in \mathcal{F} would have a certificate of size $r < C_1(f)$, which contradicts the definition of $C_1(f)$.

A standard certificate $\{\{1, m(1)\}, \dots, \{r, m(r)\}, \{i, j\}\}$ satisfies $m(1), \dots, m(r) \notin [r]$, since otherwise Lemma 4.3 would imply that \mathcal{F} is $(r - 1)$ -covered (since some of the pairs $\{t, m(t)\}$ coincide), which is impossible. This allows us to color the pairs in a standard certificate unambiguously: a pair $\{t, m(t)\}$ is colored t , and a pair $\{i, j\}$ with $i, j \notin [r]$ is colored $r + 1$. Every standard certificate contains exactly one pair of each color.

The remainder of the proof is identical to that of Lemma 3.3. \square

We can now prove Theorem 1.4. The proof is very similar to that of Theorem 1.3.

Proof of Theorem 1.4. Let \mathcal{F} be a 2-intersecting subset of \mathcal{M}_{2n} of size $(2n - 5)!!$, where $n \geq 2$, and let f be its characteristic function. Our goal is to show that

$$\mathcal{F} = \{m \in \mathcal{M}_{2n} : m(i_1) = j_1 \text{ and } m(i_2) = j_2\}$$

for some distinct $i_1, j_1, i_2, j_2 \in [2n]$.

Suppose first that $n \geq 8$. Since $n \geq 3$, Theorem 2.7 shows that \mathcal{F} is maximal and that $\deg f \leq 2$. Lemma 2.11 shows that $C_1(f) \leq n - 2$. Lemma 4.4 implies that $C_1(f) \geq 2$ and $|\mathcal{F}| \leq T(2n - 2C_1(f) - 1)!!$, where $T = 2 \lfloor C_1(f)/2 \rfloor (C_1(f) - 1)! / 2^{\lfloor C_1(f)/2 \rfloor}$. Therefore

$$\frac{(2n - 5)!!}{(2n - 2C_1(f) - 1)!!} \leq \frac{2 \lfloor C_1(f)/2 \rfloor (C_1(f) - 1)!}{2^{\lfloor C_1(f)/2 \rfloor}}.$$

The left-hand side is clearly increasing in n . Since $n \geq C_1(f) + 2$, it follows that

$$\frac{(2C_1(f))!}{3 \cdot 2^{C_1(f)} C_1(f)!} = \frac{(2C_1(f) - 1)!!}{3} \leq \frac{2 \lfloor C_1(f)/2 \rfloor (C_1(f) - 1)!}{2^{\lfloor C_1(f)/2 \rfloor}},$$

and so

$$(2C_1(f))! \leq 3 \cdot 2^{\lfloor C_1(f)/2 \rfloor} C_1(f)! (2 \lfloor C_1(f)/2 \rfloor (C_1(f) - 1)!).$$

If $C_1(f)$ is even then this reads

$$(2C_1(f))! \leq 3 \cdot 2^{C_1(f)/2} C_1(f)!^2.$$

When $C = C_1(f)$ increases by 1, the left-hand side increases by a factor of $(2C + 2)(2C + 1)$, while the right-hand side increases by a factor of $\sqrt{2}(C + 1)^2$, which is smaller for $C \geq 2$. When $C = 3$, the inequality fails. Therefore in this case, $C_1(f) = 2$.

If $C_1(f)$ is odd then the inequality reads

$$(2C_1(f))! \leq 3 \cdot 2^{(C_1(f)+1)/2} C_1(f)!(C_1(f) - 1)(C_1(f) - 1)! < 3 \cdot 2^{(C_1(f)+1)/2} C_1(f)!^2.$$

As before, if we increase $C = C_1(f)$ by 1, the left-hand side increases by a larger factor than the (far) right-hand side. When $C = 3$, the inequality fails, and so this case cannot happen.

Now let $m \in \mathcal{F}$ be arbitrary. Since $C_1(f) = 2$, there is a certificate of the form $\{\{i_1, j_1\}, \{i_2, j_2\}\}$ for m . The number of permutations satisfying this certificate is $(2n - 5)!!$, and we conclude that \mathcal{F} consists of all perfect matchings satisfying the certificate, completing the proof in this case.

In order to complete the proof, we consider the case $n \leq 7$. It suffices to show that for $n \in \{2, 3, 4, 5, 6, 7\}$, any 2-intersecting family of size $(2n - 5)!!$ which contains the perfect matching $\{1, n + 1\}, \dots, \{n + 2n\}$ is of the form

$$\{m \in \mathcal{M}_{2n} : m(i_1) = n + i_1, m(i_2) = n + i_2\}$$

for some $i_1 \neq i_2 \in [n]$.

If $n = 2$ or $n = 3$ then $(2n - 5)!! = 1$, and the claim can be verified directly. In order to verify the remaining cases $n \in \{4, 5, 6, 7\}$, we use exhaustive search just as in the proof of Theorem 1.3. The relevant code appears in Appendix A.3. \square

APPENDIX A. SAGE CODE

The code snippets below form part of the proofs of Lemma 2.5, Theorem 1.3, and Theorem 1.4. They are also available as part of the arXiv version of the paper.

A.1. CODE FOR LEMMA 2.5.

```
import itertools

def degree_2_functions():
    """Generate vector space of degree 2 functions,
    represented as truth table"""
    return span(QQ, [vector(int((x & m) == m) for x in range(16)) \
                     for m in [0, 1, 2, 4, 8, 3, 5, 9, 6, 10, 12]])

def verify_no_sensitivity_4():
    """Verify that no function from {0,1}^4 to {0,1} has
    sensitivity 4 at zero"""
    degree_2 = degree_2_functions()
    return not any(f for f in itertools.product([0,1], repeat=16) \
                  if f[0] != f[1] == f[2] == f[4] == f[8] and \
                  vector(f) in degree_2)
```

A.2. CODE FOR THEOREM 1.3.

```
def intersection_size(a, b):
    """Size of intersection of two permutations"""
    return sum(i==j for (i,j) in zip(a,b))

def generate_G_n(n, t):
```

```

    """Generate graph whose cliques are t-intersecting families of S_n
    containing id"""
    idp = list(range(n))
    V = [a for a in Permutations(range(n)) if \
         intersection_size(a, idp) >= t]
    E = [(a,b) for a in V for b in V if \
         t <= intersection_size(a, b) < n]
    return Graph([V, E])

def verify_main_theorem_n(n):
    """Verify that all 2-intersecting families containing id
    are 2-cosets for given n"""
    G_n = generate_G_n(n, 2)
    return G_n.clique_number() == factorial(n - 2) and \
           len(G_n.cliques_maximum()) == binomial(n, 2)

def verify_main_theorem():
    """Verify that all 2-intersecting families containing id
    are 2-cosets for 4 <= n <= 7"""
    return all(verify_main_theorem_n(n) for n in [4,5,6,7])

```

A.3. CODE FOR THEOREM 1.4.

```

def intersection_size(a, b):
    """Size of intersection of two perfect matchings"""
    return len(a.intersection(b))

def generate_G_n(n, t):
    """Generate graph whose cliques are t-intersecting families of M_2n
    containing a fixed PM"""
    idm = frozenset(PerfectMatchings(2*n).__iter__().__next__())
    V = [frozenset(a) for a in PerfectMatchings(2*n) if \
         intersection_size(frozenset(a), idm) >= t]
    E = [(a,b) for a in V for b in V if \
         t <= intersection_size(a, b) < n]
    return Graph([V, E])

def verify_main_theorem_n(n):
    """Verify that all 2-intersecting families containing a fixed PM are
    2-cosets for given n"""
    G_n = generate_G_n(n, 2)
    return G_n.clique_number() == \
           factorial(2*n-4)/(2^(n-2) * factorial(n-2)) and \
           len(G_n.cliques_maximum()) == binomial(n, 2)

def verify_main_theorem():
    """Verify that all 2-intersecting families containing a fixed PM are
    2-cosets for 4 <= n <= 7"""
    return all(verify_main_theorem_n(n) for n in [4,5,6,7])

```

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