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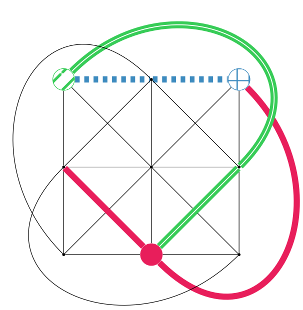
Abstract A triality is a sort of super-symmetry that exchanges the types of the elements of an incidence geometry in cycles of length three. Although geometries with trialities exhibit fascinating behavior, their construction is challenging, making them rare in the literature. To understand trialities more deeply, it is crucial to have a wide variety of examples at hand. In this article, we introduce a general method for constructing various rank-three incidence systems with trialities. Specifically, for any rank two incidence system Γ, we define its triangle complex $\Delta(\Gamma)$, a rank three incidence system whose elements consist of three copies of the flags (pairs of incident elements) of Γ. This triangle complex always admits a triality that cyclically permutes the three copies. We then explore in detail the properties of the triangle complex when Γ is a linear space, including flag-transitivity, the existence of dualities, and connectivity properties. As a consequence of our work, this construction yields the first infinite family of thick, flag-transitive and residually connected geometries with trialities but no dualities.



1. Introduction

Incidence geometries are geometric objects formed of elements of different types together with an incidence relation between them. In the more classical examples, incidence geometries are formed of points, lines, faces, etc. Some incidence geometries allow for some type of super-symmetry that exchanges the role of their types. These

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symmetries are formally called correlations. The most classical example is the one of a duality or a polarity, which exchanges the role of points and hyperplanes in projective geometries or the role of points and faces in polyhedra. Higher order collineations have not been systematically studied. That being said, order three correlations, also called trialities did make some very noticeable appearances in the history of mathematics.

The notion of triality appeared in papers of Study (see [14, Page 435], see also [15] and [16][(1)]), where he used a quadric in a seven dimensional projective plane to describe physical motions. Cartan later gave the official name of triality [6] to this phenomenon, and defined it in the broader context of Lie groups. This quadric was later investigated by Tits in [17]. In [12], it is shown how to obtain interesting geometries from the absolute points and the moving lines of this same quadric under the actions of the triality. Freudenthal further explored triality in the context of Lie algebras [7]. The notion of triality also appears in the study of maps on surfaces, where a triality is obtained as a composition of the dual operator and the Petrie dual operator (see [1, 8, 18]). Recently, Leemans and Stokes [10] showed how to construct coset geometries having trialities by using a group G that has outer automorphisms of order three. They used this technique in [11] to construct the first infinite family of coset geometries admitting trialities but no dualities. In [13], the first infinite family of flag-transitive geometries admitting trialities but no dualities is given. The construction uses coset geometries arising from Suzuki groups $\mathrm{Sz}(q)$, where $q = 2^{2e+1}$ with e a positive integer and $2e+1$ divisible by 3. We refer to the introduction of [13] for a more detailed account of the history of trialities.

In this article, we present a new construction that produces a rank three geometry $\Delta(\Gamma)$ with trialities from any given rank two geometry Γ (see Construction 1). The elements of this new geometry $\Delta(\Gamma)$ are three copies of the flags of Γ and the trialities exchange cyclically these three copies. Since the chambers of $\Delta(\Gamma)$ correspond to triangles of Γ, we call $\Delta(\Gamma)$ the triangle complex of Γ. We then study in detail the case where Γ is a linear space. We show that $\Delta(\Gamma)$ is flag-transitive if and only if Γ is transitive on its set of non-collinear triples of points (Theorem 3.13). This prompts us to give a classification of such linear spaces. We achieve this in Theorem 1.1, using the previous classification of locally two-transitive linear spaces given in [5]. We then obtain the following result, which may also be of interest to researchers studying linear spaces.

Theorem 1.1. *Assume Γ is a flag-transitive linear space of v points with a group G acting transitively on the set $\mathcal{T}(\Gamma)$ of ordered non-collinear triples of points of Γ. Then one of the following occurs:*

(1) $\Gamma = \mathrm{PG}(n,q)$, $v = \frac{q^{n+1}-1}{q-1}$, $\mathrm{PSL}(n+1,q) \trianglelefteq G \leqslant \mathrm{P\Gamma L}(n+1,q)$ *with* $n \geqslant 2$.
(2) $\Gamma = \mathrm{PG}(3,2)$, $v = 15$ *with* $G \cong \mathrm{A}_7$.
(3) $\Gamma = \mathrm{AG}(n,q)$, $v = p^d = q^n$, $G = \mathbb{F}_p^d : G_0$ *with* $\mathrm{SL}(n,q) \trianglelefteq G_0$, $q \geqslant 3$ *and* $n \geqslant 3$ *or* $\mathrm{GL}(n,q) \trianglelefteq G_0$, $q \geqslant 3$ *and* $n = 2$.
(4) Γ *is a hermitian unital* $\mathrm{U_H}(q)$, $v = q^3+1$, $G = \mathrm{P\Gamma U}(3,q)$ *and* $q = 2$ *or* 4.
(5) Γ *is a circle and G is a 3-transitive permutation group:*
  (a) $G = \mathrm{A}_v, G = \mathrm{S}_v$, $v \geqslant 3$.
  (b) $G \trianglerighteq \mathrm{PSL}(2,q)$, $v = q+1$ *and G normalizes a sharply 3-transitive permutation group.*
  (c) $G = \mathrm{M}_v$, $v = 11, 12, 22, 23, 24$ *or* $G = \mathrm{Aut}(\mathrm{M}_{22})$, $v = 22$.
  (d) $G = \mathrm{M}_{11}$, $v = 12$.
  (e) $G = \mathrm{A}_7$, $v = 15$.

(1) See `http://neo-classical-physics.info/uploads/3/4/3/6/34363841/study-analytical_kinematics.pdf` for an english translation of [16].

(f) $G = \mathbb{F}_2^n : G_0$, $v = 2^n$, $G_0 \trianglerighteq \mathrm{SL}(n,2)$ *with* $n \geqslant 2$.
(g) $G = \mathbb{F}_2^4 : \mathrm{A}_7$, $v = 16$.

The converse is also true. For any pair (Γ, G) *satisfying the conditions of one of the cases given above, the action of* G *is transitive on* $\mathcal{T}(\Gamma)$.

We then investigate which of the geometries Γ appearing in the classification theorem above yields a firm, residually connected and flag-transitive geometry $\Delta(\Gamma)$. We obtain the following classification theorem.

Theorem 1.2. *Assume* Γ *is a linear space of* v *points. The geometry* $\Delta(\Gamma)$ *obtained from Construction* 1 *is firm, residually connected and flag-transitive if and only if* Γ *is one of the following:*

(1) *A projective plane* $\mathrm{PG}(2,q)$ *with* $q \geqslant 2$ *and* $v = q^2+q+1$*;*
(2) *An affine plane* $\mathrm{AG}(2,q)$ *with* $q \geqslant 3$ *and* $v = q^2$*;*
(3) *A hermitian unital* $\mathrm{U_H}(q)$ *with* $q = 2$ *or* 4 *and* $v = q^3+1$.

Moreover, in each of these cases, $\Delta(\Gamma)$ *admits dualities if and only if* Γ *does.*

In particular, applying our construction to finite affine planes results in an infinite family of flag-transitive, residually connected and thick geometries with trialities but no dualities. While examples of infinite families of geometries with trialities but no dualities were given [10] and [13], in both cases, the geometries are constructed as coset geometries. This means that the elements of these geometries are defined as coset of some systems $(G_i)_{i\in I}$ of subgroups of some group G, and do not have simple geometric interpretation. Moreover, the existence of a triality and the lack of dualities for these geometries also relies on group theoretical arguments. Indeed, the triality is induced by an outer automorphism of G of order three, and the absence of outer automorphisms of order two in G implies the impossibility for the geometry to have dualities. In our case, the definition of $\Delta(\Gamma)$, the existence of the triality and the lack of dualities are all based on geometric constructions and arguments.

The geometry $\Delta(\mathrm{AG}(2,3))$ is the smallest geometry with trialities but no dualities arising from our construction. It is a geometry of rank three with 108 elements, 36 of each type. It has 432 maximal flags and can be considered as an orientable proper hypermap on a surface of genus 55. This hypermap is the hypermap RPH55.89 on Marston Conder's list of orientable proper hypermaps[(2)].

The paper is organized as follows. In Section 2, we give the basic notions needed to understand the paper. In Section 3, we give the construction of the triangle complex of a rank two incidence system and we study the link between automorphisms and correlations of Γ and those of $\Delta(\Gamma)$ in detail. In Section 4, we prove that a linear space Γ yields a flag-transitive triangle complex $\Delta(\Gamma)$ if and only if there exists $G \leqslant \mathrm{Aut}(\Gamma)$ that acts transitively on the set of ordered non-collinear triples of points of Γ. This leads us to prove Theorem 1.1 which classifies the linear spaces Γ that are such that their triangle complex $\Delta(\Gamma)$ is flag-transitive. In Section 5, we study the connectedness and residual connectedness of $\Delta(\Gamma)$, ultimately allowing us to prove Theorem 1.2. In Section 6, we give the Buekenhout diagrams of the triangle complexes obtained in Theorem 1.2. Finally, Section 7 contains some concluding remarks and suggestions for future works.

2. Preliminaries

Many geometric structures are composed of objects and a relation that specifies how these objects are related to one another. The notion of incidence system formalizes

(2)See `https://www.math.auckland.ac.nz/~conder/OrientableProperHypermaps101.txt` this concept by providing an abstract framework to study these configurations. We refer to [4] for an introduction to this subject and more generally to the subject of diagram geometry.

Let I be a finite non empty set. A triple $\Gamma = (X, \star, t)$ is called an *incidence system* over I if

(1) X is a non empty set whose elements are called the *elements* of Γ;
(2) $\star$ is a symmetric and reflexive relation on X. It is called the *incidence relation* of Γ;
(3) t is a map from X to I, called the *type map* of Γ, such that distinct elements $x, y \in X$ with $x \star y$ satisfy $t(x) \neq t(y)$. Elements of $t^{-1}(i)$ are called elements of type i.

The *rank* of Γ is the cardinality of the type set I. A *flag* in an incidence system Γ over I is a set of pairwise incident elements. The type of a flag F is $t(F)$, that is the set of types of the elements of F. A *chamber* is a flag of type I. An incidence system Γ is an *incidence geometry* if all its maximal flags are chambers.

Let F be a flag of Γ. An element $x \in X$ is *incident* to F if $x \star y$ for all $y \in F$. The *residue* of Γ with respect to F, denoted by Γ_F, is the incidence system formed by all the elements of Γ incident to F but not in F. The *rank* of a residue is equal to rank(Γ) - $|F|$. For an element $x \in X$, we denote by $\mathrm{Res}_\Gamma(x)$ the set of elements of $\Gamma_{\{x\}}$.

A geometry Γ is *firm* (respectively, *thick*) if every flag of type other than I is contained in at least two (respectively, three) distinct chambers of Γ. It is called *thin* if every flag of type $I \backslash \{i\}$ for some $i \in I$ is contained in exactly two chambers of Γ.

The *incidence graph* of Γ is a graph with vertex set X and where two elements x and y are connected by an edge if and only if $x \star y$ and $t(x) \neq t(y)$. Whenever we talk about the distance between two elements x and y of a geometry Γ, we mean the distance in the incidence graph of Γ and simply denote it by $d_\Gamma(x, y)$, or even $d(x, y)$ if the context allows. The geometry Γ is *residually connected* when the incidence graphs of all of its residues of rank at least 2 are connected.

Let $\Gamma = \Gamma(X, \star, t)$ and $\Lambda = (Y, *, t')$ be incidence systems over the same type set I. A *morphism* from Γ to Λ is a map $\phi\colon X \to Y$ such that, for any x, x' in X, we have $\phi(x) * \phi(x')$ if $x \star x'$ and $t'(\phi(x)) = t'(\phi(y))$ if and only if $t(x) = t(y)$. If the map ϕ is a bijection and ϕ^{-1} is also a morphism, then ϕ is called a *correlation.* If, moreover, ϕ fixes the types of every element (that is $t'(\phi(x)) = t(x)$ for all $x \in X$), then ϕ is said to be an *isomorphism.* The *type* of a correlation ϕ is the permutation it induces on the type set I. A correlation of type (i, j) is called a *duality.* Additionally, if ϕ has order 2, it is then called a *polarity.* A correlation of type (i, j, k) is called a *triality.* The group of all correlations of Γ to itself is denoted by $\mathrm{Cor}(\Gamma)$ and the group of isomorphisms from Γ to itself is denoted by $\mathrm{Aut}(\Gamma)$. An isomorphism $\phi\colon \Gamma \to \Gamma$ is usually called an *automorphism* of Γ. Remark that $\mathrm{Aut}(\Gamma)$ is a normal subgroup of $\mathrm{Cor}(\Gamma)$ since it is the kernel of the action of $\mathrm{Cor}(\Gamma)$ on I.

A geometry Γ such that the action of $\mathrm{Aut}(\Gamma)$ on the set of chambers if transitive is called *flag-transitive* or *chamber-transitive.* Notice that being transitive on the chambers implies being transitive on the set of flags of any given type since, in a geometry, every flag can be extended to a chamber. Let $G \leqslant \mathrm{Aut}(\Gamma)$. As in [9], the pair (Γ, G) is called $(2T)_1$ if for every flag F of rank $|I| - 1$, the stabilizer of F in G acts two-transitively on the elements of the residue Γ_F .

Consider a rank 2 geometry Γ over $I = \{P, L\}$ where the elements of P are called points, the elements of L are called lines and such that the following axioms are fulfilled.

(1) Every line is incident to at least two points.

(2) Every point is incident to at least two lines.
(3) Every pair of distinct points p, q is incident to one and only one line.

Such an incidence geometry $\Gamma = (\mathcal{P} \sqcup \mathcal{L}, \star, t)$ over $I = \{P, L\}$ is called a *linear space* where t is defined in the obvious way (see [3, Chapter 1, Section 2.5]). We decided to add axiom (2) to the definition of [3, Chapter 1, Section 2.5] in order to avoid the degenerate case where Γ would have a unique line.

3. The triangle complex of a rank two geometry

In this section, we define the main construction of this article, that is, the triangle complex $\Delta(\Gamma)$ of a rank two incidence system Γ. We first establish some of its general properties and then study in detail the geometry $\Delta(\Gamma)$ when Γ is a thick linear space. In particular, we pay close attention to the automorphisms and correlations of $\Delta(\Gamma)$. It turns out that, except for the trialities, the correlations of $\Delta(\Gamma)$ are all induced by correlations of Γ.

Let $\Gamma = (X, \star, t)$ be an incidence system of rank two over the type set I. Without loss of generality, we can always assume that $X = \mathcal{P} \sqcup \mathcal{L}$. We will then call the elements of $\mathcal{P}$ the points of Γ and the elements of $\mathcal{L}$ the lines of Γ.

Construction 1. *Let $\Gamma = (\mathcal{P} \sqcup \mathcal{L}, \star, t)$ be a rank two geometry. The* triangle complex *$\Delta(\Gamma) = (X_{\Delta(\Gamma)}, \star_{\Delta(\Gamma)}, t_{\Delta(\Gamma)})$ over $I = \{0, 1, 2\}$ is the rank three incidence system constructed from Γ in the following way:*

(1) *The set $X_{\Delta(\Gamma)}$ of elements of $\Delta(\Gamma)$ is the set of all the triples (p, L, i) with $p \in \mathcal{P}$, $L \in \mathcal{L}$, $i \in \{0, 1, 2\}$ satisfying $p \star L$.*
(2) *The incidence relation $\star_{\Delta(\Gamma)}$ is defined by $(p, L, i) \star_{\Delta(\Gamma)} (p', L', i+1 \bmod 3)$ if and only if $p \star L'$ and we do not have $p' \star L$.*
(3) *The type function $t_{\Delta(\Gamma)} \colon X_{\Delta(\Gamma)} \mapsto \{0, 1, 2\}$ is defined by $t_{\Delta(\Gamma)}((p, L, i)) = i$.*

In order to simplify the notation, we set once and for all that the operations on the types are always taken modulo 3. In particular, an element (p, L, i) with $i \in \mathbb{Z}$ will be understood as $(p, L, i \bmod 3)$ in the rest of the article.

Simply put, the elements of $\Delta(\Gamma)$ are three disjoint copies of the flags of Γ. With the incidence relation defined above, it is easy to see that chambers of $\Delta(\Gamma)$ correspond to triangles in Γ. This motivates the name triangle complex. Here is a simple example to illustrate the concept of a triangle complex. This example will be used later in a proof.

Example 3.1. Let $v > 2$ an integer. Consider the complete graph K_v as an incidence geometry with v points and $\binom{v}{2}$ lines. The triangle complex $\Delta(\mathrm{K}_v)$ consists of $v(v-1)$ elements of each type. In this case, we have $(a, \{a, b\}, i) \star_\Delta (c, \{c, d\}, i+1))$ if and only if $d = a$ and $b \neq c$ and $(a, \{a, b\}, i) \star_\Delta (c, \{c, d\}, i-1))$ if and only if $b = c$ and $d \neq a$. The chambers of $\Delta(\mathrm{K}_v)$ are then the sets $\{(a, \{c, a\}, 0), (b, \{a, b\}, 1), (c, \{b, c\}, 2)\}$ where $\{a, b, c\}$ is any subset of size 3 in the point set of K_v.

Let Γ and Λ be two rank two geometries and let $\phi \colon \Gamma \to \Lambda$ be a morphism. We can define $\Delta(\phi) \colon \Delta(\Gamma) \to \Delta(\Lambda) : (p, L, i) \mapsto (\phi(p), \phi(L), i)$. Abusing notation, we denote the map $\phi \to \Delta(\phi)$ by Δ also. The map Δ then becomes a functor that transforms rank two geometries into rank three geometries. In particular, the map $\Delta \colon \mathrm{Aut}(\Gamma) \to \mathrm{Aut}(\Delta(\Gamma))$ is always a group homomorphism. We will show that this map is an isomorphism when Γ is a thick linear space, but this need not hold in general.

Example 3.2. Let Γ be the rank two geometry of a square. The geometry Γ is made of four points and four lines and its automorphism group is D_8, the dihedral group of order 8. The triangle complex $\Delta(\Gamma)$ consists of 24 elements, divided into three sets of 8 of each type $i = 0, 1, 2$. It is easy to check that $\Delta(\Gamma)$ is not a geometry (there are no triangles in Γ) and that its incidence graph has two connected components that are each isomorphic to 12-gons whose vertices' types cycle between $0, 1$ and 2. The rotations of Γ act on both connected components as rotations and the axial symmetries of Γ exchange these two components. Hence, $\mathrm{Aut}(\Gamma)$ does act as automorphisms of $\Delta(\Gamma)$. But $\mathrm{Aut}(\Delta(\Gamma))$ is of order $16 = 4 \times 4$ and also contains automorphisms that fix one connected component while rotating the other.

Before proceeding, we prove a few elementary results about $\Delta(\Gamma)$. We first give a necessary condition for the incidence system $\Delta(\Gamma)$ to be a geometry.

Proposition 3.3. *Let $\Gamma = (\mathcal{P} \sqcup \mathcal{L}, \star, t)$ be a rank two geometry. If $\Delta(\Gamma)$ is a geometry, then the gonality of Γ is at most three.*

Proof. Let $p_0 \in \mathcal{P}$ be any point of Γ. Since Γ is a geometry there must exist a line $L_0 \in \mathcal{L}$ such that $\{p_0, L_0\}$ is a flag in Γ. Thus, the triple $(p_0, L_0, 0)$ is an element of $\Delta(\Gamma)$. If $\Delta(\Gamma)$ is a geometry, the flag $\{(p_0, L_0, 0)\}$ can be completed in a chamber $\{(p_0, L_0, 0), (p_1, L_1, 1), (p_2, L_2, 2)\}$. Notice that this implies that $p_0, L_1, p_1, L_2, p_2, L_0, p_0$ is an incidence chain in Γ, so that the gonality of Γ is at most three as claimed. □

The previous proposition and its proof motivate us to look at linear spaces. Indeed, in order for $\Delta(\Gamma)$ to be a geometry, we see that Γ must not only have gonality three, but in some sense it must have triangles everywhere. This is a quite restrictive property that is satisfied for linear spaces. They are thus a potential good source of inputs for Construction 1.

Proposition 3.4. *If Γ is a linear space, the incidence system $\Delta(\Gamma)$ is an incidence geometry.*

Proof. It suffices to observe that, by definition, a linear space has always at least two lines and all its lines have at least two points. Hence any flag of $\Delta(\Gamma)$ can be completed in a chamber. □

There are nonetheless rank two geometries Γ that are not linear spaces and for which $\Delta(\Gamma)$ is an interesting geometry.

Example 3.5. Let $\mathcal{D}$ be the Desargues configuration. The triangle complex $\Delta(\mathcal{D})$ is a geometry that consists of $10 \times 3 = 30$ elements of each type. Each element of type 0 is incident to four elements of type 1 and four elements of type 2. Every rank two flag can be completed in a unique chamber (see Figure 1), which in particular means that $\Delta(\Gamma)$ is not residually connected. Computation using Magma [2] shows that $\mathrm{Aut}(\Delta(\mathcal{D})) \cong \mathrm{Aut}(\mathcal{D}) \cong \mathrm{S}_5$ and that it acts strictly transitively on the 120 chambers of $\Delta(\mathcal{D})$.

The code for the triangle complex of the Desargues configuration can be found in

`https://github.com/RDelaby/GeometriesAdmittingTrialitiesFromLinearSpaces`.

The repository also contains code to build triangle complexes of other rank two geometries, such as the Pappus configuration, Hughes planes and the Möbius-Kantor configuration.

A key property of triangle complexes is that they always admit a correlation of order three that simply permutes cyclically the indices of its elements.

Proposition 3.6. *Let Γ be a rank two geometry. The triangle complex $\Delta(\Gamma)$ always admits a triality of order three.*

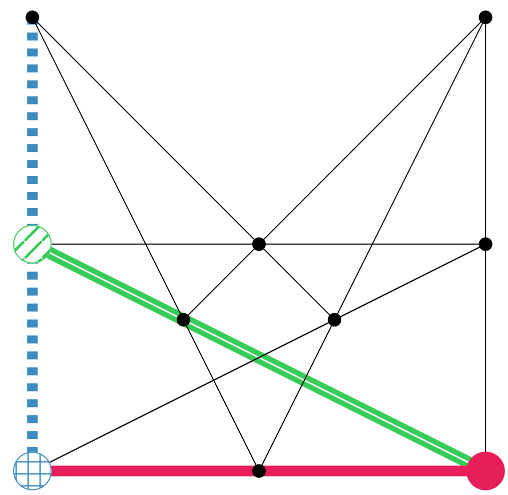


FIGURE 1. A chamber in $\Delta(\mathcal{D})$.

Proof. The application $\tau\colon \Delta(\Gamma) \to \Delta(\Gamma)$ defined by $\tau((p, L, i)) = (p, L, i+1)$ is a triality of order three. □

We call the triality τ above the *canonical triality* of $\Delta(\Gamma)$.

3.1. AUTOMORPHISMS AND CORRELATIONS OF $\Delta(\Gamma)$. We now start our investigation of automorphisms and correlations of $\Delta(\Gamma)$ when Γ is a linear space. The main technicality is to show that an automorphism $\phi \in \mathrm{Aut}(\Delta(\Gamma))$ acts on $\Delta(\Gamma)$ in a geometric way. For example, we will show that the images by ϕ of two elements (p, L_1, i) and (p, L_2, i) of $\Delta(\Gamma)$ that share a common point p must still be two elements sharing a common point. While it is very tempting to assume such a behavior, it has to be carefully proved. This is achieved in Lemmas 3.8 and 3.9.

Let $\Gamma = (\mathcal{P} \sqcup \mathcal{L}, \star, t)$ be a linear space and let $\Delta(\Gamma)$ be the geometry constructed from Γ using Construction 1. From now on, we identify each line of Γ with its residue, also called its point row. In other words, we consider a line and the set of points that are incident to the line to be the same thing. We thus write $p \in L$ to mean $p \star L$ and say that the point p is contained in the line L or that L contains p. Also, given two points $p, q \in \mathcal{P}$ with $p \neq q$ we define the line $L := [p, q]$ to be the unique line containing both p and q. Dually, the pencil of lines through a point $p \in \mathcal{P}$ is denoted by $\mathcal{L}_p$ and is the set of all lines containing p. To make this point of view and notations apparent, we will write $\Gamma = (\mathcal{P}, \mathcal{L}, \in)$.

We say that Γ *has parameters* m, n if for every point $p \in \mathcal{P}$, we have $m = |\mathcal{L}_p|$ and for every line $L \in \mathcal{L}$, we have $|L| = n$. In other words, there are n points on each line and there are m lines through each point. Note that a linear space with parameters m, n is also called a $2-(|\mathcal{P}|, n, 1)$ design, or a Steiner system $S(2, n, |\mathcal{P}|)$.

The following lemma shows that the flag-transitivity of $\Delta(\Gamma)$ implies that Γ has parameters.

LEMMA 3.7. *Let $\Gamma = (\mathcal{P} \sqcup \mathcal{L}, \star, t)$ be a finite linear space such that $\Delta(\Gamma)$ is flag-transitive. Then every line of Γ is incident to the same number of points and every point of Γ is incident to the same number of lines.*

Proof. Choose two points $p, p' \in \mathcal{P}$ with $p \neq p'$ and let $L = \big[p, p'\big]$. Counting the number of elements of type 2 incident to $(p, L, 0)$ and $(p', L, 0)$ we get

$$|\mathcal{L}_{p'}| - 1 + \sum_{q \in L \setminus \{p,p'\}} (|\mathcal{L}_q| - 1) = |\mathcal{L}_p| - 1 + \sum_{q \in L \setminus \{p,p'\}} (|\mathcal{L}_q| - 1).$$

It follows that $|\mathcal{L}_p| = |\mathcal{L}_{p'}| = m$.

Let $L, L' \in \mathcal{L}$ with $L \neq L'$ and choose $p \in L \setminus L'$ and $p' \in L' \setminus L$. Counting the number of type 2 elements incident to $(p, L, 0)$ and $(p', L', 0)$ we get

$$m(|L| - 1) = m(|L'| - 1).$$ □

For any linear space Γ, we introduce the following two projection maps

$$\pi \colon X_{\Delta(\Gamma)} \to \mathcal{P} : (p, L, i) \mapsto p$$

$$\lambda \colon X_{\Delta(\Gamma)} \to \mathcal{L} : (p, L, i) \mapsto L$$

that will be used constantly in the proofs of this section. The next lemma states that, given two linear spaces Γ and Δ, the isomorphisms from $\Delta(\Gamma)$ to $\Delta(\Lambda)$ are well behaved in the sense that they send pencils of lines to pencils of lines and a set of points belonging to a given line to a set of points belonging to a line.

Lemma 3.8. *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \in)$ and $\Lambda = (\mathcal{P}', \mathcal{L}', \in)$ be two linear spaces with equal parameters $m \geqslant 3, n \geqslant 2$ and let $\phi \colon \Delta(\Gamma) \to \Delta(\Lambda)$ be an isomorphism. Then,*

1. *For every $p \in \mathcal{P}$ and for every $L_1, L_2 \in \mathcal{L}_p$, we have $\pi(\phi((p, L_1, i))) = \pi(\phi((p, L_2, i)))$.*
2. *For every $L \in \mathcal{L}$ and for every $p_1, p_2 \in L$, we have $\lambda(\phi((p_1, L, i))) = \lambda(\phi((p_2, L, i)))$.*

Proof. By Proposition 3.6, without loss of generality, we will assume for the whole proof that $i = 0$.

Assume by contradiction that (1) does not hold. Then, there must exist two elements $F_1 = (p, L_1, 0)$ and $F_2 = (p, L_2, 0)$ in $\Delta(\Gamma)$ such that their images under the isomorphism ϕ do not share a common point. In other words, we have $\pi(\phi(F_1)) \neq \pi(\phi(F_2))$. As ϕ is an isomorphism, there are as many type 1 elements incident to both F_1 and F_2 as there are type 1 elements incident to both $\phi(F_1)$ and $\phi(F_2)$. We will use a counting argument to show that this is in fact impossible when $\pi(\phi(F_1)) \neq \pi(\phi(F_2))$. The number of elements of type 1 that are incident to both F_1 and F_2 is $(m-2)(n-1)$. The same should then be true for the images $\phi(F_1)$ and $\phi(F_2)$. But the number of type 1 elements incident to both $\phi(F_1)$ and $\phi(F_2)$ is either $n-2$ or 0. Indeed, since $\phi(F_1)$ and $\phi(F_2)$ do not share a common vertex, an element $F = (p', L', 1) \in \Delta(\Lambda)$ is incident to $\phi(F_1)$ and $\phi(F_2)$ if and only if $L' = \big[\pi(\phi(F_1)), \pi(\phi(F_2))\big] \neq \lambda(\phi(F_1)) \neq \lambda(\phi(F_2))$ and $p' \notin \{\pi(\phi(F_1)), \pi(\phi(F_2))\}$. Hence, there are either $(n-2)$ or 0 elements of type 1 incident to both $\phi(F_1)$ and $\phi(F_2)$. Since there were $(m-2)(n-1)$ elements of type 1 incident to both F_1 and F_2, this would imply that $(m-2)(n-1)$ is equal to either $n-2$ or 0. As $m \geqslant 3$ and $n \geqslant 2$, we get $(m-2)(n-1) \geqslant (n-1) > (n-2) \geqslant 0$, yielding a contradiction. Hence, we conclude that $\pi(\phi(F_1)) = \pi(\phi(F_2))$.

Assume now by contradiction that (2) is not true. There must then exist $F_1 = (p_1, L, 0)$ and $F_2 = (p_2, L, 0)$ in $\Delta(\Gamma)$ such that $\lambda(\phi(F_1)) \neq \lambda(\phi(F_2))$. The number of elements of type 2 incident to both F_1 and F_2 is $(n-2)(m-1)$. If there is no point in Λ incident to both $\lambda(\phi(F_1))$ and $\lambda(\phi(F_2))$, there are no elements of type 2 incident to both $\lambda(\phi(F_1))$ and $\lambda(\phi(F_2))$. Suppose instead that p' is the only point of Λ incident to both $\lambda(\phi(F_1))$ and $\lambda(\phi(F_2))$. If $p' = \pi(\phi(F_1))$ or $\pi(\phi(F_2))$, we again have no elements of type 2 incident to both $\lambda(\phi(F_1))$ and $\lambda(\phi(F_2))$. If not, the elements of type 2 incident to both $\lambda(\phi(F_1))$ and $\lambda(\phi(F_2))$ are precisely the $(p', L', 2)$ with $L' \neq \phi(L_1), \phi(L_2)$. There are $m-2$ such elements. This would then mean that $(n-2)(m-1)$ is equal to either $m-2$ or 0. If $n > 2$, we have that $(n-2)(m-1) \geqslant (m-1) > (m-2) \geqslant 0$, which again leads to a contradiction.

Suppose instead that $n = 2$. This means that every line contains exactly two points. Let $\phi(F_1) = (s, Y, 0)$ and $\phi(F_2) = (t, Z, 0)$. Recall that our assumption is that $Y \neq Z$. Suppose first that $Y \cap Z \neq \varnothing$. Since $m \neq 2$ there must be at least four points in $\mathcal{P}'$. Let v be a point not on Y nor Z. There are essentially three possible configurations : $Y \cap Z = \{s\} = \{t\}$, $Y \cap Z = \{s\} \neq \{t\}$, and the last configuration being $Y \cap Z \neq \{t\}$, $Y \cap Z \neq \{s\}$ with $s \neq t$. If $Y \cap Z = \{s\} = \{t\}$, there is an element of type 1 incident to both $(s, Y, 0)$ and $(t, Z, 0)$, namely $(v, [s, v], 1)$. If $Y \cap Z \neq \{t\}$, $Y \cap Z \neq \{s\}$ with $s \neq t$ there is an element of type 2 incident to both $(s, Y, 0)$ and $(t, Z, 0)$. If $Y \cap Z = \{s\} \neq \{t\}$, $(s, Y, 0)$ and $(t, Z, 0)$ are at distance 3 in the incidence graph of $\Delta(\Lambda)$. Indeed, let $Y = \{s, u\}$, $Z = \{s, t\}$. We can then form the following chain of incidences.

$$(s, Y, 0) = (s, \{s, u\}, 0) \star_\Delta (u, \{u, v\}, 2) \star_\Delta (v, \{v, t\}, 1) \star_\Delta (t, \{t, s\}, 0) = (t, Z, 0)$$

In each case, we get a contradiction since F_1 and F_2 do not have any element of type 1 or 2 incident to both of them and are not at distance 3 in the incidence graph of $\Delta(\Gamma)$. Hence, we can assume that $Y \cap Z$ is empty. There must then exist unique and distinct elements $u, v \in \mathcal{P}'$ such that $Y = \{s, u\}$ and $Z = \{t, v\}$.

If $\mathcal{P}' = \{s, t, u, v\}$ contains only these four points, then Γ and Λ are isomorphic to K_4, the complete graph on four vertices. Let us now rewrite $\{s, t, u, v\}$ as $\{0, 1, 2, 3\}$ for clarity. Consider the following setting. We want to find all chains $(0, \{0, 1\}, 0) \star_\Delta (a, \{0, a\}, 1) \star_\Delta (b, \{a, b\}, 2) \star_\Delta (c, \{b, c\}, 0) \star_\Delta (d, \{c, d\}, 1) \star_\Delta (e, \{d, e\}, 2) \star_\Delta (z, M, 0)$ for $(y, M) = (1, \{0, 1\})$ and $(y, M) = (2, \{2, 3\})$. Figures 2 and 3 show that there are four such chains when $(y, M) = (1, \{0, 1\})$ and five when $(y, M) = (2, \{2, 3\})$. Applied to our case, this contradicts the fact that $Y \neq Z$.

If, instead, there exists $z \in \mathcal{P}' \backslash \{s, t, u, v\}$, we can form the following chain of incidences.

$$(s, Y, 0) = (s, \{s, u\}, 0) \star_\Delta (u, \{u, z\}, 2) \star_\Delta (z, \{z, t\}, 1) \star_\Delta (t, \{t, v\}, 0) = (t, Z, 0)$$

Applying ϕ^{-1}, we find elements $a, b, c, d \in \mathcal{P}$ such that the following holds:

$$(p_1, L, 0) = (p_1, \{p_1, p_2\}, 0) \star_\Delta (a, \{a, b\}, 2) \star_\Delta (c, \{c, d\}, 1) \star_\Delta (p_2, \{p_1, p_2\}, 0) = (p_2, L, 0)$$

From this, we conclude that $a = p_2$, $c = b$ and $p_2 = d$, which in turn leads to $(p_2, \{p_2, b\}, 2) \star_\Delta (b, \{b, p_2\}, 1)$, a contradiction.

Having obtained contradictions in each case, we can conclude that $\lambda(\phi(F_1)) = \lambda(\phi(F_2))$, as desired. □

Lemma 3.8 suggests that given any isomorphism $\phi\colon \Delta(\Gamma) \to \Delta(\Lambda)$, we can recover a well-defined isomorphism from Γ to Λ which fully characterizes ϕ. For $p \in \mathcal{P}$, let us define $f_{\phi,i}(p) := \pi(\phi((p, L, i)))$ where L is any line such that $p \in L$. Similarly, define $g_{\phi,i}(L) := \lambda(\phi((p, L, i)))$ where $p \in L$ is any point incident to L. By Lemma 3.8, the functions $f_{\phi,i}$ and $g_{\phi,i}$ are well defined.

We now want to prove that the action of ϕ does not depend on the type i and that it comes from an isomorphism from Γ to Λ. This is achieved by the next three results.

Lemma 3.9. *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \in)$ and $\Lambda = (\mathcal{P}', \mathcal{L}', \in)$ be two linear spaces with equal parameters $m \geqslant 3, n \geqslant 2$. The functions $f_{\phi,i}\colon \mathcal{P} \to \mathcal{P}'$ and $g_{\phi,i}\colon \mathcal{L} \to \mathcal{L}'$ induced by an isomorphism $\phi\colon \Delta(\Gamma) \to \Delta(\Lambda)$ as above are bijections.*

Proof. We will show that $f_{\phi,i}$ is a bijection. Note that both $\mathcal{P}$ and $\mathcal{P}'$ have cardinality $1 + (n-1)m$, because they are both linear spaces with parameters m and n. Hence,

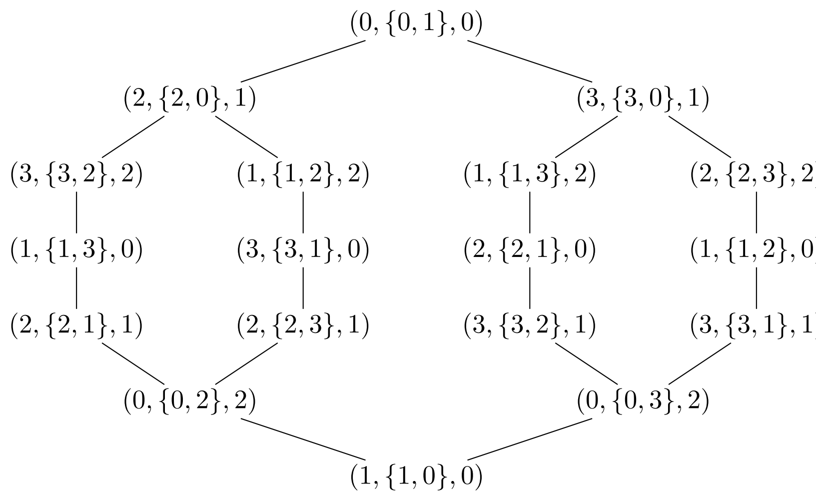


Figure 2. Chains from $(0,\{0,1\},0)$ to $(1,\{1,0\},0)$ in $\Delta(\mathrm{K}_4)$.

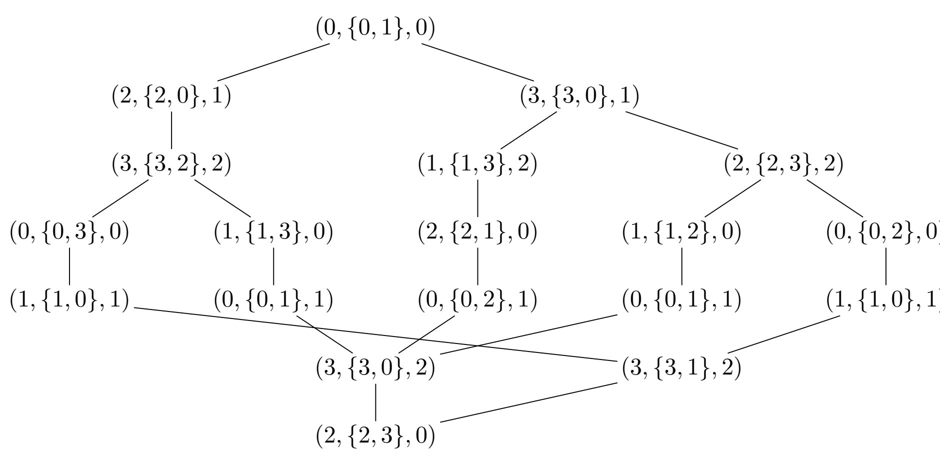


Figure 3. Chains from $(0,\{0,1\},0)$ to $(2,\{2,3\},0)$ in $\Delta(\mathrm{K}_4)$.

it suffices to show that $f_{\phi,i}$ is surjective. Let $p' \in \mathcal{P}'$ be any point of Λ. Choose $L' \in \mathcal{L}'$ such that $p' \in L'$. Then $(p', L', i) \in X_{\Delta(\Lambda)}$. Since ϕ is an isomorphism, there exists $(p, L, i) \in \Delta(\Gamma)$ such that $\phi((p, L, i)) = (p', L', i)$ and thus $f_{\phi,i}(p) = p'$, showing surjectivity.

The same argument shows that $g_{\phi,i}$ is also a bijection. □

Proposition 3.10. *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \in)$ and $\Lambda = (\mathcal{P}', \mathcal{Q}', \in)$ be two linear spaces with equal parameters $m \geqslant 3, n \geqslant 2$. Then we have $f_{\phi,0} = f_{\phi,1} = f_{\phi,2}$ and $g_{\phi,0} = g_{\phi,1} = g_{\phi,2}$.*

Proof. Let $p \in \mathcal{P}$. For all $L \in \mathcal{L}$ such that $p \in L$, let $q \in L\backslash\{p\}$ and D be a line such that $D \neq L$ and $q \in D$. Then $(p, L, 0) \star_{\Delta(\Gamma)} (q, D, 2)$. Applying ϕ, we get $\phi((p, L, 0)) \star_{\Delta(\Lambda)} \phi((q, D, 2))$. In particular, $f_{\phi,2}(q) = \pi(\phi((q, D, 2))) \in \lambda(\phi((p, L, 0))) = g_{\phi,0}(L)$. Since this is true for all $q \in L\backslash\{p\}$, the restriction of $f_{\phi,2}\colon \{v \in L\backslash\{p\}\} \to \{w \in g_{\phi,0}(L)\backslash\{f_{\phi,0}(p)\}\}$ is a bijection. Indeed, $f_{\phi,2}$ is itself a bijection from $\mathcal{P}$ to $\mathcal{P}$ by 3.9 and the image of an element in $\{v \in L\backslash\{p\}\}$ has to be in $\{w \in g_{\phi,0}(L)\backslash\{f_{\phi,0}(p)\}\}$. Because Γ and Λ have the same parameters, the size of these sets is the same and this is thus a bijection. Moreover, consider $q \in L\backslash\{p\}$ and $Z \in \mathcal{L}\backslash\{L\}$ such that $p \in Z$. Then $(q, L, 0) \star_{\Delta(\Gamma)} (p, Z, 2)$, applying ϕ we get in particular that $f_{\phi,2}(p) \in g_{\phi,0}(L)$. Combined with the fact that $f_{\phi,2}$ is a bijection from $\{v \in L\backslash\{p\}\}$ to $\{w \in g_{\phi,0}(L)\backslash\{f_{\phi,0}(p)\}\}$, this forces $f_{\phi,2}(p) = f_{\phi,0}(p)$. The same reasoning allows us to prove that $f_{\phi,0} = f_{\phi,1} = f_{\phi,2}$ and $g_{\phi,0} = g_{\phi,1} = g_{\phi,2}$, where for the latter three functions, we need to dualize the arguments of this proof by considering pencils of points. ☐

Proposition 3.11. *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \in)$ and $\Lambda = (\mathcal{P}', \mathcal{L}', \in)$ be two linear spaces with equal parameters $m \geqslant 3, n \geqslant 2$. The function $F_\phi \colon \mathcal{P} \sqcup \mathcal{Q} \mapsto \mathcal{P}' \sqcup \mathcal{Q}'$ defined by*

$$F_\phi(x) = \begin{cases} f_{\phi,0}(x) & \text{if } x \in \mathcal{P} \\ g_{\phi,0}(x) & \text{if } x \in \mathcal{L} \end{cases}$$

is an isomorphism of Γ onto Λ. Moreover, for all $p \in \mathcal{P}$, $L \in \mathcal{L}$ and $0 \leqslant i \leqslant 2$, we have $\phi(p, L, i) = (F_\phi(p), F_\phi(L), i)$.

Proof. We already know F_ϕ is a bijection by Lemma 3.9. Let $p \in \mathcal{P}$ and $L \in \mathcal{L}$ such that $p \in L$. Choose $L' \neq L$ such that $p \in L'$ and $p' \in L$ with $p \neq p'$. We then have $(p, L', 0) \star_{\Delta(\Gamma)} (p', L, 1)$. Applying ϕ, we get $f_{\phi,0}(p) \in g_{\phi,1}(L)$ but by Proposition 3.10, we can conclude that $f_{\phi,0}(p) \in g_{\phi,0}(L)$ and hence $F_\phi(p) \in F_\phi(L)$.

For the second part, assume that p and L are not incident in Γ. Pick $p' \in L$ and construct the line $L' = [p, p']$. Then $(p', L, 0) \star_{\Delta(\Gamma)} (p, L', 1)$, thus $f_{\phi,1}(p)$ is not incident with $g_{\phi,0}(L)$. Hence, $f_{\phi,0}(p)$ is not incident with $g_{\phi,0}(L)$ which allows us to conclude that F_ϕ is an isomorphism. ☐

Corollary 3.12. *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \in)$ and $\Lambda = (\mathcal{P}', \mathcal{L}', \in)$ be two linear spaces with equal parameters $m \geqslant 3, n \geqslant 2$. Then*

1. *For each isomorphism $\phi \colon \Delta(\Gamma) \to \Delta(\Lambda)$ there exists a unique isomorphism $\bar{\phi} \colon \Gamma \to \Lambda$ such that $\Delta(\bar{\phi}) = \phi$.*
2. *Mapping $\bar{\phi}$ to ϕ in such a way is compatible with composition, and inverts the map Δ.*
3. *The map $\Delta\colon \operatorname{Aut}(\Gamma) \to \operatorname{Aut}(\Delta(\Gamma))$ is an isomorphism of groups.*

Note that all the statements in this section deal only with isomorphisms. We have thus not proven that the functor Δ is full and faithful. Armed with the results obtained so far, we are now ready to prove one of the main results of this section.

Theorem 3.13. *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \in)$ be a finite linear space with at least two points on each line and at least three lines through any point. Then $\Delta(\Gamma)$ is a flag-transitive geometry if and only if $\operatorname{Aut}(\Gamma)$ is transitive on $\mathcal{T}(\Gamma)$, the set of ordered non-collinear triples of points of Γ.*

Proof. Assume $\operatorname{Aut}(\Gamma)$ acts transitively on $\mathcal{T}(\Gamma)$. Let $C_1 = \{(p_0, L_0, 0), (p_1, L_1, 1), (p_2, L_2, 2)\}$ and $C_2 = \{(p'_0, L'_0, 0), (p'_1, L'_1, 1), (p'_2, L'_2, 2)\}$ be two chambers of $\Delta(\Gamma)$. The associated triples of points of both chambers are in $\mathcal{T}(\Gamma)$. Take $g \in \operatorname{Aut}(\Gamma)$ satisfying $g(p_j) = p'_j$ for $j = 0, 1, 2$. The induced automorphism $\phi_g \in \operatorname{Aut}(\Delta(\Gamma))$ defined by $\phi_g(p, L, i) = (g(p), g(L), i)$ sends C_1 to C_2. Hence $\Delta(\Gamma)$ is flag-transitive.

Assume $\Delta(\Gamma)$ is flag-transitive. By Lemma 3.7, the incidence system Γ has parameters. Let $T_1 = (p_0, p_1, p_2)$ and $T_2 = (p'_0, p'_1, p'_2)$ in $\mathcal{T}(\Gamma)$. Construct two chambers C_1 and C_2 of $\Delta(\Gamma)$ in the following way:

$$C_1 = \{(p_0, [p_0, p_2], 0), (p_1, [p_1, p_0], 1), (p_2, [p_1, p_2], 2)\},$$
$$C_2 = \{(p'_0, [p'_0, p'_2], 0), (p'_1, [p'_1, p'_0], 1), (p_2, [p'_1, p'_2], 2)\}.$$

Let $\phi \in \mathrm{Aut}(\Delta(\Gamma))$ such that $\phi(C_1) = C_2$. By Proposition 3.11 applied to $\Lambda = \Gamma$, there exists $F_\phi \in \mathrm{Aut}(\Gamma)$ such that $\phi(p, L, i) = (F_\phi(p), F_\phi(L), i)$. Thus, $F_\phi(p_j) = p'_j$ for $j = 0, 1, 2$ and $\mathrm{Aut}(\Gamma)$ is transitive on $\mathcal{T}(\Gamma)$. □

So far, we have focused only on automorphisms of both Γ and $\Delta(\Gamma)$. We now take a look at dualities in both these geometries.

Proposition 3.14. *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \in)$ be a thick linear space with parameters. Then Γ admits dualities if and only if $\Delta(\Gamma)$ admits dualities.*

Proof. Suppose first that there exists a duality α of Γ. This means that α sends points to lines and lines to points while preserving incidence. We define a correlation β of $\Delta(\Gamma)$ as follows:

$$\beta(p, L, i) = \begin{cases} (\alpha(L), \alpha(p), 0), & \text{if } i = 0 \\ (\alpha(L), \alpha(p), 2), & \text{if } i = 1 \\ (\alpha(L), \alpha(p), 1), & \text{if } i = 2 \end{cases}$$

It is rather straightforward to check that β is indeed a correlation that fixes type 0 while exchanging types 1 and 2, and hence a duality. For example, we have $(p, L, 0) \star_{\Delta(\Gamma)} (p', L', 1)$ if and only if p is different from p', and p is on both L and L'. This holds if and only if $\alpha(L)$ is different from $\alpha(L')$, and $\alpha(L')$ is on both $\alpha(p)$ and $\alpha(p')$. Hence, we get $(p, L, 0) \star_{\Delta(\Gamma)} (p', L', 1)$ if and only if $(\alpha(L'), \alpha(p'), 2) \star_{\Delta(\Gamma)} (\alpha(L), \alpha(p), 0)$.

Suppose now that there exists a duality β of $\Delta(\Gamma)$. Without loss of generality, suppose that β exchanges elements of type 1 and 2 of $\Delta(\Gamma)$. We will also denote by β the map on $\{0, 1, 2\}$ induced by the duality, meaning that β fixes 0 and exchanges 1 and 2. The statements and proofs of Lemmas 3.8 and 3.9 and Proposition 3.11 can be adapted to the setting of dualities. More precisely, it can be shown that if there exists a duality β of $\Delta(\Gamma)$ as above, then the number m of lines per point in Γ must be equal to the number n of points per line and for $p \in \mathcal{P}$, and for any $L, L' \in \mathcal{L}_p$, we have $\lambda(\beta((p, L, \beta(i)))) = \lambda(\beta((p, L', \beta(i))))$ and for $L \in \mathcal{L}$ and any $p, p' \in L$, we have $\pi(\beta((p, L, \beta(i)))) = \pi(\beta((p', L, \beta(i))))$. Intuitively, this means that β sends pencils of lines to collinear points and vice versa. The proof of this statement is almost identical to the proof of Lemma 3.8. The only difference is that since β exchanges the roles of elements of type 1 and 2, we need to compare the number of elements of type 1 incident to pairs F_1, F_2 of elements of Γ with the number of elements of type 2 (not of type 1) of their images $\beta(F_1)$ and $\beta(F_2)$. We can then proceed to show that there are bijections $f_{\beta,i} \colon \mathcal{P} \to \mathcal{L}$ and $g_{\beta,i} \colon \mathcal{L} \to \mathcal{P}$ and that these functions actually do not depend on i. Finally, we can show that $F_\beta(x) = \begin{cases} f_{\beta,0}(x) & \text{if } x \in \mathcal{P} \\ g_{\beta,0}(x) & \text{if } x \in \mathcal{L} \end{cases}$ is a duality of Γ. □

Proposition 3.15. *Let Γ be a finite thick flag-transitive linear space such that $\Delta(\Gamma)$ is flag-transitive, and let $\langle\tau\rangle \cong \mathrm{C}_3$ be the subgroup of $\mathrm{Cor}(\Delta(\Gamma))$ generated by the canonical triality τ. Then $\langle\tau\rangle$ is a normal subgroup of $\mathrm{Cor}(\Delta(\Gamma))$. Moreover, we have $\mathrm{Cor}(\Delta(\Gamma)) \cong \mathrm{C}_3 \times \mathrm{Aut}(\Gamma)$ if Γ does not admit dualities, and $\mathrm{Cor}(\Delta(\Gamma)) \cong \mathrm{C}_3 \rtimes \mathrm{Cor}(\Gamma)$ if and only if Γ admits a polarity.*

Proof. Let τ as in Proposition 3.6 and α a duality of Γ, if it exists. First of all, τ commutes with any automorphism ϕ of $\Delta(\Gamma)$. Indeed, by Proposition 3.10, we have $\phi(p, L, i) = (f_\phi(p), g_\phi(L), i)$ for some permutations f_ϕ and g_ϕ that do not depend on the index i. As τ only acts on the indices, it is then clear that $\tau \circ \phi \circ \tau^{-1} = \phi$. If there are no dualities in $\Delta(\Gamma)$, this shows that τ is central and therefore that $\langle\tau\rangle$ is normal.

Suppose thus that $\Delta(\Gamma)$ has dualities. By Proposition 3.14, we know that this implies that Γ also has a duality α and that

$$\beta(p,L,i) = \begin{cases} (\alpha(L),\alpha(p),0), & \text{if } i=0 \\ (\alpha(L),\alpha(p),2), & \text{if } i=1 \\ (\alpha(L),\alpha(p),1), & \text{if } i=2 \end{cases}$$

is a duality of $\Delta(\Gamma)$. Let β' be any duality of $\Delta(\Gamma)$. Without loss of generality, we can assume that β' acts on the type set I in the same way as β (if not, we compose it by τ or τ^2). Then, β' can be written as $\beta' = \phi \circ \beta$ with $\phi \in \mathrm{Aut}(\Delta(\Gamma))$. The fact that $\langle(0,1,2)\rangle \trianglelefteq S_3$ then ensures that $\beta' \circ \tau \circ \beta'(p,L,i) = (p,L,i-1) = \tau^2(p,L,i)$ for any $(p,L,i) \in \Delta(\Gamma)$. This then shows that $\langle\tau\rangle$ is a normal subgroup of $\mathrm{Cor}(\Delta(\Gamma))$. If Γ does not admit dualities, we have the following short exact sequence

$$1 \longrightarrow \mathrm{Aut}(\Delta(\Gamma)) \longrightarrow \mathrm{Cor}(\Delta(\Gamma)) \longrightarrow \mathrm{C}_3 \longrightarrow 1$$

where the group C_3 is the image of the action of correlations on the type set I. The map $s\colon \mathrm{C}_3 \to \mathrm{Cor}(\Delta(\Gamma))$ sending the permutation $(0,1,2)$ to the canonical triality τ is a splitting for this sequence. Hence, we obtain that $\mathrm{Cor}(\Delta(\Gamma)) \cong \mathrm{Aut}(\Delta(\Gamma)) \times \mathrm{C}_3$.

If Γ admits dualities then the short exact sequence becomes as follows:

$$1 \longrightarrow \mathrm{Aut}(\Delta(\Gamma)) \longrightarrow \mathrm{Cor}(\Delta(\Gamma)) \longrightarrow \mathrm{S}_3 \longrightarrow 1$$

A splitting for this sequence exists exactly when we can find a correlation of type (i,j) and of order 2 in $\mathrm{Cor}(\Gamma)$. That is, if there exists a polarity π in $\mathrm{Cor}(\Delta)$, the map $s\colon \mathrm{S}_3 \to \mathrm{Cor}(\Delta(\Gamma))$ defined by $s(0,1,2) = \tau$ and $s(i,j) = \pi$ is a splitting. Conversely, any splitting s will need to send (i,j) to a polarity. Thus $\mathrm{Cor}(\Delta(\Gamma)) \cong \mathrm{Aut}(\Delta(\Gamma)) \rtimes \mathrm{S}_3$ if and only if $\Delta(\Gamma)$, and hence Γ, admits a polarity. Moreover, since $\langle\tau\rangle$ is normal, $\mathrm{Cor}(\Delta(\Gamma)) = \langle\tau\rangle \cdot \mathrm{Cor}(\Gamma)$ and $\langle\tau\rangle \cap \mathrm{Cor}(\Gamma) = \{1\}$, we also have $\mathrm{Cor}(\Delta(\Gamma)) \cong \mathrm{C}_3 \rtimes \mathrm{Cor}(\Gamma)$. □

3.2. The cases where $n = 2$ or $m = 2$. As far as linear spaces with parameters go, the theorems of the section cover almost every case. In fact, the only excluded cases are those of complete graphs K_n. We show that for the complete graph K_3, nothing holds true, while for the complete graphs K_v with $v \geqslant 4$, everything holds except the statement on dualities.

First, notice that if $m = 2$, there are exactly two lines through every point. Hence, Γ must be the geometry of a triangle, which is a complete graph with three vertices. Here, we have $\mathrm{Aut}(\Gamma) \cong \mathrm{S}_3$, but $\mathrm{Aut}(\Delta(\Gamma)) \cong \mathrm{S}_6$ because $\Delta(\Gamma)$ consists of six disjoint chambers.

If $n = 2$, Γ is a complete graph. Hence we can assume from now on that Γ is the geometry of a complete graph K_v where v is the number of points of Γ. We will show that, even though Γ has no dualities when $v \geqslant 4$, $\Delta(\Gamma)$ has dualities. Indeed, consider the map β that sends $(p,L,0)$ to $(p^L,L,0)$, $(p,L,1)$ to $(p^L,L,2)$ and $(p,L,2)$ to $(p^L,L,1)$, where, in every case, p^L is the unique point of L which is not p.

Lemma 3.16. *Let Γ be the geometry of the complete graph K_v, with $v \geqslant 3$. The map β defined above is a duality of $\Delta(\Gamma)$.*

Proof. Let $C = \{(p_0,L_0,0),(p_1,L_1,1),(p_2,L_2,2)\}$ be a chamber of $\Delta(\Gamma)$. Note that ${p_0}^{L_0} = p_2, {p_1}^{L_1} = p_0, {p_2}^{L_2} = p_1$. Then

$$\beta(C) = \{({p_0}^{L_0},L_0,0),({p_1}^{L_1},L_1,2),({p_2}^{L_2},L_2,1)\} = \{(p_2,L_0,0),(p_1,L_2,1),(p_0,L_1,2)\}$$

is a chamber of $\Delta(\Gamma)$. This suffices to show that β is a bijection on the elements of $\Delta(\Gamma)$ that preserves incidence. □

Since K_v does not have any dualities when $v \geqslant 4$, this shows that the thickness hypothesis in Proposition 3.14 is essential.

When $v = 3$, Γ has a canonical polarity. Let $\{a, b, c\}$ be the three vertices of Γ and let A, B, C be the three edges of Γ, labeled such that $a \notin A, b \notin B$ and $c \notin C$. Then, Γ the map α defined by $\alpha(a) = A, \alpha(b) = B$ and $\alpha(c) = C$ is a polarity. The composition $\beta \circ \alpha$ is then an automorphism of $\Delta(K_3)$ that is not of the form $\Delta(\phi)$ with $\phi \in \mathrm{Aut}(\Delta(K_3))$.

4. Flag-transitive linear spaces and triangles

In this section we want to prove Theorem 1.1, that classifies linear spaces having an automorphism group acting transitively on the set of non-collinear triples of points. The motivation here is given by Theorem 3.13.

Lemma 4.1. *If Γ is a linear space and $G \leqslant \mathrm{Aut}(\Gamma)$ is transitive on the set $\mathcal{T}(\Gamma)$ of ordered non-collinear triples of points, then Γ is a flag-transitive linear space and (Γ, G) is $(2T)_1$.*

Proof. Take a flag $\{p, L\}$ of Γ where p is a point of Γ and L is a line of Γ. As (Γ, G) is transitive on the set of ordered non-collinear triples of points, the stabilizer of p and L in G, namely $G_{p,L}$, must act transitively of the lines incident to p and distinct from L. Hence the action of the stabilizer G_p must be two-transitive on the lines containing p. Similarly, $G_{p,L}$ must be transitive on the points of L distinct from p. Hence G_L must act two-transitively on the points incident to L. □

This lemma permits us to rely on the classification of $(2T)_1$ flag-transitive linear spaces that we recall below.

Theorem 4.2. [5] *Assume Γ is a flag-transitive linear space of v points with a group G acting on it. Then, if (Γ, G) is $(2T)_1$, one of the following occurs:*

1. $\Gamma = \mathrm{AG}(2, 4)$, *with* $G = \mathrm{A\Gamma L}(1, 16)$.
2. $\Gamma = \mathrm{PG}(n, q)$, $v = \frac{q^{n+1}-1}{q-1}$, $\mathrm{PSL}(n+1, q) \trianglelefteq G \leqslant \mathrm{P\Gamma L}(n+1, q)$ *with* $n \geqslant 2$.
3. $\Gamma = \mathrm{PG}(3, 2)$, $v = 15$ *with* $G \cong \mathrm{A}_7$.
4. $\Gamma = \mathrm{AG}(n, q)$, $v = p^d = q^n$, $G = \mathbb{F}_p^d : G_0$ *with* $\mathrm{SL}(n, q) \trianglelefteq G_0$, $q \geqslant 3$ *and* $n \geqslant 2$.
5. Γ *is a hermitian unital* $\mathrm{U_H}(q)$, $v = q^3 + 1$, $\mathrm{PSU}(3, q) \trianglelefteq G$.
6. Γ *is a circle and G is a 3-transitive permutation group:*
   - (a) $G = \mathrm{A}_v, G = S_v$, $v \geqslant 3$.
   - (b) $G \trianglerighteq \mathrm{PSL}(2, q)$, $v = q + 1$ *and G normalizes a sharply 3-transitive permutation group.*
   - (c) $G = \mathrm{M}_v$, $v = 11, 12, 22, 23, 24$ *or* $G = \mathrm{Aut}(\mathrm{M}_{22})$, $v = 22$.
   - (d) $G = \mathrm{M}_{11}$, $v = 12$.
   - (e) $G = \mathrm{A}_7$, $v = 15$.
   - (f) $G = \mathbb{F}_2^n : G_0$, $v = 2^n$, $G_0 \trianglerighteq \mathrm{SL}(n, 2)$ *with* $n \geqslant 2$.
   - (g) $G = \mathbb{F}_2^4 : \mathrm{A}_7$, $v = 16$.

The converse is also true. For any pair (Γ, G) satisfying the conditions of one of the cases given above, the action of G is $(2T)_1$.

By Lemma 4.1, in order to prove Theorem 1.1, we only need to look at all the cases appearing in Theorem 4.2 and check which cases give linear spaces with a group acting transitively on the set of ordered non-collinear triples of points.

Proof of Theorem 1.1. Let Γ be a linear space. Let $\mathcal{T}(\Gamma)$ be the set of ordered non-collinear triples of points of Γ. We check the cases of Theorem 4.2 one by one to see which ones remain.

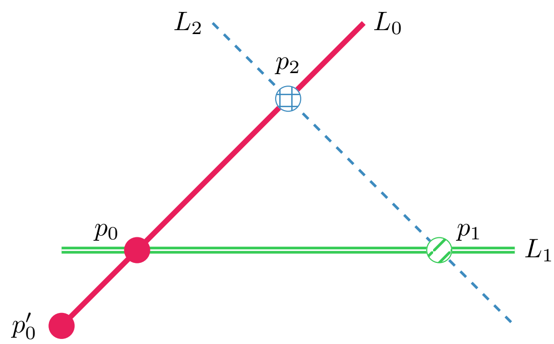


Figure 4. A path from $(p_0, L_0, 0)$ to $(p'_0, L_0, 0)$.

(1) In this case, Γ has 16 points and $\mathrm{A\Gamma L}(1, 16)$ is of order 960. The set $\mathcal{T}(\Gamma)$ has size $16 \cdot 15 \cdot 12 = 2880$. Hence this case cannot occur.

(2) In this case, the group $\mathrm{PSL}(n + 1, q)$ obviously acts transitively on $\mathcal{T}(\Gamma)$ and so do all groups G with $\mathrm{PSL}(n+1, q) \trianglelefteq G \leqslant \mathrm{P\Gamma L}(n+1, q)$. This becomes case (1) of Theorem 1.1.

(3) The cardinality of $\mathcal{T}(\Gamma)$ is $15 \cdot 14 \cdot 12 = 2520$ and the group A_7 acts transitively on $\mathcal{T}(\Gamma)$. This becomes case (2) of Theorem 1.1.

(4) In that case, we need G_0 to be large enough so that the group $p^n : G_0$ acts transitively on non-collinear triples of points; This implies that $G_0 \geqslant \mathrm{GL}(n, q)$ when $n = 2$. This is case (3) of Theorem 1.1.

(5) The cardinality of $\mathcal{T}(\Gamma)$ is $(q^3 + 1) \cdot q^3 \cdot (q^3 - q)$ while $|\,\mathrm{P\Gamma U}(3, q)| = (q^3 + 1) \cdot q^3 \cdot (q^2 - 1) \cdot 2e$ where $q = p^e$. Hence the only possibilities for G to be transitive on $\mathcal{T}(\Gamma)$ is for $q = 2$ or 4 and $G = \mathrm{P\Gamma U}(2, q)$. This is case (4) of Theorem 1.1.

(6) In all these cases, the group is 3-transitive on the points and lines have two points, hence no three points are collinear and G acts transitively on $\mathcal{T}(\Gamma)$. This is case (5) of Theorem 1.1. □

5. Connectedness and residual connectedness

In this section, we want to prove Theorem 1.2. Hence, for each linear space Γ appearing in Theorem 1.1, we now check which ones give a residually connected geometry using Construction 1.

Observe that the input for Construction 1 is the incidence geometry. The group acting is not important. Therefore, Case (2) of Theorem 1.1 will be dealt with Case (1).

Proposition 5.1. *If Γ is a finite affine or projective plane then $\Delta(\Gamma)$ is a connected geometry.*

Proof. The fact that $\Delta(\Gamma)$ is a geometry is obvious. Hence we can check connectivity by checking that any two elements of type 0 of $\Delta(\Gamma)$ are connected.

Let $(p_0, L_0, 0) \in X_{\Delta(\Gamma)}$. We claim that for any $p'_0 \in L_0$ and for any $L'_0 \in \mathcal{L}_{p_0}$ with $p_0 \neq p'_0$ and $L_0 \neq L'_0$, there is an incidence chain from $(p_0, L_0, 0)$ to $(p'_0, L_0, 0)$ and a chain from $(p_0, L_0, 0)$ to $(p_0, L'_0, 0)$.

For the first claim, choose a point p_1 not incident to L_0 and a point $p_2 \notin \{p'_0, p_0\}$ with $p_2 \in L_0$. Such a point exists as Γ has at least three points per line. Let L_1 be the line incident to p_0 and p_1 and let L_2 be the line incident to p_1 and p_2 (see Figure 4). Then $(p_0, L_0, 0), (p_1, L_1, 1), (p_2, L_2, 2), (p'_0, L_0, 0)$ is an incidence chain in $\Delta(\Gamma)$.

For the second claim, take p_1 a point not incident to L_0 nor L'_0. Let $p_2 \in L'_0$ with p_2 not incident to $L_1 = \big[p_0, p_1\big]$. Finally let L_2 be the line of Γ incident to p_1 and p_2

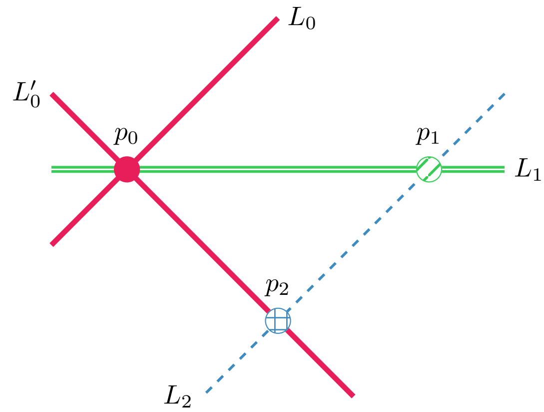


FIGURE 5. A path from $(p_0, L_0, 0)$ to $(p_0, L_0', 0)$.

(see Figure 5). Then $(p_0, L_0, 0), (p_1, L_1, 1), (p_2, L_2, 2), (p_0, L_0', 0)$ is an incidence chain in $\Delta(\Gamma)$.

Since $\Delta(\Gamma)$ is a geometry, every element of a given type is incident to two elements of the other types. Since Γ is itself a connected incidence geometry, using the previous observations we can construct an incidence chain from any element to any other element of $\Delta(\Gamma)$ by composing the previous operations. Hence, $\Delta(\Gamma)$ is connected. ☐

PROPOSITION 5.2. *Let $n \geqslant 3$. If $\Gamma = \mathrm{AG}(n, q)$ or $\Gamma = \mathrm{PG}(n, q)$, then $\Delta(\Gamma)$ is not residually connected*

Proof. Let $F = \{(p_0, L_0, 0), (p_1, L_1, 1)\}$ be a rank two flag of $\Delta(\Gamma)$. An element $(p_2, L_2, 2)$ incident to F will have L_2 contained in the plane spanned by L_0 and L_1. Since there are multiple planes containing L_0, the residue of $(p_0, L_0, 0)$ will split in distinct connected components, one for each plane containing L_0. ☐

PROPOSITION 5.3. *If Γ is a finite linear or projective plane with three points per line or more, then $\Delta(\Gamma)$ is residually connected.*

Proof. By Proposition 3.6, $\Delta(\Gamma)$ has a triality, hence it suffices to show that residues of a flag of type 1 are connected. Moreover, since $\Delta(\Gamma)$ is a geometry, it suffices to show that, given any two elements of type 1 in the residue of a flag of type 0, there is a chain in that residue that connects them.

Let $\Delta(\Gamma)_{(p_0, L_0, 0)}$ be the residue of $(p_0, L_0, 0)$. Let $(p_1, L_1, 1) \in \Delta(\Gamma)_{(p_0, L_0, 0)}$ and $(p_1', L_1', 1) \in \Delta(\Gamma)_{(p_0, L_0, 0)}$. Note that if $p_1 = p_1'$ then $L_1 = L_1'$ so nothing needs to be checked.

Assume first that $L_1 = L_1'$ and $p_1 \neq p_1'$. Then the elements $(p_1, L_1, 1)$ and $(p_1', L_1, 1)$ clearly cannot be at distance 2 from each other in the incidence graph of $\Delta(\Gamma)_{(p_0, L_0, 0)}$. Suppose there exist $q_2, q_2' \in L_0$ so that the lines $L_2 = \left[p_1, q_2\right]$ and $L_2' = \left[p_1', q_2'\right]$ intersect. Name that intersection p_1'' (see Figure 6). The chain

$$(p_1, L_1, 1), (q_2, L_2, 2), (p_1'', L_1'', 1), (q_2', L_2', 2), (p_1', L_1, 1)$$

is an incidence chain in $\Gamma_{(p_0, L_0, 0)}$. We conclude that the distance between $(p_1, L_1, 1)$ and $(p_1', L_1, 1)$ in the incidence graph of $\Delta(\Gamma)_{(p_0, L_0, 0)}$ is 4.

Suppose the above construction fails, that is we cannot find p_2 and p_2' as requested since $L_2 = \left[p_1, p_2\right]$ is parallel to $L_2' = \left[p_1', p_2'\right]$. In that case, we are dealing with $\mathrm{AG}(2, 3)$ (as if there were more than three points per line, we could take another point p_2' to make sure L_2 and L_2' were not parallel). This case is small enough to be

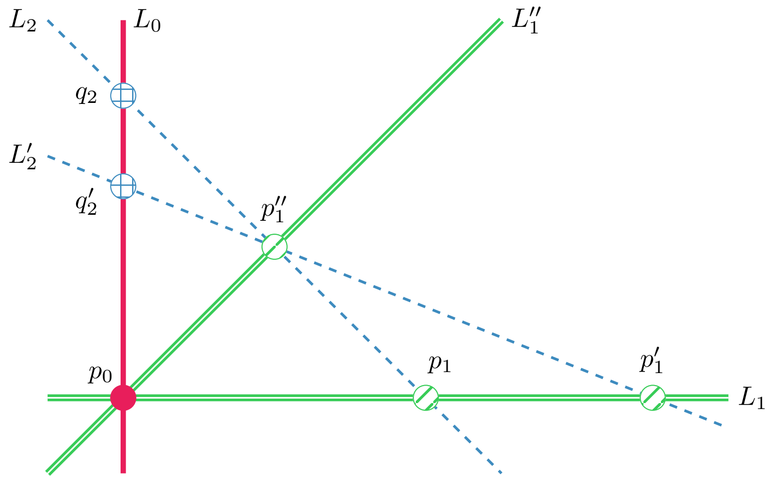


Figure 6. The case where $L_1 = L_1'$.

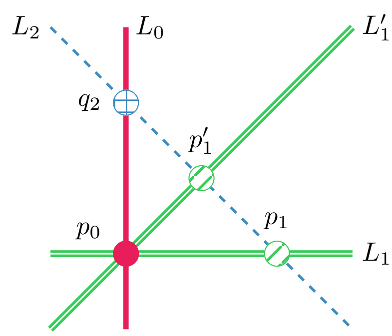


Figure 7. The case where $[p_1, p_1'] \cap L_0 \neq \varnothing$ and $p_1' \notin L_1$.

fully analysed by hand and check that the residue is then the incidence graph of a 6-gon.

Assume finally that $L_1 \neq L_1'$ and $p_1 \neq p_1'$. Take $L_2 = [p_1, p_1']$. If L_2 and L_0 have a point in common, let $q_2 = L_2 \cap L_0$ (see Figure 7). Then $(p_1, L_1, 1), (q_2, L_2, 2), (p_1', L_1', 1)$ forms an incidence chain so $(p_1, L_1, 1)$ and $(p_1', L_1', 1)$ are at distance 2.

In an affine plane, it could happen that $[p_1, p_1']$ is parallel to L_0 and the previous construction fails. In that case, pick a third point on $[p_1, p_1']$, say p_1'' and take the line $L_1'' = [p_0, p_1'']$. On that line, take a third point q_1. Now, the line $L_2 = [p_1, q_1]$ necessarily intersects L_0 in, say q_2. Moreover, the line $L_2' = [p_1', q_1]$ also intersects L_0 in, say q_2'. We have that $(p_1, L_1, 1), (q_2, L_2, 2), (q_1, L_1'', 1), (q_2', L_2', 2), (p_1', L_1', 1)$ is a chain in $\Delta(\Gamma)_{(p_0, L_0, 0)}$, as illustrated in Figure 8.

□

Corollary 5.4. *If* $\Gamma = \mathrm{AG}(2, q)$ *with* $q \geqslant 4$ *or* $\Gamma = \mathrm{PG}(2, q)$ *with* $q \geqslant 3$*, then the gonality of any rank two residue of* $\Delta(\Gamma)$ *is equal to* 3 *and the point and line diameters are equal to* 4.

Proof. By Propositon 3.6, $\Delta(\Gamma)$ has a triality, hence we know that all rank two residues will be isomorphic. Suppose we look at a residue of a flag of type 0. By the proof of Proposition 5.3, we already know the distance between two elements of type 2 in

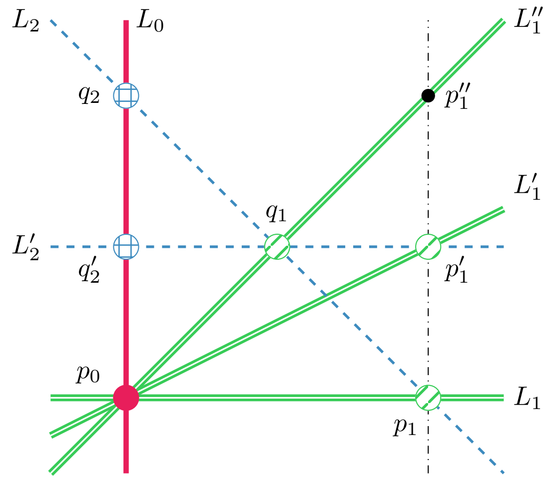


Figure 8. The case where $\left[p_1, p_1'\right]$ is parallel to L_0.

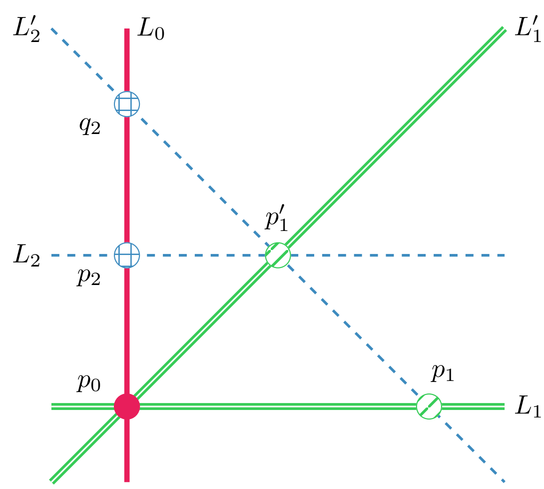


Figure 9. A path of length three.

$\Delta(\Gamma)_{(p_0,L_0,0)}$ is less than or equal to 4. Let $(p_1, L_1, 1), (p_2, L_2, 2) \in \Delta(\Gamma)_{(p_0,L_0,0)}$. We claim that $d_{\Delta(\Gamma)_{(p_0,L_0,0)}}((p_1, L_1, 1), (p_2, L_2, 2)) \leqslant 3$. Indeed, take $q_2 \in L_0$ with $q_2 \notin \{p_0, p_2\}$ such that $\left[p_1, q_2\right]$ is not parallel to L_2. (note that this is not possible in $\mathrm{AG}(2,3)$). Construct in order $L_2' = \left[p_1, q_2\right]$, $p_1' = L_2' \cap L_2$, $L_1' = \left[p_1', p_0\right]$. Then

$$(p_1, L_1, 1), (q_2, L_2', 2), (p_1', L_1', 1), (p_2, L_2, 2)$$

is an incidence chain in $\Delta(\Gamma)_{(p_0,L_0,0)}$ (see Figure 9).

Hence the point and line diameters are equal to 4.

To compute the gonality, notice we have at least four points per lines. Choose three points $q_2, q_2', q_2'' \in L_0$ and $q_2, q_2', q_2'' \neq p_0$. Construct the lines $L_2 = \left[q_2, p_1\right]$ and $L_2' = \left[q_2', p_1\right]$. There are at least two points in $L_2' \backslash \{q_2', p_1\}$ so at least one of the two lines passing by q_2'' and the first point and q_2'' and the second point will be non parallel to L_2 (one does not need to worry about this case in the projective plane). Name that

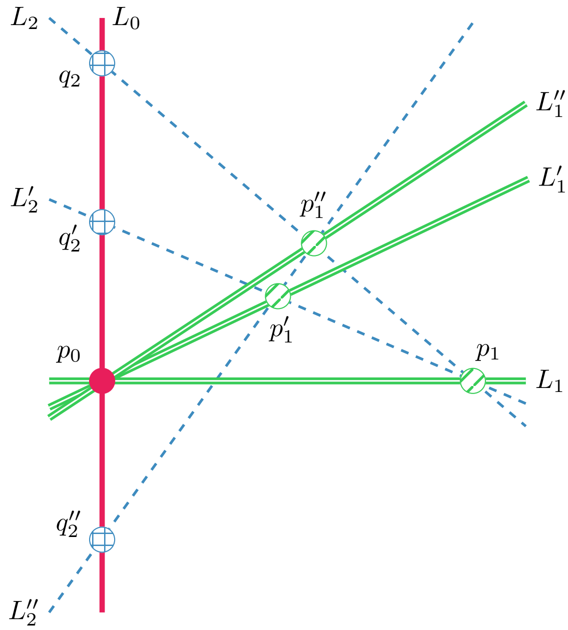


FIGURE 10. A cycle with basepoint $(p_1, L_1, 1)$ of length six in $\Delta(\Gamma)_{(p_0,L_0,0)}$.

line L_2''. Let $p_1' = L_2' \cap L_2''$ and $p_1'' = L_2 \cap L_2''$. Then

$$(p_1, L_1, 1), (q_2, L_2, 2), (p_1'', L_1'', 1), (q_2'', L_2'', 2), (p_1', L_1', 1), (q_2', L_2', 2), (p_1, L_1, 1)$$

is an incidence chain of length 6 in $\Delta(\Gamma)_{(p_0,L_0,0)}$ (see Figure 10).

One can easily verify that cycles based in $(p_1, L_1, 1)$ of length 4 cannot exist in $\Delta(\Gamma)_{(p_0,L_0,0)}$. Moreover, the same construction can be used for cycles with a base point $(p_2, L_2, 2)$. We conclude that the gonality is 3 which completes the proof. □

PROPOSITION 5.5. *If* $\Gamma = \mathrm{K}_v$*,* $v \geqslant 4$ *then* $\Delta(\Gamma)$ *is connected and not residually connected.*

Proof. Since every line has exactly two points, $\Delta(\Gamma)$ is not residually connected. We first prove there is an incidence chain from (p, L, i) to $(p, L, i+1)$ in $\Delta(\Gamma)$. Secondly, we prove there is an incidence chain from (p, L, i) to (p^L, L, i) in $\Delta(\Gamma)$, where p^L is the unique point of L different from p. Without loss of generality, take $(p_0, L_0, 0) \in X_{\Delta(\Gamma)}$. Let $p_1, p_2 \notin L_0$.

For the first claim, construct $L_1 = [p_0, p_1], L_2 = [p_1, p_2], L_0' = [p_2, {p_0}^{L_0}]$. Then,

$$(p_0, L_0, 0), (p_1, L_1, 1), (p_2, L_2, 2), ({p_0}^{L_0}, L_0', 0), (p_0, L_0, 1)$$

is an incidence chain in $\Delta(\Gamma)$ (see Figure 11).

For the second claim, construct $L_1 = [p_0, p_1], L_2 = [p_1, p_2], L_0' = [p_2, p_0]$. Then,

$$(p_0, L_0, 0), (p_1, L_1, 1), (p_2, L_2, 2), (p_0, L_0', 0), ({p_0}^{L_0}, L_0, 1)$$

is an incidence chain in $\Delta(\Gamma)$ (see Figure 12). By then using the first claim two times we get an incidence chain from $(p_0, L_0, 0)$ to $({p_0}^{L_0}, L_0, 0)$.

Using these two moves, it is readily seen that $\Delta(\Gamma)$ is connected. □

Proof of Theorem 1.2. Theorem 1.2 is a summary of the results obtained in the Propositions and Corollary of this section. □

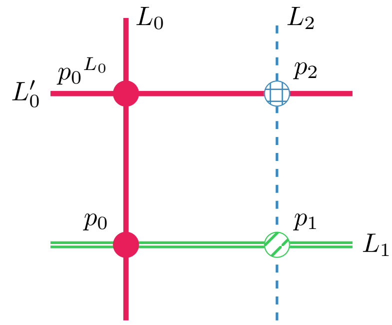


FIGURE 11. A path from $(p_0, L_0, 0)$ to $(p_0, L_0, 1)$.

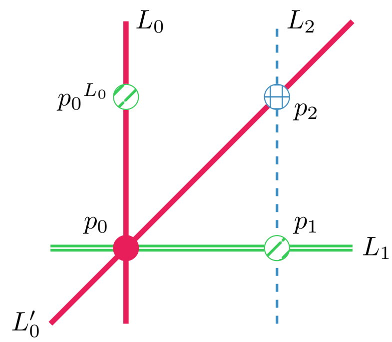


FIGURE 12. A path from $(p_0, L_0, 0)$ to $(p_0^{L_0}, L_0, 1)$.

COROLLARY 5.6. *The family $\Delta(\mathrm{AG}(2,q))$ for $q > 3$, q a power of a prime, is an infinite family of thick, residually connected and flag-transitive geometries with trialities but no dualities.*

Proof. This is a direct consequence of Theorem 1.2 and Proposition 3.15. □

6. FIRM, RESIDUALLY CONNECTED AND FLAG-TRANSITIVE GEOMETRIES WITH A TRIALITY

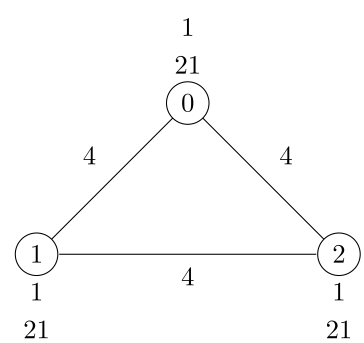
In this section we synthesize some important features of the geometries constructed in the previous section by computing their Buekenhout diagrams. We start by giving their precise definition, which requires to first define some properties of rank 2 geometries.

Francis Buekenhout introduced a diagram associated to incidence geometries [4]. His idea was to associate to each rank two residue a set of three integers giving information on its incidence graph. Let Γ be a rank two geometry. We can consider Γ to have type set $I = \{P, L\}$, where P and L stand for points and lines. The *point-diameter*, denoted by $d_P(\Gamma) = d_P$, is the largest integer k such that there exists a point $p \in P$ and an element $x \in \Gamma$ with $d(p,x) = k$. Similarly the *line-diameter*, denoted by $d_L(\Gamma) = d_L$, is the largest integer k such that there exists a line $l \in L$ and an element $x \in \Gamma$ with $d(l,x) = k$. Finally, the *gonality* of Γ, denoted by $g(\Gamma) = g$ is half the length of the smallest circuit in the incidence graph of Γ.

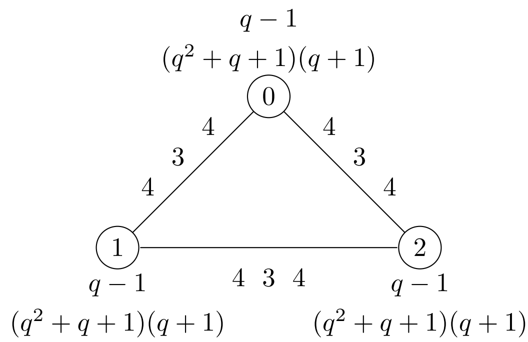
If a rank two geometry Γ has $d_P = d_L = g = n$ for some natural number n, we say that it is a *generalized n-gon.* Generalized 2-gons are also called generalized digons. They are in some sense trivial geometries since all points are incident to all lines. Their incidence graphs are complete bipartite graphs. Generalized 3-gons are projective planes.

Let Γ be a geometry over I. The *Buekenhout diagram* (or diagram for short) D for Γ is a graph whose vertex set is I. Each edge $\{i,j\}$ is labeled with a collection D_{ij} of rank two geometries. We say that Γ belongs to D if every residue of rank two of type $\{i,j\}$ of Γ is one of those listed in D_{ij} for every pair of $i \neq j \in I$. In most cases, we use conventions to turn a diagram D into a labeled graph. The most common convention is to not draw an edge between two vertices i and j if all residues of type $\{i,j\}$ are generalized digons, and to label the edge $\{i,j\}$ by a natural integer n if all residues of type $\{i,j\}$ are generalized n-gons. It is also common to omit the label when $n=3$. If the edge $\{i,j\}$ is labeled by a triple (d_{ij}, g_{ij}, d_{ji}) it means that every residue of type $\{i,j\}$ had $d_P = d_{ij}, g = g_{ij}, d_L = d_{ji}$. We can also add information to the vertices of a diagram. We can label the vertex i with the number n_i of elements of type i in Γ. Moreover, if for all flags F of co-type i, we have that $|\Gamma_F| = s_i + 1$, we will also label the vertex i with the integer s_i. We now give the Buekenhout diagrams of the geometries appearing in Theorem 1.2.

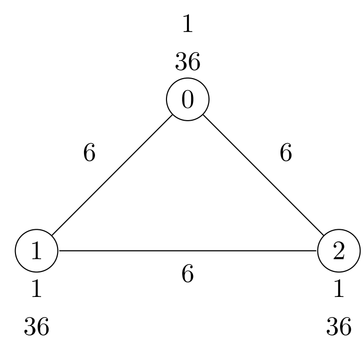
For $\Gamma = \mathrm{PG}(2,2)$, the diagram of $\Delta(\Gamma)$ can be computed either by hand or using MAGMA [2] and is as follows:



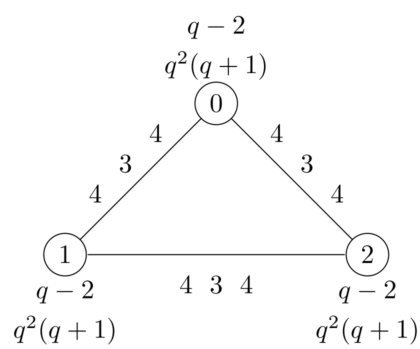
For $\Gamma = \mathrm{PG}(2,q)$, with $q \geqslant 3$ the diagram of $\Delta(\Gamma)$ is computed thanks to Corollary 5.4.



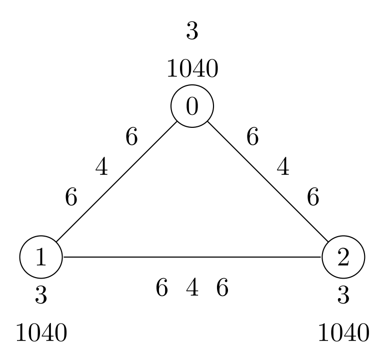
For $\Gamma = \mathrm{AG}(2,3)$, the diagram of $\Delta(\Gamma)$ can again be computed either by hand or using Magma [2].



For $\Gamma = \mathrm{AG}(2,q), q \geqslant 4$ the diagram of $\Delta(\Gamma)$ is the following thanks to Corollary 5.4.



For $\Gamma = \mathrm{U_H}(2)$, as $\mathrm{U_H}(2) \cong \mathrm{AG}(2,3)$, the diagram of $\Delta(\Gamma)$ is already given above. For $\Gamma = \mathrm{U_H}(4)$, the diagram of $\Delta(\Gamma)$ diagram can be computed thanks to Magma. The rank two residues do not have dualities.



We conclude this section with Table 1. This table lists all the geometries Γ we discussed and tells us whether $\Delta(\Gamma)$ is connected, residually connected or thin. It also contains the description of the automorphism group and correlation group of $\Delta(\Gamma)$.

7. Conclusion

In this paper, we described a geometric construction that permits to get, from a rank two geometry Γ, a rank three incidence system $\Delta(\Gamma)$ that admits trialities. The gonality of Γ needs to be at most three as proven in Proposition 3.3 in order to have that $\Delta(\Gamma)$ be a geometry. We then focused on the case where the gonality is three and this led us naturally to investigate which linear spaces Γ give interesting geometries

Γ	Connected	RC	Thin	$\mathrm{Aut}(\Delta(\Gamma))$	$\mathrm{Cor}(\Delta(\Gamma))$
$\mathrm{AG}(2,3)$	Yes	Yes	Yes	$\mathrm{AGL}(2,3)$	$\mathrm{AGL}(2,3)\times \mathrm{C}_3$
$\mathrm{AG}(n,3)$, $n\geqslant 3$	Yes	No	Yes	$\mathrm{AGL}(n,3)$	$\mathrm{AGL}(n,3)\times \mathrm{C}_3$
$\mathrm{AG}(n,q)$, $n\geqslant 3, q\geqslant 4$	Yes	No	No	$\mathrm{AGL}(n,q)$	$\mathrm{AGL}(n,q)\times \mathrm{C}_3$
$\mathrm{AG}(2,q)$, $q\geqslant 4$	Yes	Yes	No	$\mathrm{AGL}(2,q)$	$\mathrm{AGL}(2,q)\times \mathrm{C}_3$
$\mathrm{PG}(2,2)$	Yes	Yes	Yes	$\mathrm{P\Gamma L}(3,2)$	$\mathrm{P\Gamma L}(3,2)\rtimes \mathrm{S}_3$
$\mathrm{PG}(2,q)$, $q\geqslant 3$	Yes	Yes	No	$\mathrm{P\Gamma L}(3,q)$	$\mathrm{P\Gamma L}(3,q)\rtimes \mathrm{S}_3$
$\mathrm{PG}(n,q)$, $n\geqslant 3, q\geqslant 3$	Yes	No	No	$\mathrm{P\Gamma L}(n+1,q)$	$\mathrm{P\Gamma L}(n+1,q)\times \mathrm{C}_3$
$\mathrm{PG}(3,2)$	Yes	No	Yes	$\mathrm{P\Gamma L}(4,2)$	$\mathrm{P\Gamma L}(4,2)\times \mathrm{C}_3$
K_3	No	No	No	S_6	$\mathrm{S}_6\times \mathrm{S}_3$
$\mathrm{K_n}$, $n\geqslant 4$	Yes	No	No	$\mathrm{S_n}$	$\mathrm{S_n}\times \mathrm{S}_3$
$\mathrm{U_H}(4)$	Yes	Yes	No	$\mathrm{P\Gamma U}(3,4)$	$\mathrm{P\Gamma U}(3,4)\times \mathrm{C}_3$

Table 1. Table of geometries.

$\Delta(\Gamma)$. We thus determined which linear spaces Γ give "interesting geometries" $\Delta(\Gamma)$, that is geometries that are firm, residually connected and flag-transitive.

We did not investigate the case where the gonality of Γ is two. However, experiments suggest that $\Delta(\Gamma)$ will not be residually connected in that case.

We did not find any examples of rank two geometries Γ that have gonality three but are not linear spaces and that would give a $\Delta(\Gamma)$ that is firm, residually connected and flag-transitive. The construction suggests however that there is no restriction on the diameter of Γ, but simply that elements at distance more than four of each other will not end up in common rank two residues.

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