ALGEBRAIC
COMBINATORICS

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Volume 9, issue 2 (2026), p. 327-355.
https://doi.org/10.5802/alco.483

*Algebraic Combinatorics is published by The Combinatorics Consortium
and is a member of the Centre Mersenne for Open Scientific Publishing*
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

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Towards the classification of maximum scattered linear sets of $\mathrm{PG}(1, q^5)$

Stefano Lia, Giovanni Longobardi & Corrado Zanella

Abstract Every maximum scattered linear set in $\mathrm{PG}(1, q^5)$ is the projection of an $\mathbb{F}_q$-subgeometry Σ of $\mathrm{PG}(4, q^5)$ from a plane Γ external to the secant variety to Σ [20]. The pair (Γ, Σ) will be called a projecting configuration for the linear set. The projecting configurations for the only known maximum scattered linear sets in $\mathrm{PG}(1, q^5)$, namely those of pseudoregulus and LP type, have been characterized in the literature [6, 30]. Let (Γ, Σ) be a projecting configuration for a maximum scattered linear set in $\mathrm{PG}(1, q^5)$. Let σ be a generator of $\mathbb{G} = \mathrm{P\Gamma L}(5, q^5)_\Sigma$, and $A = \Gamma \cap \Gamma^{\sigma^4}$, $B = \Gamma \cap \Gamma^{\sigma^3}$. If A and B are not both points, then the projected linear set is of pseudoregulus type [6]. Suppose that they are points. The rank of a point X is the vectorial dimension of the span of the orbit of X under the action of $\mathbb{G}$. In this paper, by investigating the geometric properties of projecting configurations, it is proved that if at least one of the points A and B has rank 5, the associated maximum scattered linear set must be of LP type. Then, if a maximum scattered linear set of a new type exists, it must be such that $\operatorname{rk} A = \operatorname{rk} B = 4$. In this paper we derive two possible polynomial forms that such a linear set must have. An exhaustive analysis by computer shows that for $q \leqslant 25$ no new maximum scattered linear set exists.

1. Introduction

1.1. Linear sets in finite projective spaces.

Let $\mathrm{PG}(d, q^n)$ be the d-dimensional projective space over the vector space $\mathbb{F}_{q^n}^{d+1}$. If U is an r-dimensional $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^n}^{d+1}$, then

$$L_U = \{\langle v\rangle_{\mathbb{F}_{q^n}} : v \in U,\, v \neq 0\} \subseteq \mathrm{PG}(d, q^n)$$

is called $\mathbb{F}_q$*-linear set* of rank r. Despite the simple definition, linear sets are very rich structures, connected to interesting objects, such as blocking sets, two-intersection sets, complete caps, translation spreads of the Cayley Generalized Hexagon, translation ovoids of polar spaces, semifield flocks, finite semifields and rank metric codes, see for example [14, 25, 26] and their references.

The linear set L_U is *scattered* if $\dim_{\mathbb{F}_q}\left(\langle v\rangle_{\mathbb{F}_{q^n}} \cap U\right) \leqslant 1$ for any $v \in \mathbb{F}_{q^n}^{d+1}$ or, equivalently, if it has size $(q^r-1)/(q-1)$. A nontrivial scattered $\mathbb{F}_q$-linear set L_U of $\mathrm{PG}(d, q^n)$ of highest possible rank is a *maximum scattered* $\mathbb{F}_q$*-linear set* (MSLS for short); in this case U is called a *maximum scattered subspace* of $\mathbb{F}_{q^n}^{d+1}$. The rank r of an MSLS depends on d and n. It holds $r \leqslant \frac{(d+1)n}{2}$, see [2], with equality when $(d+1)n$ is even [1, 2, 5]. Two $\mathbb{F}_q$-linear sets are called *equivalent* (resp. *projectively equivalent*)

Manuscript received 31st July 2024, revised 11th December 2025 and 15th January 2026, accepted 15th January 2026.

Keywords. finite projective space, linear set, linearized polynomial, scattered polynomial.

ISSN: 2589-5486 `http://algebraic-combinatorics.org/`

if there exists a collineation (resp. a *projectivity*) of $\mathrm{PG}(d, q^n)$ that maps one onto the other.

The maximum scattered $\mathbb{F}_q$-linear sets in $\mathrm{PG}(1, q^n)$ can be effectively described by means of linearized polynomials. In fact, every linear set in $\mathrm{PG}(1, q^n)$ of rank n is projectively equivalent to

$$L_f = \{\langle (x, f(x)) \rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\},$$

where $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ is some $\mathbb{F}_q$-linearized polynomial. This L_f is a scattered linear set if and only if each ratio $f(x)/x$ occurs for at most $q-1$ distinct non-zero values of $x \in \mathbb{F}_{q^n}^*$; in other words, if and only if for all $y, z \in \mathbb{F}_{q^n}^*$

$$\frac{f(y)}{y} = \frac{f(z)}{z} \implies \frac{y}{z} \in \mathbb{F}_q. \tag{1}$$

Any linearized polynomial fulfilling (1) is called *scattered polynomial.* There are only four known families of scattered polynomials which are defined for infinitely many values of n [10, 16, 18, 23, 28]. Of these families, only the following two occur in $\mathrm{PG}(1, q^5)$, the focus of this work:

- (i) x^{q^s}, $(s, n) = 1$ that is called scattered polynomial of *pseudoregulus type* [2];
- (ii) $x^{q^s} + \delta x^{q^{n-s}}$, $n > 3$, $(s, n) = 1$, $\mathrm{N}_{q^n/q}(\delta) \neq 0, 1$ that is called scattered polynomial of *LP type* from the names of Lunardon and Polverino who introduced it [19, 27].

The corresponding linear sets are called linear sets of *pseudoregulus* and *LP type*, respectively. Further details on scattered polynomials can be found in [17].

1.2. Projecting configurations.

A *canonical* $\mathbb{F}_q$*-subgeometry* of $\mathrm{PG}(d, q^n)$ is a set projectively equivalent to the linear set of rank $d+1$ associated with the $\mathbb{F}_q$-subspace $\mathbb{F}_q^{d+1}$ (the set of points with coordinates rational over $\mathbb{F}_q$). The relevance of subgeometries lies in the following result.

Theorem 1.1 ([20]). *If L_U is an $\mathbb{F}_q$-linear set of rank r in $\Lambda = \mathrm{PG}(d, q^n)$ such that $\langle L_U \rangle = \Lambda$, then there exists a projective space $\mathrm{PG}(r-1, q^n) \supseteq \Lambda$, an $\mathbb{F}_q$-canonical subgeometry Σ of $\mathrm{PG}(r-1, q^n)$, and a complement Γ of Λ in $\mathrm{PG}(r-1, q^n)$, such that $\Gamma \cap \Sigma = \varnothing$, and L_U is the projection of Σ from the vertex Γ onto Λ. Conversely, any such projection is a linear set.*

In this paper any such pair (Γ, Σ) is called *projecting configuration* for the linear set L_U. The properties of L_U can be described in terms of such projecting configuration, the subspace Λ being immaterial. The corresponding projection map is denoted by $\wp_\Gamma$. In [7], it was proved that for some linear sets the projecting configuration is not unique up to collineations. This issue is further formalized in [4], where the definition of $\Gamma\mathrm{L}$-*class* of a linear set L_U is seen from the perspective of projecting configurations. For a linear set L_U of $\Gamma\mathrm{L}$-class one the projecting configuration is unique up to the action of $\Gamma\mathrm{L}(r, q^n)$; in this case L_U is called *simple.* It has been recently proved in [11] that, for any $n \geqslant 2$, the $\Gamma\mathrm{L}$-class of an MSLS in $\mathrm{PG}(1, q^n)$ not of pseudoregulus type is at most two.

According to Theorem 1.1, the projecting configuration for a maximum linear set L_U in $\mathrm{PG}(1, q^n)$, i.e. a linear set of rank n, is contained in $\mathrm{PG}(n-1, q^n)$, and it has as a vertex an $(n-3)$-dimensional projective subspace Γ projecting a subgeometry Σ, $\Sigma \cap \Gamma = \varnothing$, on a line ℓ such that $\Gamma \cap \ell = \varnothing$.

In this paper, the focus is on the mutual position of Γ and Σ; in particular, on the behavior of the vertex Γ under the action of the pointwise stabilizer $\mathbb{G} = \mathrm{P}\Gamma\mathrm{L}(n, q^n)_\Sigma$. This is a cyclic group of order n.

Assume that (Γ,Σ) is a projecting configuration in $\mathrm{PG}(n-1,q^n)$ for a linear set L_U. Let σ be a generator of $\mathbb{G}=\mathrm{P\Gamma L}(n,q^n)_\Sigma$. The *rank* of a point $P\in\mathrm{PG}(n-1,q^n)$ with respect to the $\mathbb{F}_q$-subgeometry Σ is

$$\operatorname{rk}P = 1+\dim\langle P,P^\sigma,P^{\sigma^2},\dots,P^{\sigma^{n-1}}\rangle,$$

where, from this point on, the angle brackets $\langle\ \rangle$ without a field as an index will be used to denote the projective closure of a point set in a projective space. Equivalently, $\operatorname{rk}P$ is the minimum size of a subset $\mathcal{I}\subseteq\Sigma$ such that $P\in\langle\mathcal{I}\rangle$. The following result holds.

Theorem 1.2. *Let* Σ *be an* $\mathbb{F}_q$*-canonical subgeometry of* $\mathrm{PG}(n-1,q^n)$ *and let* Γ *and* Λ *be an* $(n-3)$*-dimensional projective space and a line, respectively such that* $\Gamma\cap\Sigma=\varnothing=\Gamma\cap\Lambda$. *Let consider the projection map of* Σ *from* Γ *into* Λ*:*

$$\wp_\Gamma : P\in\Sigma \longrightarrow \langle P,\Gamma\rangle\cap\Lambda\in\Lambda. \tag{2}$$

Then following assertions are equivalent:

- (i) $\wp_\Gamma(\Sigma)$ *is scattered;*
- (ii) *the restriction of* $\wp_\Gamma$ *to* Σ *is injective;*
- (iii) *every point of* Γ *has rank greater than two.*

In [6, 8, 11, 14, 22, 30], several results on the characterization of known maximum scattered linear sets of the projective line have been achieved. These results yield a classification of MSLSs of $\mathrm{PG}(1,q^n)$ for $n\leqslant 4$. In what follows, we collect some of these and outline the current state of the art.

Firstly, since all maximum scattered $\mathbb{F}_2$-linear sets of $\mathrm{PG}(1,2^n)$ have size 2^n-1 and are equivalent, they are of pseudoregulus type, see also [24].

It is also straightforward to see that any maximum scattered $\mathbb{F}_q$-linear set of $\mathrm{PG}(1,q^2)$ is a Baer subline, i.e., a (canonical) subgeometry of $\mathrm{PG}(1,q^2)$.

In [6], a characterization result for linear sets of pseudoregulus type, obtained via their projecting configurations, is presented.

Theorem 1.3 ([6]). *Assume that* $\wp_\Gamma(\Sigma)$ *is scattered. Then the following assertions are equivalent:*

- (i) $\wp_\Gamma(\Sigma)$ *is of pseudoregulus type;*
- (ii) $\dim(\Gamma\cap\Gamma^\sigma)=n-4$ *for some generator* σ *of* $\mathbb{G}$*;*
- (iii) *there exists a point* P *and a generator* σ *of* $\mathbb{G}$ *such that* $\operatorname{rk}P=n$ *and*

$$\Gamma=\langle P,P^\sigma,\dots,P^{\sigma^{n-3}}\rangle.$$

Since a maximum $\mathbb{F}_q$-linear set of $\mathrm{PG}(1,q^3)$ has a rank-three point of $\mathrm{PG}(2,q^3)$ as its projecting vertex, Condition (ii) of the theorem above is trivially fulfilled. Hence, we obtain the following.

Theorem 1.4. *Any maximum scattered* $\mathbb{F}_q$*-linear set of* $\mathrm{PG}(1,q^3)$ *is of pseudoregulus type.*

This result had already been obtained using properties of the stabilizer of a subgeometry Σ of $\mathrm{PG}(2,q^3)$; see [4, Example 5.1, Section 5.2 and Remark 5.6] and [13].

Since the projecting configuration of an $\mathbb{F}_q$-linear set of rank 4 in $\mathrm{PG}(1,q^4)$ is unique up to collineations ([4, Theorem 4.5]), Csajbók and Zanella proved in [8] the following classification result.

Theorem 1.5 ([8]). *The only maximum scattered linear sets in* $\mathrm{PG}(1,q^4)$ *are those of pseudoregulus type or of LP type.*

In the same spirit of Theorem 1.3, in [30] the authors provided a characterization of LP type linear sets. Their result, which will be crucial for this article, is the following.

THEOREM 1.6 ([11, 30]). *Let $n \geqslant 4$ and $q > 2$. Assume that $\wp_\Gamma(\Sigma)$ is scattered and not of pseudoregulus type. Then $\wp_\Gamma(\Sigma)$ is of LP type if and only if there exists a generator σ of $\mathbb{G}$ such that:*

(*i*) $\dim(\Gamma \cap \Gamma^\sigma \cap \Gamma^{\sigma^2}) > n - 7$;
(*ii*) *there exist points P and Q such that*

$$\Gamma = \langle P, P^\sigma, \ldots, P^{\sigma^{n-4}}, Q \rangle,$$

and the line $\langle P^{\sigma^{n-3}}, P^{\sigma^{n-1}} \rangle$ meets Γ.

Moreover, if $\wp_\Gamma(\Sigma)$ is of LP type, then the point P is unique and has rank n.

1.3. OUTLINE OF THE PAPER. In this paper, we study the case $n = 5$. The projecting configuration (Γ, Σ) consists of a plane Γ and an $\mathbb{F}_q$-subgeometry $\Sigma \cong \mathrm{PG}(4, q)$ in $\mathrm{PG}(4, q^5)$. In [9], the projecting configurations for non-scattered linear sets are investigated. Section 2 deals with the geometric properties of a projecting configuration (Γ, Σ) for an MSLS in $\mathrm{PG}(1, q^5)$.

For any point $X \in \mathrm{PG}(4, q^5)$, define $X_i = X^{\sigma^i}$, $i \in \mathbb{N}$. If $\wp_\Gamma(\Sigma) \subseteq \mathrm{PG}(1, q^5)$ is an MSLS not of pseudoregulus type, and σ is a generator of $\mathbb{G}$, then

(*i*) the intersection points of Γ with Γ^{σ^4}, Γ^σ, Γ^{σ^3}, Γ^{σ^2} are of type A, A_1, B, B_2 (cf. Lemma 2.1), respectively;
(*ii*) no three of them are collinear (Proposition 2.9);
(*iii*) $\mathrm{rk}\, A \geqslant 4$, $\mathrm{rk}\, B \geqslant 4$ (Proposition 2.7);
(*iv*) $\wp_\Gamma(\Sigma)$ is of LP type iff $\langle A_2, A_4 \rangle \cap \Gamma \neq \varnothing$ and $\mathrm{rk}\, A = 5$, or $\langle B_3, B_4 \rangle \cap \Gamma \neq \varnothing$ and $\mathrm{rk}\, B = 5$ (Theorems 1.6 and 2.3). If $\mathrm{rk}\, A = \mathrm{rk}\, B = 4$, then we have a scattered LS of a new type, i.e. not projectively equivalent to the known ones.

In Section 3 algebraic relations between representative vectors for the points A_i and B_j are proved. Taking in $\mathrm{PG}(4, q^5)$ coordinates such that some A_is and B_js are base points, we derive algebraic conditions characterizing the MSLSs of LP type.

The investigation is strongly influenced by the rank of the points A and B. Thus, in Section 4 the case of $\max\{\mathrm{rk}\, A, \mathrm{rk}\, B\} = 5$ is studied.

This work builds on the results in [22] concerning maximum scattered linear sets of $\mathrm{PG}(1, q^5)$ that are not of pseudoregulus type. In that paper, a necessary and sufficient condition was proven for these sets to be of type LP under the assumption that the point A, as described in Section 2, has rank 5. As in [22], an algebraic curve $\mathcal{V}$ of degree at most four is associated to $\wp_\Gamma(\Sigma)$. Up to some special cases, this $\mathcal{V}$ turns out to be an absolutely irreducible curve of genus at most three. In the general case, the Hasse-Weil Theorem guarantees the existence of a point with coordinates rational over $\mathbb{F}_q$, while the remaining cases have been analyzed with a case-by-case analysis. It follows that for $\mathrm{rk}\, A = 5$ or $\mathrm{rk}\, B = 5$ any such MSLS is of LP type.

The case $\mathrm{rk}\, A = \mathrm{rk}\, B = 4$, for which $\wp_\Gamma(\Sigma)$ would be of a new type, remains open and is partially treated in Section 6. In Subsection 6.1 it is proved that this is equivalent to the existence of a nucleus of a special scattered linear set in $\mathrm{PG}(2, q^5)$. In Subsection 6.2 the possible new sets are described algebraically (Theorem 6.3), Proposition 6.4). This allowed to run a `GAP` script showing that there is no new scattered linear set up to $q = 25$.

The results of this work are summarized in Theorem 7.1.

2. Some geometric properties

A point $P \in \mathrm{PG}(d,q^n)$ of homogeneous coordinates $a_0, a_1, \ldots, a_d$ (with respect to some given frame) will be denoted by $P = [a_0, a_1, \ldots, a_d]$. In this section we suppose that

- $\Sigma \cong \mathrm{PG}(4,q)$ is a canonical $\mathbb{F}_q$-subgeometry in $\mathrm{PG}(4,q^5)$. When necessary coordinates will be chosen such that
$$\Sigma = \{[t, t^q, t^{q^2}, t^{q^3}, t^{q^4}] \colon t \in \mathbb{F}_{q^5}^*\}.$$
- Γ is a plane in $\mathrm{PG}(4,q^5)$ such that the projection $\mathbb{L} = \wp_\Gamma(\Sigma)$ is a maximum scattered linear set of $\mathrm{PG}(1,q^5)$.
- $1 \neq \sigma \in \mathrm{P\Gamma L}(5,q^5)$ belongs to the subgroup of order five $\mathbb{G}$ that fixes Σ pointwise.
- The linear set $\mathbb{L} = \wp_\Gamma(\Sigma)$ is not of pseudoregulus type.

Lemma 2.1. *There exist two points $A, B \in \Gamma$ such that $A^\sigma, B^{\sigma^2} \in \Gamma$ and these are unique.*

Proof. Consider the subspaces $\Gamma \cap \Gamma^{\sigma^4}$ and $\Gamma \cap \Gamma^{\sigma^3}$. By Theorem 1.3, these are necessarily points, say A and B, otherwise $\mathbb{L}$ is a linear set of pseudoregulus type. Now, suppose that X is a point in Γ such that $X^\sigma \in \Gamma$, then $X \in \Gamma \cap \Gamma^{\sigma^4}$, so $X = A$. Replacing σ by σ^2, the same argument shows the uniqueness of point B. $\square$

Hereafter, for the remainder of the article, the points $A, B \in \Gamma$ are defined by
$$A = \Gamma \cap \Gamma^{\sigma^4} \quad \text{and} \quad B = \Gamma \cap \Gamma^{\sigma^3}.$$
Moreover, the notation $X_i = X^{\sigma^i}$, $i = 0, 1, \ldots, 4$, will be adopted for any point X in $\mathrm{PG}(4,q^5)$, and similarly $\Gamma_i = \Gamma^{\sigma^i}$, $i = 0, 1, \ldots, 4$. Note that, for any $i, j \in \{0,1,2,3,4\}$ with $i \neq j$, $\Gamma_i \cap \Gamma_j = A_h$ or $\Gamma_i \cap \Gamma_j = B_h$ for some $h \in \{0,1,2,3,4\}$.

Proposition 2.2. *The ten points A_i, B_i for $i = 0,1,2,3,4$ are distinct.*

Proof. First of all, we show that $A_i \neq A_j$ and $B_i \neq B_j$ for $i, j \in \{0,1,2,3,4\}$ distinct. Indeed, if $A_i = A_j$ with $0 \leqslant i < j \leqslant 4$, then $A_i^{\sigma^{j-i}} = A_i$. So A_i, and hence A belong to Σ, since they are fixed by $\mathbb{G}$, which leads to a contradiction. The same argument holds for the points of type B_i. Next, it is enough to prove that A is distinct from B_i for $i = 0,1,2,3,4$.

1. if $A = B$, then $\Gamma \cap \Gamma_4 = \Gamma \cap \Gamma_3$; so, $A = \Gamma_3 \cap \Gamma_4 = (\Gamma \cap \Gamma_4)^{\sigma^4} = A_4$, a contradiction.
2. if $A = B_1$ then $\Gamma \cap \Gamma_4 = \Gamma_1 \cap \Gamma_4$; so, $A = \Gamma \cap \Gamma_1 = (\Gamma \cap \Gamma_4)^\sigma = A_1$, a contradiction.
3. if $A = B_2$ then $\Gamma \cap \Gamma_4 = \Gamma \cap \Gamma_2$; so, $B_2 = \Gamma_2 \cap \Gamma_4 = (\Gamma \cap \Gamma_3)^{\sigma^4} = B_4$, a contradiction.
4. if $A = B_3$ then B, B_2 and B_3 belong to Γ, and $B, B_2 \in \Gamma_2$. Then, $\langle B, B_2 \rangle$, which is a line as proved before, is contained in $\Gamma \cap \Gamma_2$. By Theorem 1.3, this is a contradiction.
5. if $A = B_4$ then $\Gamma \cap \Gamma_4 = \Gamma_2 \cap \Gamma_4$: so, $B_4 = \Gamma \cap \Gamma_2 = (\Gamma \cap \Gamma_3)^{\sigma^2} = B_2$, a contradiction.

Then it follows that the ten points A_i and B_i, $i = 0,1,2,3,4$, are distinct. $\square$

Theorem 2.3. *The maximum scattered linear set $\mathbb{L} = \wp_\Gamma(\Sigma)$ is of LP type if and only if $\langle A_2, A_4 \rangle \cap \Gamma \neq \varnothing$, or $\langle B_3, B_4 \rangle \cap \Gamma \neq \varnothing$.*

Proof. The result is a direct consequence of Theorem 1.6. Indeed, the linear set $\mathbb{L}$ is of LP type if and only if there exists a generator τ of the group $\mathbb{G}$ and a unique point, say Z, such that $Z, Z^\tau \in \Gamma$ and

$$\langle Z^{\tau^2}, Z^{\tau^4}\rangle \cap \Gamma \neq \varnothing. \tag{3}$$

Now, if $\tau \in \{\sigma, \sigma^4\}$ then $Z \in \{A, A_1\}$ and $\langle Z^{\tau^2}, Z^{\tau^4}\rangle = \langle A_2, A_4\rangle$; if $\tau \in \{\sigma^2, \sigma^3\}$ then $Z \in \{B, B_2\}$ and $\langle Z^{\tau^2}, Z^{\tau^4}\rangle = \langle B_3, B_4\rangle$. ☐

Definition 2.4. *If* $\langle A_2, A_4\rangle \cap \Gamma \neq \varnothing$, *then* (Γ, Σ) *has* configuration I. *If* $\langle B_3, B_4\rangle \cap \Gamma \neq \varnothing$, *then* (Γ, Σ) *has* configuration II.

Proposition 2.5. *The points* A, A_1, B, B_2 *are not collinear.*

Proof. Suppose that there exists a line ℓ containing the points A, A_1, B, B_2. First of all, $\ell^\sigma \neq \ell$ since otherwise ℓ would meet Σ in $q+1$ points. Since $A_1 \in \ell \cap \ell^\sigma$, the span $\pi = \langle \ell \cup \ell^\sigma\rangle$ is a plane. The points $A, A_1, A_2, B, B_1, B_2, B_3$ lie on π; so, $\ell^{\sigma^2} = \langle A_2, B_2\rangle \subseteq \pi$. It follows $\pi^\sigma = \langle \ell^\sigma \cup \ell^{\sigma^2}\rangle = \pi$. Then $\pi \cap \Sigma$ is plane of Σ. Hence, π contains a line m such that $|m \cap \Sigma| = q+1$. Therefore, m meets $\ell \subseteq \Gamma$, which, by Theorem 1.2 (*iii*), yields a contradiction. ☐

As consequence of proposition above, we have $\Gamma = \langle A, A_1, B, B_2\rangle$.

Lemma 2.6. *If* $B \in \langle A, A_1\rangle$, *then*

i) $\operatorname{rk} A = 5$,
ii) $\mathbb{L}$ *is not of LP type.*

Proof. If $\operatorname{rk} A < 5$, then there is a solid (i.e. a subspace of projective dimension three) T such that $A_i, B_i \in T$ for all $i = 0,1,2,3,4$, and T also contains all planes Γ_i, a contradiction.

Now, since $\operatorname{rk} A = 5$, $S = \langle A, A_1, A_2, A_3\rangle$ is a solid. Since $B \in \langle A, A_1\rangle$, $B_2 \in \langle A_2, A_3\rangle$ and by Proposition 2.5, the plane Γ is contained in S. Since $\langle A_2, A_4\rangle \cap S = A_2$, and $A_2 \notin \Gamma$ (cf. Lemma 2.1), $\langle A_2, A_4\rangle \cap \Gamma = \varnothing$.

By Theorem 2.3, it remains to show that $\langle B_3, B_4\rangle \cap \Gamma = \varnothing$. For any $i = 1,2,3,4$, the intersection $\Gamma_i \cap S$ is a line. By hypothesis $B \in \langle A, A_1\rangle$, then the points A_3, A_4, B_3 are collinear. Taking into account $A_4 \notin S$, $A_3 \in S$, we have that B_3 is not in S. In the same way, since $A \in S$, $A_4 \notin S$ and A_4, A, B_4 are collinear, B_4 is not in S. Consider now a plane π containing A, A_3, A_4, B_3, B_4. Such π meets S in the line $\langle A, A_3\rangle$ and $\pi \cap \Gamma = A$. The line $\langle B_3, B_4\rangle \nsubseteq \Gamma_4$ is contained in π, is distinct from $\langle A, B_4\rangle \subseteq \Gamma_4$, hence $\langle B_3, B_4\rangle \cap \Gamma = \varnothing$. ☐

Proposition 2.7. *Any four points among* A, A_1, A_2, A_3, A_4 *are independent. Any four points among* B, B_1, B_2, B_3, B_4 *are independent.*

Proof. Since $A_2 \notin \langle A, A_1\rangle$, the points A, A_1, A_2 are independent. If $A_3 \in \langle A, A_1, A_2\rangle$, then the lines $\langle A, A_1\rangle \subseteq \Gamma$ and $\langle A_2, A_3\rangle \subseteq \Gamma_2$ have a point in common which must be B_2, the intersection point of $\Gamma \cap \Gamma_2$. Let $S = \langle A, A_1, A_2, A_3, A_4\rangle$. The dimension of S is at most three. From $B_2 \in S$ and $S^\sigma = S$, it follows that S contains all ten points of type A_i and B_i. Therefore, by Proposition 2.5, S contains all planes Γ_i, $i = 0,1,2,3,4$, a contradiction. This implies that A, A_1, A_2, A_3 are independent and the first assertion follows.

Similarly, B, B_2 and B_4 are independent. Assume $B_1 \in \langle B, B_2, B_4\rangle$. Then the lines $\langle B, B_2\rangle \subseteq \Gamma$ and $\langle B_4, B_1\rangle \subseteq \Gamma_4$ meet in the point A. This leads to a contradiction similar to the previous one. ☐

Throughout this paper, we denote the *trace* and the *norm* functions of $\mathbb{F}_{q^5}$ over $\mathbb{F}_q$ by

$$\mathrm{Tr}_{q^5/q}(x) = x + x^q + x^{q^2} + x^{q^3} + x^{q^4} \text{ and } \mathrm{N}_{q^5/q}(x) = x^{1+q+q^2+q^3+q^4},$$

respectively. For more details on their properties, see [15, Chapter 2].

In [29, Proposition 3.8], the third author showed that $f_b(x) = x^q + bx^{q^2} \in \mathbb{F}_{q^5}[x]$ is non-scattered for any $b \in \mathbb{F}^*_{q^5}$. The following result allows us to complete the analysis of scatteredness of binomials in $\mathbb{F}_{q^5}[X]$.

Proposition 2.8. *For any* $b \in \mathbb{F}^*_{q^5}$*, the linearized polynomial* $g_b(x) = x^{q^2} + bx^{q^4}$ *is not scattered.*

Proof. By Propositions 3.8 and 3.10 in [29], since the polynomial $x^q - b^{-1}x^{q^2} \in \mathbb{F}_{q^5}[x]$ is not scattered for any $b \in \mathbb{F}^*_{q^5}$, the algebraic curve

$$\chi'_b : -bX^{q-1} + Y^{q-1} + 1 = 0 \tag{4}$$

has a point (α_0, β_0) in $\mathrm{AG}(2,q^5)$ with coordinates in $\mathbb{F}^*_{q^5}$ such that $\mathrm{Tr}_{q^5/q}(\beta_0) = 0$. Then by (4),

$$\alpha_0^{q-1} = b^{-1}(\beta_0^{q-1} + 1). \tag{5}$$

Since $\mathrm{Tr}_{q^5/q}(\beta_0) = 0$, there exists $z_0 \in \mathbb{F}_{q^5} \setminus \mathbb{F}_q$ such that $\beta_0 = z_0^q - z_0$. Moreover, since for any $x \in \mathbb{F}_{q^5}$, $\mathrm{N}_{q^5/q}(x)^q = \mathrm{N}_{q^5/q}(x)$,

$$\mathrm{N}_{q^5/q}\left(-b^{-1}\frac{z_0^{q^2} - z_0}{z_0^{q^4} - z_0}\right) = \mathrm{N}_{q^5/q}\left(b^{-1}\frac{z_0^{q^2} - z_0}{z_0^{q} - z_0}\right) = \mathrm{N}_{q^5/q}(b^{-1}(\beta_0^{q-1} + 1))$$

and by (5), we get

$$\mathrm{N}_{q^5/q}\left(-b\frac{z_0^{q^4} - z_0}{z_0^{q^2} - z_0}\right) = 1.$$

Hence, there exists $x_0 \in \mathbb{F}^*_{q^5}$ such that $x_0^{q-1} = -b\frac{z_0^{q^4} - z_0}{z_0^{q^2} - z_0}$. Now, putting $y_0 = z_0^{q^2} - z_0$ we have that

$$x_0^{q-1} = -b\frac{z_0^{q^4} - z_0^{q^2} + z_0^{q^2} - z_0}{z_0^{q^2} - z_0} = -b(y_0^{q^2-1} + 1).$$

Then, (x_0, y_0) is a point of the algebraic curve

$$\chi_b : b^{-1}X^{q-1} + Y^{q^2-1} + 1 = 0$$

in $\mathrm{AG}(2,q^5)$ with coordinates in $\mathbb{F}^*_{q^5}$ and $\mathrm{Tr}_{q^5/q}(y_0) = 0$. By [29, Proposition 3.10], the polynomial $g_b(x)$ is not scattered. ☐

Proposition 2.9. *The quadruple* $\mathcal{F} = (A, A_1, B, B_2)$ *is a frame for* Γ*, i.e., no three points of* $\mathcal{F}$ *are collinear.*

Proof. Let $\bar\sigma \in \mathbb{G}$ be the map

$$\bar\sigma : [x_0, x_1, x_2, x_3, x_4] \longmapsto [x_4^q, x_0^q, x_1^q, x_2^q, x_3^q].$$

Define $X_{(j)} = X^{\bar\sigma^j}$ for $j = 0,1,2,3,4$. Since $\bar\sigma$ is a generator of $\mathbb{G}$, we have $\sigma = \bar\sigma^i$ for some $i \in \{1,\dots,4\}$. Hence, the set of points $\{A, A_1, B, B_2\}$ is equal to $\{\bar A, \bar A_{(1)}, \bar B, \bar B_{(2)}\}$, with $\bar A = \Gamma \cap \Gamma^{\bar\sigma^4}$ and $\bar B = \Gamma \cap \Gamma^{\bar\sigma^3}$. By way of contradiction, suppose that $(\bar A, \bar A_{(1)}, \bar B, \bar B_{(2)})$ is not a frame for Γ. Then it is enough to analyze four cases:

- *Case 1.* $\bar{B} \in \langle \bar{A}, \bar{A}_{(1)} \rangle$. Note that a similar result to Lemma 2.6 also holds for the points $\bar{A}$ and $\bar{B}$ defined by means of $\bar{\sigma}$; indeed, it is enough to follow the same steps of the proof, substituting σ by $\bar{\sigma}$. Therefore, $\operatorname{rk} \bar{A} = 5$. Since the setwise stabilizer of Σ in $\mathrm{PGL}(5, q^5)$ acts transitively on the points of $\mathrm{PG}(4, q^5)$ of rank 5, [3, Proposition 3.1], it may be assumed that $\bar{A}$ and $\bar{A}_{(1)}$ are $[0,1,0,0,0]$ and $[0,0,1,0,0]$, respectively. Moreover, since $\bar{B} \in \langle \bar{A}, \bar{A}_{(1)} \rangle$ and it is distinct from $\bar{A}$ and $\bar{A}_{(1)}$, w.l.o.g. we may assume that $\bar{B} = [0, 1, -a^{q^3}, 0, 0]$ with $a \in \mathbb{F}^*_{q^5}$. Then
$$\mathbb{L} = \wp_\Gamma(\Sigma) = \{\langle (u, u^{q^4} + a u^{q^3}) \rangle_{\mathbb{F}_{q^5}} : u \in \mathbb{F}^*_{q^5}\}.$$
Next, $\mathbb{L} = L_{\hat{f}}$, where $\hat{f}(u) = u^q + a^{q^2} u^{q^2}$ is the adjoint polynomial of $f(u) = u^{q^4} + a u^{q^3}$, [4, Lemma 3.1]. So,
$$\mathbb{L} = \{\langle (u, u^q + a^{q^2} u^{q^2}) \rangle_{\mathbb{F}_{q^5}} : u \in \mathbb{F}^*_{q^5}\},$$
obtaining a contradiction by [29, Proposition 3.8].
- *Case 2.* $\bar{B} \in \langle \bar{A}, \bar{B}_{(2)} \rangle$. Then $\bar{A} \in \langle \bar{B}, \bar{B}_{(2)} \rangle$, and $\operatorname{rk} \bar{B} = 5$, for otherwise the ten points of type $\bar{A}_{(j)}$ and $\bar{B}_{(j)}$ would belong to a common solid in $\mathrm{PG}(4, q^5)$. As before we may choose $\bar{B}_{(2)} = [0,0,0,1,0]$ and $\bar{B} = [0,1,0,0,0]$, respectively, and $\bar{A} = [0, -a^{q^4}, 0, 1, 0]$ with $a \in \mathbb{F}^*_{q^5}$. Then, we get
$$\mathbb{L} = \{\langle (u, u^{q^2} + a u^{q^4}) \rangle_{\mathbb{F}_{q^5}} : u \in \mathbb{F}^*_{q^5}\},$$
and by Proposition 2.8, $\mathbb{L}$ is not scattered, a contradiction.

A similar argument can be applied to the cases $\bar{B} \in \langle \bar{A}_{(1)}, \bar{B}_{(2)} \rangle$ and $\bar{B}_{(2)} \in \langle \bar{A}, \bar{A}_{(1)} \rangle$ carrying out to a contradiction. Then $\mathcal{F}$ is a frame of Γ. ☐

Theorem 2.10. *Let $E = \langle A, B \rangle \cap \langle A_1, B_2 \rangle$ and $F = \langle A, B_2 \rangle \cap \langle A_1, B \rangle$. The linear set $\mathbb{L}$ is of LP type if and only if $E \in \langle A_2, A_4 \rangle$, or $F \in \langle B_3, B_4 \rangle$.*

Proof. The sufficiency of the condition follows directly from Theorem 2.3.

Now assume that $\mathbb{L}$ is of LP type. Then at least one among $X = \Gamma \cap \langle A_2, A_4 \rangle$ and $Y = \Gamma \cap \langle B_3, B_4 \rangle$ is a point.

Suppose that X is a point, and $X \neq E$; that is, $\Gamma = \langle A, B, X \rangle$, or $\Gamma = \langle A_1, B_2, X \rangle$. Define the solid $S_1 = \langle \Gamma, A_2, A_4 \rangle$. If $\Gamma = \langle A, B, X \rangle$, then, since $X^{\sigma^2} \in \langle A_4, A_1 \rangle \subseteq S_1$, also $\Gamma_2 = \langle A_2, B_2, X^{\sigma^2} \rangle$ is contained in S_1, hence $\Gamma \cap \Gamma_2$ is a line: a contradiction. Similarly, if $\Gamma = \langle A_1, B_2, X \rangle$, then S_1 contains A_4, B, and $X^{\sigma^3} \in \langle A, A_2 \rangle$, hence $\Gamma_3 \subseteq S_1$, a contradiction again. We reach analogous contradictions assuming $Y \neq F$. ☐

Any maximum scattered linear set in $\mathrm{PG}(1, q^5)$ is, up to projectivities, of type L_{f_0} or L_{f_1}, with $f_0(x) = x^q + a_2 x^{q^2} + a_3 x^{q^3} + a_4 x^{q^4}$ and $f_1(x) = x^{q^2} + a_3 x^{q^3}$, see [22, Proposition 2.1]. Indeed, if $f(x) = \sum_{i=0}^4 a_i x^{q^i}$ and $\hat{f}(x) = \sum_{i=0}^4 a_i^{q^{5-i}} x^{q^{5-i}}$, then $L_f = L_{\hat{f}}$ [4, Lemma 3.1]. The scattered polynomials of type $f_1(x)$ are of pseudoregulus (when $a_3 = 0$) or LP (when $a_3 \neq 0$) type. On the other hand, if $f(x) \in \mathbb{F}_{q^5}[x]$ is an $\mathbb{F}_q$-linearized polynomial and $\Sigma = \{[t, t^q, t^{q^2}, t^{q^3}, t^{q^4}] : t \in \mathbb{F}^*_{q^5}\}$, then we can explicitly show a vertex Γ_f such that $\wp_{\Gamma_f}(\Sigma) \cong L_f$. In particular,
$$\Gamma_{f_0} = \langle [0, -a_4, 0, 0, 1], [0, -a_3, 0, 1, 0], [0, -a_2, 1, 0, 0] \rangle.$$
For $q = 3, 4, 5$ all planes Γ_{f_0} have been analyzed by a `GAP` script, and when they give rise to MSLSs they always have either the pseudoregulus configuration (that is, they satisfy the hypotheses of Theorem 1.3), or configuration I, or configuration II. The scripts for the cases $q = 3, 4$ and $q = 5$ are available at

https://pastebin.com/GaV9PTYp and https://pastebin.com/TMA5y0Ez, respectively. This implies that for $q \leqslant 5$, any maximum scattered linear set in $\mathrm{PG}(1,q^5)$ is either of pseudoregulus or of LP type. This result is contained in our main result, Theorem 7.1, which will be proved with different arguments.

3. CHARACTERIZATION BY EQUATIONS

In this section, we maintain the notation from the previous one. We will denote by $\mathbb{L}$ the projection $\wp_\Gamma(\Sigma)$ with the plane Γ as vertex. Recall that $\mathbb{L}$ is a maximum scattered linear set not of pseudoregulus type. Moreover, with a slight abuse of notation, by σ we will denote both

i) a generator of the subgroup $\mathbb{G}$ of $\mathrm{P\Gamma L}(5,q^5)$ fixing Σ pointwise;

ii) its underlying semilinear map with companion automorphism $x \mapsto x^{q^s}$, $1 \leqslant s \leqslant 4$.

Note that if $P \in \Sigma$, then there exists a vector $v \in \mathbb{F}_{q^5}^5$ such that $P = \langle v \rangle_{\mathbb{F}_{q^5}}$ and $v^\sigma = v$.

PROPOSITION 3.1. *There exist* $u, v \in \mathbb{F}_{q^5}^5$ *and* $\lambda, \mu \in \mathbb{F}_{q^5}^*$ *such that* $A = \langle u \rangle_{\mathbb{F}_{q^5}}$, $B = \langle v \rangle_{\mathbb{F}_{q^5}}$ *and*

$$v = u - \lambda u^\sigma + \mu v^{\sigma^2}. \tag{6}$$

If $\mathrm{N}_{q^5/q}(\mu) = 1$ *or* $\mathrm{N}_{q^5/q}(\lambda) = 1$*, it is possible to choose* u *and* v *such that* $\mu = 1$ *or* $\lambda = 1$*, respectively.*

Proof. Let $(u, v_0) \in \mathbb{F}_{q^5}^5 \times \mathbb{F}_{q^5}^5$ such that $\langle u \rangle_{\mathbb{F}_{q^5}} = A$ and $\langle v_0 \rangle_{\mathbb{F}_{q^5}} = B$. By Proposition 2.9, there exist $\ell, m, n \in \mathbb{F}_{q^5}^*$ such that

$$v_0 = \ell u + m u^\sigma + n v_0^{\sigma^2}.$$

Putting $v = \ell^{-1} v_0$, we have $v^{\sigma^2} = \ell^{-q^{2s}} v_0^{\sigma^2}$ and

$$v = u + m\ell^{-1} u^\sigma + n \ell^{q^{2s}-1} v^{\sigma^2}$$

So the first part of the statement follows. Moreover, by Hilbert's Theorem 90, if $\mathrm{N}_{q^5/q}(\mu) = 1$ then $\mu = \rho^{\sigma^2 - 1}$ for some $\rho \in \mathbb{F}_{q^5}^*$, then putting $u' = \rho u$ and $v' = \rho v$, by (6), we obtain

$$v' = u' - \lambda' u'^\sigma + v'^{\sigma^2}, \qquad \lambda' \in \mathbb{F}_{q^5}^*.$$

A similar argument can be applied in case $\mathrm{N}_{q^5/q}(\lambda) = 1$. ☐

Hereafter, u, v, λ, μ are as specified by (6), and $u_i = u^{\sigma^i}$ and $v_i = v^{\sigma^i}$ for $i \in \{0,1,2,3,4\}$.

PROPOSITION 3.2. *The following identities hold:*

$$(1 - \mathrm{N}_{q^5/q}(\mu))v = \sum_{i=0}^{4} a_i u_i \tag{7}$$

$$(1 - \mathrm{N}_{q^5/q}(\lambda))u = \sum_{i=0}^{4} b_i v_i \tag{8}$$

where

$$\begin{aligned} a_0 &= 1 - \lambda^{q^{4s}} \mu^{q^{2s}+1} \\ a_1 &= \mu^{q^{4s}+q^{2s}+1} - \lambda \\ a_2 &= \mu(1 - \lambda^{q^s} \mu^{q^{4s}+q^{2s}}) \end{aligned} \qquad \begin{aligned} a_3 &= \mu(\mu^{q^{4s}+q^{2s}+q^s} - \lambda^{q^{2s}}) \\ a_4 &= \mu^{q^{2s}+1}(1 - \lambda^{q^{3s}} \mu^{q^{4s}+q^s}) \end{aligned}$$

$$b_0 = 1 - \lambda^{q^{2s}+q^s+1}\mu^{q^{3s}} \qquad b_3 = \lambda(\lambda^{q^{2s}+q^s} - \mu^{q^s})$$
$$b_1 = \lambda(1 - \lambda^{q^{3s}+q^{2s}+q^s}\mu^{q^{4s}}) \qquad b_4 = \lambda^{q^s+1}(\lambda^{q^{3s}+q^{2s}} - \mu^{q^{2s}})$$
$$b_2 = \lambda^{q^s+1} - \mu$$

Proof. By composing

$$v_i = u_i - \lambda^{q^{is}}u_{i+1} + \mu^{q^{is}}v_{i+2}, \qquad i = 0,2,4,1,3, \tag{9}$$

(indices taken mod 5) a relation containing v and u_i, $i = 0,1,2,3,4$ arises, equivalent to (7). Similarly, to obtain (8) compose $u_i = v_i - \mu^{q^{is}}v_{i+2} + \lambda^{q^{is}}u_{i+1}$ for $i = 0,1,2,3,4$. ☐

Let take into account the basis $\mathcal{B} = \{u, u_1, u_2, u_3, v_4\}$ of $\mathbb{F}_{q^5}^5$. From now on, if a vector $x \in \mathbb{F}_{q^5}^5$ has coordinates $(a_0, a_1, a_2, a_3, a_4)$ with respect to $\mathcal{B}$, we will denote it by $x \equiv (a_0, a_1, a_2, a_3, a_4)_{\mathcal{B}}$. Then,

$$u_4 \equiv (a, b, c, d, e)_{\mathcal{B}}.$$

for some $a, b, c, d, e \in \mathbb{F}_{q^5}$.

Proposition 3.3. *The following equation holds:*

$$(1 - \mu^{q^{4s}+q^s}\lambda^{q^{3s}})e = 1 - \mathrm{N}_{q^5/q}(\mu). \tag{10}$$

Proof. By repeated usage of (9), in coordinates with respect to the base $\mathcal{B}$, we have

$$v_2 = u_2 - \lambda^{q^{2s}}u_3 + \mu^{q^{2s}}v_4 \equiv (0, 0, 1, -\lambda^{q^{2s}}, \mu^{q^{2s}})_{\mathcal{B}} \tag{11}$$

$$v = u - \lambda u_1 + \mu v_2 \equiv (1, -\lambda, \mu, -\lambda^{q^{2s}}\mu, \mu^{q^{2s}+1})_{\mathcal{B}}, \tag{12}$$

and

$$u_4 = v_4 + \lambda^{q^{4s}}u - \mu^{q^{4s}}\left(u_1 - \lambda^{q^s}u_2 + \mu^{q^s}\left(u_3 - \lambda^{q^{3s}}u_4 + \mu^{q^{3s}}(u - \lambda u_1 + \mu v_2)\right)\right). \tag{13}$$

The assertion follows by taking into account the fifth coordinates in both expressions in (13). ☐

Proposition 3.4. *The linear set* $\mathbb{L}$ *is of LP type if and only if*

$$(\lambda^{q^{2s}}e + \mu^{q^{2s}}d)(1 - \lambda^{q^{3s}}d - \lambda^{q^{3s}+q^{2s}}c) = 0 \tag{14}$$

Proof. By Theorem 2.3, the linear set $\mathbb{L}$ is of LP type if and only if u, u_1, v_2, u_2, u_4 or u, u_1, v_2, v_3, v_4 are linearly dependent. From (12),

$$\begin{aligned} &v_3 = u_3 - \lambda^{q^{3s}}u_4 + \mu^{q^{3s}}v \equiv \\ &(-\lambda^{q^{3s}}a + \mu^{q^{3s}}, -\lambda^{q^{3s}}b - \lambda\mu^{q^{3s}}, -\lambda^{q^{3s}}c + \mu^{q^{3s}+1}, 1 - \lambda^{q^{3s}}d - \lambda^{q^{2s}}\mu^{q^{3s}+1}, \\ &-\lambda^{q^{3s}}e + \mu^{q^{3s}+q^{2s}+1})_{\mathcal{B}}. \end{aligned} \tag{15}$$

So, $\mathbb{L}$ is of LP type if and only if one of the following determinants is zero:

$$\begin{vmatrix} -\lambda^{q^{2s}} & \mu^{q^{2s}} \\ d & e \end{vmatrix}, \qquad \begin{vmatrix} 1 & -\lambda^{q^{2s}} \\ -\lambda^{q^{3s}}c + \mu^{q^{3s}+1} & 1 - \lambda^{q^{3s}}d - \lambda^{q^{2s}}\mu^{q^{3s}+1} \end{vmatrix}. \tag{16}$$

☐

Remark 3.5. Consider

$$v_3 = u_3 - \lambda^{q^{3s}}u_4 + \mu^{q^{3s}}v \tag{17}$$

and

$$v_3 = \mu^{-q^s}(v_1 - u_1 + \lambda^{q^s}u_2) = \mu^{-q^{4s}-q^s}(v_4 - u_4 + \lambda^{q^{4s}}u) - \mu^{-q^s}u_1 + \lambda^{q^s}\mu^{-q^s}u_2. \tag{18}$$

By difference we get that the vector

$$(\lambda^{q^{3s}}-\mu^{-q^{4s}-q^s})u_4$$

has coordinates in $\mathcal{B}$

$$\begin{aligned}(\mu^{q^{3s}}-\lambda^{q^{4s}}\mu^{-q^{4s}-q^s},-\lambda\mu^{q^{3s}}+\mu^{-q^s},\mu^{q^{3s}+1}-\lambda^{q^s}\mu^{-q^s},1-\lambda^{q^{2s}}\mu^{q^{3s}+1},\\ \mu^{q^{3s}+q^{2s}+1}-\mu^{-q^{4s}-q^s})_{\mathcal{B}}.\end{aligned} \tag{19}$$

In case $\lambda\mu^{q^{3s}+q^s}-1\neq 0$, this allows to obtain the coordinates of u_4 in terms of λ and μ.

Next, we find relations between the parameters a, b, c, d, e, λ and μ based on the fact that the semilinear map σ^i, $i=0,1,2,3,4$, acts on the coordinates as $X\mapsto M_iX^{q^{is}}$. It holds

$$M_3=\begin{pmatrix}0 & a & 1 & 0 & 0\\ 0 & b & 0 & 1 & 0\\ 0 & c & 0 & 0 & 1\\ 1 & d & 0 & 0 & -\lambda^{q^{2s}}\\ 0 & e & 0 & 0 & \mu^{q^{2s}}\end{pmatrix}, \tag{20}$$

$$M_2=\begin{pmatrix}0 & 0 & a & 1 & \mu^{-q^{4s}}(\lambda^{q^{4s}}-a)\\ 0 & 0 & b & 0 & -\mu^{-q^{4s}}b\\ 1 & 0 & c & 0 & -\mu^{-q^{4}}c\\ 0 & 1 & d & 0 & -\mu^{-q^{4s}}d\\ 0 & 0 & e & 0 & \mu^{-q^{4s}}(1-e)\end{pmatrix}. \tag{21}$$

By equating the second columns of $M_2M_3^{q^{2s}}$ and the identity matrix one obtains the following equations, where $r=c^{q^{2s}}-\mu^{-q^{4s}}e^{q^{2s}}$:

$$\begin{cases} ar+d^{q^{2s}}+\lambda^{q^{4s}}\mu^{-q^{4s}}e^{q^{2s}} & =0\\ br & =1\\ cr+a^{q^{2s}} & =0\\ dr+b^{q^{2s}} & =0\\ er+\mu^{-q^{4s}}e^{q^{2s}} & =0.\end{cases} \tag{22}$$

If $\operatorname{rk} A=5$, or, equivalently, $e\neq 0$, the system above is equivalent to

$$\begin{cases} a=\mu^{-q^{2s}-1}e^{1-q^s}(e^{q^s}-1)\\ b=-\mu^{q^{4s}}e^{1-q^{2s}}\\ c=\mu^{-q^{2s}}e^{1-q^{3s}}(e^{q^{3s}}-1)\\ d=-\mu^{q^{4s}+q^s}e^{1-q^{4s}}.\end{cases} \tag{23}$$

4. CLASSIFICATION IN THE CASE $\max\{\operatorname{rk} A,\operatorname{rk} B\}=5$

Let us define the following linear set:

$$L_{\alpha,\beta,s}=\{\langle(x-\alpha x^{q^{2s}},x^{q^s}-\beta x^{q^{2s}})\rangle_{\mathbb{F}_{q^5}}:x\in\mathbb{F}_{q^5}^*\}\ (\alpha,\beta\in\mathbb{F}_{q^5},\ s\in\{1,2,3,4\}).$$

The result stated in [22, Proposition 2.5] holds more in general for any $1\leqslant s\leqslant 4$. More precisely,

PROPOSITION 4.1.

(*i*) *The linear set* $L_{\alpha,\beta,s}$ *has rank less than five if and only if*

$$\alpha^{q^s}=\beta^{q^s+1}\ \text{and}\ \mathrm{N}_{q^5/q}(\alpha)=\mathrm{N}_{q^5/q}(\beta)=1. \tag{24}$$

(*ii*) *If* $\alpha^{q^s}\neq\beta^{q^s+1}$, *then* $L_{\alpha,\beta,s}$ *is not of pseudoregulus type.*

(iii) *If* $\alpha^{q^s}=\beta^{q^s+1}$ *and* $(\mathrm{N}_{q^5/q}(\alpha),\,\mathrm{N}_{q^5/q}(\beta))\neq(1,1)$*, then* $L_{\alpha,\beta,s}$ *is of pseudoregulus type.*

We point out that the results of [22, Section 4] hold true more generally by replacing α^q with α^{q^s} and β^q with β^{q^s} in their arguments. We state here two of them in this general setting.

Proposition 4.2. *Let* $\alpha,\beta\in\mathbb{F}_{q^5}$ *with* $(\alpha,\beta)\neq(0,0)$*. Then* $L_{\alpha,\beta,s}$ *is maximum scattered if and only if there is no* $z\in\mathbb{F}_{q^5}$ *such that*

$$\begin{cases}\mathrm{N}_{q^5/q}(z)=-1\\ \beta^{q^{3s}+q^s+1}z^{q^s}-\beta^{q^s+1}z^{q^{2s}+q^s+1}(1-\alpha z)^{q^{3s}}+\beta^{q^{3s}}(1-\alpha z)^{q^s+1}=0.\end{cases}$$

Proposition 4.3. *Let* $\alpha,\beta\in\mathbb{F}_{q^5}$ *with* $\beta\neq0$*,* $1\leqslant s\leqslant4$ *and* $\alpha^{q^s}/\beta^{q^s+1}\in\mathbb{F}_{q^5}\setminus\mathbb{F}_q$*. Then* $L_{\alpha,\beta,s}$ *is not maximum scattered.*

Next, we shall describe the linear set $\mathbb{L}$ using the basis of $\mathbb{F}_{q^5}^5$, $\mathcal{B}=\{u,u_1,u_2,u_3,v_4\}$ introduced in Section 3 and we will prove that, under the assumption that at least one between A and B has rank 5, $\mathbb{L}$ is of LP type. Note that for a generator σ of $\mathbb{G}$, the assumption $\max\{\operatorname{rk}A,\operatorname{rk}B\}=5$, where $A=\Gamma\cap\Gamma^{\sigma^4}$ and $B=\Gamma\cap\Gamma^{\sigma^3}$, is equivalent to the existence of $\tau\in\mathbb{G}$, $\tau\neq1$, such that $\operatorname{rk}A'=5$ with $A'=\Gamma\cap\Gamma^{\tau^4}$ and $A'\in\{A,B\}$. For this reason, it is enough to deal with the case $\operatorname{rk}A=5$. So, suppose that point $A=\Gamma\cap\Gamma^{\sigma^4}$ has rank 5. Since $\mathbb{L}$ is scattered, for any two distinct points $P_i=\langle(x_0^{(i)},x_1^{(i)},x_2^{(i)},x_3^{(i)},x_4^{(i)})_{\mathcal{B}}\rangle_{\mathbb{F}_{q^5}}$, $i=1,2$, in Σ, the points A, A_1, B_2, P_1, P_2 are independent; equivalently, by (11),

$$\begin{vmatrix}1 & -\lambda^{q^{2s}} & \mu^{q^{2s}}\\ x_2^{(1)} & x_3^{(1)} & x_4^{(1)}\\ x_2^{(2)} & x_3^{(2)} & x_4^{(2)}\end{vmatrix}\neq0.$$

This implies that the following linear set is scattered:

$$L=\{\langle(\mu^{q^{2s}}x_2-x_4,\mu^{q^{2s}}x_3+\lambda^{q^{2s}}x_4)\rangle_{\mathbb{F}_{q^5}}:\langle(x_0,\dots,x_4)_{\mathcal{B}}\rangle_{\mathbb{F}_{q^5}}\in\Sigma\}. \tag{25}$$

The linear set L is the projection of Σ from Γ to the line $x_0=x_1=x_4=0$. Therefore, $L\cong\mathbb{L}=\wp_\Gamma(\Sigma)$.

By Proposition 3.1, it may be assumed that either $\mu=1$, or $\mathrm{N}_{q^5/q}(\mu)\neq1$ and we have to distinguish two cases, that are described in the following propositions. As in the previous section, let $u_4\equiv(a,b,c,d,e)_{\mathcal{B}}$, and the semilinear map σ^i, $i=0,1,2,3,4$, acts on the coordinates as $X\mapsto M_iX^{q^{is}}$.

Proposition 4.4. *If* $\mu=1$*, then* $\mathbb{L}$ *is projectively equivalent to* $L_{1,1-e,s}$*, and* $e\in\mathbb{F}_q$*.*

Proof. By (10), one has $\lambda=1$ and

$$a=e-e^{1-q^s},\ b=-e^{1-q^{2s}},\ c=e-e^{1-q^{3s}},\ d=-e^{1-q^{4s}}.$$

The condition (14) is equivalent to

$$(e-1)(e-e^{1-q^{3s}}-e^{1-q^{4s}}-1)=0. \tag{26}$$

The equation $M_1X^{q^s}=X$ with $x_4=t$ gives

$$x_2=e^{-q^{3s}}(t^{q^{3s}}-t)+t,\quad x_3=e^{-q^{4s}}(t^{q^{4s}}-t).$$

From (25), by previous multiplication of the components for $e^{q^{3s}}$ and $e^{q^{4s}}$, respectively, and by setting $t=x^{q^{2s}}$,

$$\mathbb{L}\cong L_{1,1-e^{q^s},s}=\{\langle(x-x^{q^{2s}},x^{q^s}-(1-e^{q^s})x^{q^{2s}})\rangle_{\mathbb{F}_{q^5}}:x\in\mathbb{F}_{q^5}^*\}.$$

By Proposition 4.3, $1/(1-e^{q^s})^{q^s+1}\in\mathbb{F}_q$; this implies $1-e^{q^s}\in\mathbb{F}_q$, that is, $e\in\mathbb{F}_q$. In conclusion,

$$\mathbb{L}\cong L_{1,1-e,s}=\{\langle(x-x^{q^{2s}},x^{q^s}-(1-e)x^{q^{2s}})\rangle_{\mathbb{F}_{q^5}}:x\in\mathbb{F}^*_{q^5}\},\quad e\in\mathbb{F}_q. \tag{27}$$

□

Proposition 4.5. *If* $\mathrm{N}_{q^5/q}(\mu)\neq 1$*, then* $\mathbb{L}$ *is projectively equivalent to* $L_{\alpha,\beta,s}$*, where*

$$\alpha=\mu^{-q^{2s}},\quad \beta=\frac{\lambda^{q^{2s}}-\mu^{q^{4s}+q^{2s}+q^s}}{\mu^{q^{2s}}(\lambda^{q^{2s}}\mu^{q^{3s}+1}-1)}. \tag{28}$$

Proof. Since

$$M_1=M_3M_3^{q^{3s}}=\begin{pmatrix}0 & ab^{q^{3s}}+c^{q^{3s}} & 0 & a & 1\\ 1 & b^{q^{3s}+1}+d^{q^{3s}} & 0 & b & -\lambda\\ 0 & b^{q^{3s}}c+e^{q^{3s}} & 0 & c & \mu\\ 0 & a^{q^{3s}}+b^{q^{3s}}d-\lambda^{q^{2s}}e^{q^{3s}} & 1 & d & -\lambda^{q^{2s}}\mu\\ 0 & b^{q^{3s}}e+\mu^{q^{2s}}e^{q^{3s}} & 0 & e & \mu^{q^{2s}+1}\end{pmatrix}, \tag{29}$$

and $M_1u_1^\sigma=u_2$, one has

$$\begin{cases} ab^{q^{3s}}+c^{q^{3s}}=0\\ b^{q^{3s}+1}+d^{q^{3s}}=0\\ b^{q^{3s}}c+e^{q^{3s}}=1\\ a^{q^{3s}}+b^{q^{3s}}d-\lambda^{q^{2s}}e^{q^{3s}}=0\\ b^{q^{3s}}e+\mu^{q^{2s}}e^{q^{3s}}=0.\end{cases} \tag{30}$$

By $e\neq 0$, solving (30),

$$a=\lambda^{q^{4s}}e-\mu^{q^{4s}+q^{3s}+q^s}e^{-q^s+1},\ b=-\mu^{q^{4s}}e^{-q^{2s}+1},$$
$$c=\lambda^{q^s}\mu^{q^{4s}}e-\mu^{q^{4s}+q^{3s}+q^s+1}e^{-q^{3s}+1},\ d=-\mu^{q^{4s}+q^s}e^{-q^{4s}+1},$$

and

$$e=\frac{1-\mathrm{N}_{q^5/q}(\mu)}{1-\lambda^{q^{3s}}\mu^{q^{4s}+q^s}}, \tag{31}$$

cf. (10). The equation $M_1X^{q^s}=X$ with $x_4=t$ gives

$$\mu^{q^{2s}}x_2-t=(1-\mathrm{N}_{q^5/q}(\mu))^{-1}(1-\lambda^{q^s}\mu^{q^{4s}+q^{2s}})(\mu^{q^{2s}}t^{q^{3s}}-t),$$
$$\mu^{q^{2s}}x_3+\lambda^{q^{2s}}t=(1-\mathrm{N}_{q^5/q}(\mu))^{-1}\left((\mu^{q^{2s}}-\lambda^{q^{2s}}\mu^{q^{3s}+q^{2s}+1})t^{q^{4s}}+(\lambda^{q^{2s}}-\mu^{q^{4s}+q^{2s}+q^s})t\right).$$

By (10), $\lambda\mu^{q^{3s}+q^s}\neq 1$ and this leads to (28). □

Proposition 4.6. *If* $\mathrm{N}_{q^5/q}(\mu)\neq 1$*, then both* $\rho=\mu/\lambda^{q^s+1}$ *and*

$$e=\frac{1-\rho^5\,\mathrm{N}_{q^5/q}(\lambda)^2}{1-\rho^2\,\mathrm{N}_{q^5/q}(\lambda)} \tag{32}$$

are elements of $\mathbb{F}_q$ *with* $1\leqslant s\leqslant 4$*.*

Proof. Let α and β be as in (28). If $\beta=0$, then $\lambda^{q^s+1}=\mu\,\mathrm{N}_{q^5/q}(\mu)$; so, $\rho=\mathrm{N}_{q^5/q}(\mu)^{-1}$. Otherwise $(\beta^{q^s+1}/\alpha^{q^s})^{q^s-1}=1$ by Proposition 4.3. This is equivalent to

$$\frac{\mu^{q^{4s}+q^{2s}}(\lambda^{q^{4s}}-\mu^{q^{4s}+q^{3s}+q^s})(\lambda^{q^{2s}}\mu^{q^{3s}+1}-1)}{\mu^{q^{4s}+q^{3s}}(\lambda^{q^{4s}}\mu^{q^{2s}+1}-1)(\lambda^{q^{2s}}-\mu^{q^{4s}+q^{2s}+q^s})}=1$$

and to $(\mathrm{N}_{q^5/q}(\mu)-1)(\lambda^{q^{4s}}\mu^{q^{2s}}-\lambda^{q^{2s}}\mu^{q^{3s}})=0$, hence $\lambda^{q^{2s}-1}=\mu^{q^s-1}$. Finally, Formula (32) can be obtained by substitution of $\mu=\rho\lambda^{q^s+1}$ in (10). □

As seen in (9), $v_4 = u_4 - \lambda^{q^{4s}} u + \mu^{q^{4s}} v_1$ and this leads to coordinates $[-a + \lambda^{q^{4s}}, -b, -c, -d, 1-e]$ of B_1 with respect to the basis $\mathcal{B} = \{u, u_1, u_2, u_3, v_4\}$.

In view of Proposition 4.6, if $\mathrm{N}_{q^5/q}(\mu) \neq 1$, then (23) become

$$\begin{cases} a = \mu^{-q^{2s}-1}(e-1) \\ b = -\mu^{q^{4s}} \\ c = \mu^{-q^{2s}}(e-1) \\ d = -\mu^{q^{4s}+q^{s}}. \end{cases} \tag{33}$$

This implies the following result.

Proposition 4.7. *If $\mathrm{N}_{q^5/q}(\mu) \neq 1$, then the line A_3B_1 meets Γ in one point.*

Proof. Taking into account (11), the condition of dependence for A, A_1, A_3, B_1, B_2 is

$$\begin{vmatrix} -c & 1-e \\ 1 & \mu^{q^{2s}} \end{vmatrix} = 0.$$

Then the thesis follows from (33). ☐

4.1. An algebraic condition for being non-LP type. Until the end of this subsection we will assume that $\mathbb{L}$ is not of LP type as well as not of pseudoregulus type. Also, we will assume that $\operatorname{rk} A = 5$. This allows us to take $[x_0, x_1, x_2, x_3, x_4]$ as homogeneous coordinates of a point $\langle \sum x_i u_i \rangle_{\mathbb{F}_{q^5}}$.

Proposition 4.8. *The linear set $\mathbb{L}$ is projectively equivalent to*

$$\{\langle (x - k^{-1}x^{q^{2s}}, x^{q^{s}} - \delta k^{q^{4s}+q^{2s}} x^{q^{2s}}) \rangle_{\mathbb{F}_{q^5}} : x \in \mathbb{F}_{q^5}^*\} \tag{34}$$

for some $k \in \mathbb{F}_{q^5}^*$ and $\delta \in \mathbb{F}_q$; that is, $\mathbb{L} \cong L_{\alpha,\beta,s}$ with $\alpha = k^{-1}$, $\beta = \delta k^{q^{4s}+q^{2s}}$.

Proof. Case $\mathrm{N}_{q^5/q}(\mu) \neq 1$. Let $S = \langle \Gamma, A_3 \rangle$. The point A_4 does not belong to S, for otherwise S would also contain $\Gamma_4 = \langle A, A_4, B_1 \rangle$. By a similar argument S does not contain A_2. Then the equation of S is $x_4 = kx_2$, for some $k \neq 0$ in $\mathbb{F}_{q^5}$.

Let $B = [b_0, b_1, b_2, b_3, b_4]$. By Proposition 4.7, $B_1 \in S$. The algebraic conditions for $B, B_1, B_2 \in S$, and $B_2 \in \langle A, A_1, B \rangle$ are $b_4 = kb_2$, $b_3^{q^s} = kb_1^{q^s}$, $b_2^{q^{2s}} = kb_0^{q^{2s}}$, and

$$\operatorname{rk} \begin{pmatrix} b_2 & k^{q^{4s}} b_1 & kb_2 \\ b_0^{q^{2s}} & b_1^{q^{2s}} & b_2^{q^{2s}} \end{pmatrix} = 1$$

respectively. The coordinates b_1 and b_3 are nonzero, for otherwise $B \in \langle A, A_2, A_4 \rangle$, and $\mathbb{L}$ is of LP type; furthermore, $b_0b_2b_4 \neq 0$, for otherwise $A_3 \in A_1B \subseteq \Gamma$, a contradiction. The condition on the rank implies $b_0 = k^{-q^{3s}} b_2$ and $k^{q^{4s}-1} b_2^{q^{2s}-1} = b_1^{q^{2s}-1}$, hence $b_1 = \delta k^{q^{2s}+1} b_2$ for some $\delta \in \mathbb{F}_q^*$. Therefore $B = [k^{-q^{3s}}, \delta k^{q^{2s}+1}, 1, \delta k^{q^{4s}+q^{2s}+1}, k]$, and

$$F = AB_2 \cap A_1B = [k^{-q^{3s}-1}, k^{q^{2s}}, k^{-1}, \delta k^{q^{4s}+q^{2s}}, 1]. \tag{35}$$

By projecting a point $[x^{q^{3s}}, x^{q^{4s}}, x, x^{q^{s}}, x^{q^{2s}}]$ of Σ from the vertex

$$\langle A, A_1, B_2 \rangle = \langle [1,0,0,0,0], [0,1,0,0,0], [*, *, k^{-1}, \delta k^{q^{4s}+q^{2s}}, 1] \rangle$$

onto a complementary line, one obtains (34).

Case $\mu = 1$. The linear set $\mathbb{L}$ is described in (27), which is equivalent to (34) for $k = 1$, $\delta = 1 - e$. ☐

Theorem 4.9. *Let $\mathbb{L}$ as described in Proposition 4.8. If $\varepsilon = \mathrm{N}_{q^5/q}(k)$, then $\delta^2\varepsilon \neq 1$, $\delta^3\varepsilon^2 + (1-3\delta)\varepsilon + 1 \neq 0$, and no $x \in \mathbb{F}_{q^5}$ exists satisfying*

$$\begin{cases} \varepsilon \, \mathrm{N}_{q^5/q}(x) = -1 \\ \delta^2 \varepsilon x^{q^{s}} - \delta \varepsilon x^{q^{2s}+q^{s}+1}(1-x)^{q^{3s}} + (1-x)^{q^{s}+1} = 0. \end{cases} \tag{36}$$

Proof. CASE $\mathrm{N}_{q^5/q}(\mu) \neq 1$. By Proposition 4.1 (ii),

$$\frac{\alpha^{q^s}}{\beta^{q^s+1}} = \frac{k^{-q^s}}{\delta^2 k^{q^{4s}+q^{3s}+q^{2s}+1}} = \frac{1}{\delta^2\varepsilon}$$

is distinct from one, so $\delta^2\varepsilon \neq 1$.

By substitution of $\beta^{q^{3s}+q^s+1} = \delta^3 k^{q^{4s}+q^{3s}+2q^{2s}+2}$,$\beta^{q^{3s}} = \delta k^{q^{2s}+1}$, $\alpha^{q^{3s}} = k^{-q^{3s}}$, $\beta^{q^s+1} = \delta^2 k^{q^{4s}+q^{3s}+q^{2s}+1}$, Proposition 4.2 reads as follows: $\mathbb{L}$ is maximum scattered if and only if there is no $z \in \mathbb{F}_{q^5}$ such that

$$\begin{cases} \mathrm{N}_{q^5/q}(z) = -1 \\ \delta^3 k^{q^{4s}+q^{3s}+2q^{2s}+2} z^{q^s} - \delta^2 k^{q^{4s}+q^{3s}+q^{2s}+1} z^{q^{2s}+q^s+1}(1-k^{-1}z)^{q^{3s}} \\ \quad +\delta k^{q^{2s}+1}(1-k^{-1}z)^{q^s+1} = 0. \end{cases}$$

Dividing by $\delta k^{q^{2s}+1}$ and substituting $z = kx$, we get (36).

Since $\mathbb{L}$ is not of LP type, the points

$$\begin{aligned} B_3 &= [1, \delta k^{q^{3s}+q^{2s}+1}, k^{q^{3s}}, k^{-q^s}, \delta k^{q^{3s}+1}], \\ B_4 &= [\delta k^{q^{4s}+q^s}, 1, \delta k^{q^{4s}+q^{3s}+q^s}, k^{q^{4s}}, k^{-q^{2s}}], \end{aligned}$$

and F (cf. (35)) are not collinear. By standard row-reducing, this reads as $\delta^3\varepsilon^2 + (1-3\delta)\varepsilon + 1 \neq 0$.

CASE $\mu = 1$. Clearly condition $\delta^2\varepsilon \neq 1$ and there is no $x \in \mathbb{F}_{q^5}$ satisfying (36) hold. The condition $\delta^3\varepsilon^2 + (1-3\delta)\varepsilon + 1 = 0$ is equivalent to $e = 3$, and this by (26) implies that $\mathbb{L}$ is of LP type, a contradiction. ☐

5. AN ALGEBRAIC EQUATION

In this section, we shall show that if $\mathbb{L}$ is a maximum scattered linear set of $\mathrm{PG}(1,q^5)$ neither of pseudoregulus type nor of LP type, and $\operatorname{rk} A = 5$, the conditions in Theorem 4.9 never hold. As a consequence, we will get that there are no new maximum scattered linear sets in $\mathrm{PG}(1,q^5)$ with $\max\{\operatorname{rk} A, \operatorname{rk} B\} = 5$. More precisely, we will show the following.

THEOREM 5.1. *Let* $1 \leqslant s \leqslant 4$ *and* $\delta, \varepsilon \in \mathbb{F}_q^*$ *such that* $\delta^2\varepsilon \neq 1$. *If* $\delta^3\varepsilon^2 + (1-3\delta)\varepsilon + 1 \neq 0$, *then there exists* $x \in \mathbb{F}_{q^5}$ *satisfying* (36).

Although we will use the techniques contained in [22, Section 4], to make the work as self-contained as possible, we prefer to adapt them to our context.

Consider a normal element of $\mathbb{F}_{q^5}$ over $\mathbb{F}_q$, say γ, see e.g. [15, Theorem 2.35]. Then, for any integer $1 \leqslant s \leqslant 4$, $\{\gamma, \gamma^{q^s}, \gamma^{q^{2s}}, \gamma^{q^{3s}}, \gamma^{q^{4s}}\}$ is an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^5}$ and every element x in $\mathbb{F}_{q^5}$ can be written as $x = \sum_{i=0}^{4} x_i \gamma^{q^{si}}$, where $x_i \in \mathbb{F}_q$, $i = 0,1,2,3,4$. Moreover, we can identify $\mathbb{F}_{q^5}$ with $\mathbb{F}_q^5$ in the natural way.

Equations (36) are therefore equivalent to a system $\mathcal{S}$ of ten equations

$$C_j(x_0, x_1, x_2, x_3, x_4) = 0$$

in the new variables x_i. Denoting by $\mathcal{V} \subseteq \mathrm{AG}(5,q)$ the affine variety associated with $\mathcal{S}$, Theorem 5.1 is therefore equivalent to the statement that $\mathcal{V}$ has an $\mathbb{F}_q$-rational point. To prove this, we will show that $\mathcal{V}$ is a variety of dimension one, i.e. an algebraic curve, and then the existence of a point will be a consequence of the Hasse-Weil Theorem , in the case the curve $\mathcal{V}$ is absolutely irreducible [12, Theorem 9.18], or an $\mathbb{F}_q$-rational point of $\mathcal{V}$ will be directly exhibited.

We first apply the following change of variables in $\mathrm{AG}(5,\overline{\mathbb{F}}_q)$ (whose matrix is a so-called Moore matrix and is nonsingular)

$$\phi:\begin{cases} A=x_0\gamma+x_1\gamma^{q^s}+x_2\gamma^{q^{2s}}+x_3\gamma^{q^{3s}}+x_4\gamma^{q^{4s}}\\ B=x_4\gamma+x_0\gamma^{q^s}+x_1\gamma^{q^{2s}}+x_2\gamma^{q^{3s}}+x_3\gamma^{q^{4s}}\\ C=x_3\gamma+x_4\gamma^{q^s}+x_0\gamma^{q^{2s}}+x_1\gamma^{q^{3s}}+x_2\gamma^{q^{4s}}\\ D=x_2\gamma+x_3\gamma^{q^s}+x_4\gamma^{q^{2s}}+x_0\gamma^{q^{3s}}+x_1\gamma^{q^{4s}}\\ E=x_1\gamma+x_2\gamma^{q^s}+x_3\gamma^{q^{2s}}+x_4\gamma^{q^{3s}}+x_0\gamma^{q^{4s}}.\end{cases}$$

The $\mathbb{F}_q$-rational points in $\mathrm{AG}(5,q)$ are mapped by ϕ to those of type

$$(A,B,C,D,E)=(x,x^{q^s},x^{q^{2s}},x^{q^{3s}},x^{q^{4s}}).$$

Denote by $\mathcal{C}$ the image of $\mathcal{V}$ under ϕ. Since the dimension, the genus and the absolute irreducibility are birational invariants, we can study $\mathcal{C}$ in place of $\mathcal{V}$.

Let

$$\begin{aligned} &G(A,B,C,D,E):=ABCDE\varepsilon+1,\\ &F_0(A,B,C,D,E):=\delta^2\varepsilon B-\delta\varepsilon ABC(1-D)+(1-A)(1-B)\\ &F_{i+1}(A,B,C,D,E):=F_i(B,C,D,E,A),\quad i=0,1,2,3.\end{aligned}\tag{37}$$

By (36), we get that $\mathcal{C}$ is given by the following system:

$$\mathcal{C}:\begin{cases} G(A,B,C,D,E)=0\\ F_i(A,B,C,D,E)=0,\ i=0,1,2,3,4.\end{cases}\tag{38}$$

Now, we are in the position to prove the following.

Lemma 5.2. *Let $\delta,\varepsilon\in\mathbb{F}_q^*$ such that $\delta^2\varepsilon\neq 1$. If $\delta^3\varepsilon^2+(1-3\delta)\varepsilon+1\neq 0$, then the algebraic variety $\mathcal{C}$ (cf. (38)) is birationally equivalent to the plane curve $\mathcal{Q}: f(X,Y)=0$ of degree at most four, where*

$$f(X,Y)=f_0X^2Y^2+f_1XY(X+Y)+f_2(X^2+Y^2)+f_3XY+f_4(X+Y)+f_5,\tag{39}$$

with

$$\begin{aligned} f_0&=(-\delta^2\varepsilon+1)(\delta\varepsilon-1),\\ f_1&=(\delta^2\varepsilon+\delta\varepsilon-2)(\delta^2\varepsilon-1),\\ f_2&=-(\delta^2\varepsilon-1)^2,\\ f_3&=\delta^6\varepsilon^3-5\delta^4\varepsilon^2+6\delta^2\varepsilon+\delta\varepsilon+\delta-4,\\ f_4&=(\delta^2\varepsilon-1)(\delta^2\varepsilon+\delta-2),\\ f_5&=(\delta-1)(-\delta^2\varepsilon+1).\end{aligned}\tag{40}$$

Proof. We eliminate the variables C,D,E successively from the system (38), using $G(A,B,C,D,E)=0$ and $F_i(A,B,C,D,E)=0$, to obtain a single equation in A,B. By the first equation in (38), we get $A,B,C,D,E\neq 0$ and, hence,

$$E=-1/(\varepsilon ABCD).\tag{41}$$

Moreover, by $F_0(A,B,C,D,E)=0$ we derive

$$D=P(A,B,C):=(ABC\delta\varepsilon-AB+A-B\delta^2\varepsilon+B-1)/(ABC\delta\varepsilon).\tag{42}$$

Since $F_1(A,B,C,P(A,B,C),-1/(\varepsilon ABC\cdot P(A,B,C)))=0$, we get

$$A(B\delta\varepsilon-B-\delta^2\varepsilon+1)C=(B\delta^2\varepsilon-B-\delta+1).$$

Note that, since $AC\neq 0$, $B\delta\varepsilon-B-\delta^2\varepsilon+1$ is zero if and only if $B\delta^2\varepsilon-B-\delta+1$ is zero. In this case, since $B\neq 0$ and $\delta^2\varepsilon\neq 1$, it would follow $(\delta^2\varepsilon-1)^2-(\delta-1)(\delta\varepsilon-1)=0$, against our hypotheses. Hence, we have

$$C=Q(A,B):=(B\delta^2\varepsilon-B-\delta+1)/(A(B\delta\varepsilon-B-\delta^2\varepsilon+1)). \tag{43}$$

In order to eliminate the variable C from (42), write

$$\begin{aligned}\hat{P}(A,B):=P(A,B,Q(A,B))=&\frac{1}{B\delta\varepsilon(1-B-\delta+B\delta^2\varepsilon)}(-AB^2\delta\varepsilon+AB^2+AB\delta^2\varepsilon+\\&AB\delta\varepsilon-2AB-A\delta^2\varepsilon+A+B^2\delta^2\varepsilon-B^2+B\delta^4\varepsilon^2-\\&3B\delta^2\varepsilon+2B+\delta^2\varepsilon-1).\end{aligned}$$

Now, we will express the equations $F_i(A,B,C,D,E)=0$, $i\in\{2,3,4\}$, in terms of the variables A and B.

Substituting in $F_2(A,B,C,D,E)=0$, we obtain the equation $f(A,B)=0$, where

$$f(A,B):=F_2(A,B,Q(A,B),\hat{P}(A,B),-1/(\varepsilon ABQ(A,B)\hat{P}(A,B))).$$

This can be written as

$$\begin{aligned}f(A,B)=&A^2B^2(-\delta^2\varepsilon+1)(\delta\varepsilon-1)+\\&AB(A+B)(\delta^2\varepsilon+\delta\varepsilon-2)(\delta^2\varepsilon-1)+\\&(A^2+B^2)(-(\delta^2\varepsilon-1)^2)+\\&AB(\delta^6\varepsilon^3-5\delta^4\varepsilon^2+6\delta^2\varepsilon+\delta\varepsilon+\delta-4)+\\&(A+B)(\delta^2\varepsilon-1)(\delta^2\varepsilon+\delta-2)+\\&(\delta-1)(-\delta^2\varepsilon+1).\end{aligned}$$

Finally, note that both $F_3(A,B,C,D,E)=0$ and $F_4(A,B,C,D,E)=0$, when expressed in terms of A and B only, are satisfied as soon as $f(A,B)=0$. ☐

Lemma 5.3. *Let* $\delta,\varepsilon\in\mathbb{F}_q^*$ *satisfy* $\delta^2\varepsilon\neq 1$ *and* $\delta^3\varepsilon^2+(1-3\delta)\varepsilon+1\neq 0$. *Then, the curve* $\mathcal{Q}:f(X,Y)=0$ *(cf.* (39)*) has no linear components.*

Proof. The coefficient f_0 of the term of degree four of $f(X,Y)$ in (39) is 0 if and only if $\delta\varepsilon=1$. We split the proof in two cases: $f_0=0$ and $f_0\neq 0$.

Case 1. $f_0=0$. We have that $\delta-1\neq 0$, otherwise $\delta^2\varepsilon=1$. Then,

$$f(X,Y)=(\delta-1)^2(X^2Y-X^2+XY^2+XY\delta-3XY+2X-Y^2+2Y-1),$$

and $\mathcal{Q}$ is a cubic of the affine plane $\mathrm{AG}(2,q)$. Let $[\overline{X},\overline{Y},\overline{Z}]$ the homogeneous coordinates of a point in $\mathrm{PG}(2,q)$ and let $g(\overline{X},\overline{Y},\overline{Z}):=\overline{Z}^3f(\overline{X}/\overline{Z},\overline{Y}/\overline{Z})$. We now show that the projective variety $\overline{\mathcal{Q}}:g(\overline{X},\overline{Y},\overline{Z})=0$ has no linear components. Note that the intersection of the curve $\overline{\mathcal{Q}}$ with the coordinate axes $\overline{X}=0$, $\overline{Y}=0$ and $\overline{Z}=0$ are the point sets $I_1=\{[0,1,0],[0,1,1]\}$, $I_2=\{[1,0,0],[1,0,1]\}$ and $I_3=\{[0,1,0],[1,0,0],[1,-1,0]\}$, respectively. If a line ℓ is a component of $\overline{\mathcal{Q}}$, then ℓ must meet each of the coordinate axes in at least one point of I_i, $i=1,2,3$. This follows from the fact that any two lines in the plane meet in a point or they coincide, and that the intersection of ℓ with the coordinates axes must be a subset of the the intersection of $\overline{\mathcal{Q}}$ with the axes. In particular, ℓ must be the line joining a point one of I_1 and a point of I_2. By direct checking, a line joining a point of I_1 and a point of I_2 is one of the following:

(*i*) the line $\overline{Z}=0$. This is not a component of $\overline{\mathcal{Q}}$.

(*ii*) the line $\overline{X}-\overline{Z}=0$. If this line is a component of $\overline{\mathcal{Q}}$, then

$$\overline{X}^2\overline{Y}+\overline{XY}^2-\overline{X}^2\overline{Z}+(\delta-3)\overline{XYZ}+2\overline{XZ}^2-\overline{Y}^2\overline{Z}+2\overline{YZ}^2-\overline{Z}^3=$$
$$(\overline{Z}-\overline{X})(a\overline{X}^2+b\overline{Y}^2+c\overline{XY}+d\overline{XZ}+e\overline{YZ}+f\overline{Z}^2)$$

for some $a,b,c,d,e,f\in\mathbb{F}_q$. Then, comparing the coefficients on the left and the right hand-side, we get $a=0$, $b=c=-d=f=-1$ and $e=2$. This implies $\delta=0$, which is a contradiction with our hypotheses.

(*iii*) the line $\overline{Y}-\overline{Z}=0$. Since $g(\overline{X},\overline{Y},\overline{Z})=g(\overline{Y},\overline{X},\overline{Z})$, a contradiction follows as above.

(*iv*) the line $\overline{X}+\overline{Y}-\overline{Z}=0$. If this line is a component of $\overline{\mathcal{Q}}$, then

$$(\overline{X}+\overline{Y}-\overline{Z})(a\overline{X}^2+b\overline{Y}^2+c\overline{XY}+d\overline{XZ}+e\overline{YZ}+f\overline{Z}^2)=$$
$$\overline{X}^2\overline{Y}+\overline{XY}^2-\overline{X}^2\overline{Z}+(\delta-3)\overline{XYZ}+2\overline{XZ}^2-\overline{Y}^2\overline{Z}+2\overline{YZ}^2-\overline{Z}^3.$$

Then, comparing the coefficients on the left- and the right-hand side, we get $a=b=0$, $c+e=-d=f=1$. This leads to $\delta=1$, a contradiction again.

Case 2: $f_0\neq 0$. This is equivalent to $\delta\varepsilon\neq 1$. Consider $h(X,Y):=f_0^{-1}f(X,Y)$. If $\mathcal{Q}$ has a linear component, then

$$h(X,Y)=l(X,Y)\cdot k(X,Y),$$

with

$$l(X,Y)=aX+bY+c\quad\text{and}\quad k(X,Y)=k_1XY^2+k_2X^2Y+k_3XY+k_4X+k_5Y+k_6$$

both belonging to $\mathbb{F}_q[X,Y]$. We will show that either $a\neq 0$ and $b=0$ or $a=0$ and $b\neq 0$. Suppose first $a\neq 0$. Since $h(X,Y)$ has a term in X^2Y^2 and has no term in XY^3 and in X^3Y, we get that k_1 cannot be equal to zero and that $k_2=b=0$. A similar argument applies if $b\neq 0$, getting $a=0$. Since $h(X,Y)=h(Y,X)$, we obtain that

$$h(X,Y)=(X-r)(Y-r)\cdot m(X,Y)$$

for some $r\in\mathbb{F}_q$, and $m(X,Y)\in\mathbb{F}_q[X,Y]$ with $\deg m(X,Y)=2$. Clearly, $h(r,Y)$ is the zero polynomial.

As a consequence, all the coefficients in Y of the following polynomial are zero:

$$\begin{aligned}f(r,Y)=f_0\cdot h(r,Y)=&(1-r)(\delta^2\varepsilon-1)(\delta^2\varepsilon r-\delta-r+1)+\\&(\delta^4\varepsilon^2+\delta^3\varepsilon-3\delta^2\varepsilon-\delta+\delta^4r^2\varepsilon^2+\delta^3r^2\varepsilon^2-3\delta^2r^2\varepsilon-\delta r^2\varepsilon+\\&2r^2+\delta^6r\varepsilon^3-5\delta^4r\varepsilon^2+6\delta^2r\varepsilon+\delta r\varepsilon+\delta r-4r+2)Y+\\&(1-r)(\delta^2\varepsilon-1)(\delta r\varepsilon-\delta^2\varepsilon-r+1)Y^2.\end{aligned}$$

In particular, the constant term is zero and hence either $r=1$ or $r=(\delta-1)/(\delta^2\varepsilon-1)$. If $r=1$ then the coefficient of Y is $\varepsilon\delta^3(\delta^3\varepsilon^2-3\delta\varepsilon+\varepsilon+1)$ which cannot be zero under our hypothesis. If $r=(\delta-1)/(\delta^2\varepsilon-1)$ then the coefficient of Y is $\delta^2\left(\delta^3\varepsilon^2-3\delta\varepsilon+\varepsilon+1\right)$, which again cannot be zero. This concludes the proof. ☐

Since a cubic curve without linear components is absolutely irreducible, Lemma 5.3 implies the following result.

Proposition 5.4. *Let* $\delta,\varepsilon\in\mathbb{F}_q^*$ *such that* $\delta\varepsilon=1$ *and assume* $\delta^2-2\delta+1\neq 0$. *Then, the curve* $\mathcal{Q}:f(X,Y)=0$ *(cf.* (39)*) is an absolutely irreducible cubic.*

Lemma 5.5. *Let* $\delta,\varepsilon\in\mathbb{F}_q^*$ *satisfy* $\delta^2\varepsilon\neq 1$ *and* $\delta^3\varepsilon^2+(1-3\delta)\varepsilon+1\neq 0$. *Moreover, assume that one of the following holds:*

i) $\varepsilon\neq 1$,

ii) $\varepsilon=1$ *and either* q *is an odd power of* 2, *or* $q\equiv 0,2,3\pmod 5$ *odd.*

Then, the curve $\mathcal{Q}: f(X,Y)=0$ *(cf.* (39)*) has no quadratic components.*

Proof. If $\mathcal{Q}$ is a cubic, then the thesis follows from Proposition 5.4. So, we assume that $\mathcal{Q}: f(X,Y)=0$ is a quartic. In particular, $f_0\neq 0$, which is equivalent to $\delta\varepsilon\neq 1$. Let us suppose that $\mathcal{Q}$ splits into two absolutely irreducible conics. Hence, $f(X,Y)=a(X,Y)\cdot b(X,Y)$ where

$$\begin{aligned} a(X,Y)&=a_0Y^2+a_1XY+a_2X^2+a_3Y+a_4X+a_5\\ b(X,Y)&=b_0X^2+b_1XY+b_2Y^2+b_3X+b_4Y+b_5, \end{aligned}$$

both belonging to $\mathbb{F}_q[X,Y]$. Note that, since $f(X,Y)$ has no term in X^4,Y^4, X^3, Y^3 and in X^3Y and XY^3 then

$$\begin{cases} a_0b_2=0\\ a_2b_0=0\\ a_4b_0+a_2b_3=0\\ a_3b_2+a_0b_4=0\\ a_1b_0+a_2b_1=0\\ a_0b_1+a_1b_2=0 \end{cases} \tag{44}$$

Case 1. $a_0\neq 0$. This implies in the system above that $b_1=b_2=b_4=0$. Since $b(X,Y)$ is polynomial of degree 2, then $b_0\neq 0$. So, we get $a_1=a_2=a_4=0$ and hence

$$a(X,Y)=a_0Y^2+a_3Y+a_5 \quad\text{and}\quad b(X,Y)=b_0X^2+b_3X+b_5. \tag{45}$$

Let $h(X,Y):=f_0^{-1}\cdot f(X,Y)$. Since $f_0=a_0b_0$ (cf. (40)), we have

$$h(X,Y)=(Y^2+s_1Y+s_2)(X^2+t_1X+t_2) \tag{46}$$

where $s_i=a_{2i+1}/a_0$ and $t_i=b_{2i+1}/b_0$, $i\in\{1,2\}$. Since $h(X,Y)=h(Y,X)$, $s_1=t_1$ and $s_2=t_2$ hold. It follows that

$$s_2^2=f_5/f_0,\quad s_1^2=f_3/f_0,\quad s_1=f_1/f_0,\quad s_1s_2=f_4/f_0,\quad s_2=f_2/f_0.$$

This implies $f_2^2=f_0f_5$, namely (cf. (40))

$$\delta\left(\delta^2\varepsilon-1\right)^2\left(\delta^3\varepsilon^2-3\delta\varepsilon+\varepsilon+1\right)=0$$

which contradicts our hypotheses.

Case 2. $a_0=0$. If $a_2\neq 0$, by System (44) we get $b_0=b_1=b_3=0$. Then $b_2\neq 0$ and $a_1=a_3=0$, getting

$$a(X,Y)=a_2X^2+a_4X+a_5 \quad\text{and}\quad b(X,Y)=b_2Y^2+b_4Y+b_5. \tag{47}$$

Similarly, this leads to a contradiction as discussed in **Case 1**.

If $a_2=0$, since $a_1\neq 0$ we get $b_0=b_2=0$ and hence

$$a(X,Y)=a_1XY+a_3Y+a_4X+a_5 \quad\text{and}\quad b(X,Y)=b_1XY+b_3X+b_4Y+b_5. \tag{48}$$

Let $h(X,Y):=f_0^{-1}\cdot f(X,Y)$. Since $f_0=a_1b_1$ (cf. (40)), we have

$$h(X,Y)=(XY+s_1Y+s_2X+s_3)(XY+t_1X+t_2Y+t_3), \tag{49}$$

where $s_i = a_{i+2}/a_1$ and $t_i = b_{i+2}/b_1$, $i \in \{1,2,3\}$. Hence, the following conditions hold:

$$\begin{cases} s_1 + t_2 = h_1 \\ s_2 + t_1 = h_1 \\ s_2 t_1 = h_2 \\ s_1 t_2 = h_2 \\ s_3 + s_1 t_1 + s_2 t_2 + t_3 = h_3 \\ s_3 t_1 + s_2 t_3 = h_4 \\ s_3 t_2 + s_1 t_3 = h_4 \\ s_3 t_3 = h_5 \end{cases} \tag{50}$$

where $h_i = f_i/f_0$, $i \in \{1,2,3,4,5\}$. By (39), $f_2 \neq 0$ and hence $h_2 \neq 0$. By System (50), s_1, s_2, t_1 and t_2 are not zero.

Multiplying by s_1 and by s_2 the first and second equations yields respectively

$$s_1^2 + s_1 t_2 = h_1 s_1 \quad \text{and} \quad s_2^2 + s_2 t_1 = h_1 s_2. \tag{51}$$

Subtracting the two expressions above and using that $s_1 t_2 = h_2 = s_2 t_1$ gives

$$s_2^2 - s_1^2 = h_1(s_2 - s_1). \tag{52}$$

Similarly, by interchanging the role of s_i and t_i, we get

$$t_2^2 - t_1^2 = h_1(t_2 - t_1). \tag{53}$$

Case 2.1 $s_1 \neq s_2$. Then by (52), $s_1 + s_2 = h_1$. By the first two equations of the System in (50), we have $s_1 = t_1$ and $s_2 = t_2$. Moreover, by the third-last and the second-last equations in System (50), we get $s_3 = t_3$ and hence System (50) reduces into

$$\begin{cases} s_1 + s_2 = h_1 \\ s_1 s_2 = h_2 \\ 2s_3 + s_1^2 + s_2^2 = h_3 \\ s_3(s_1 + s_2) = h_4 \\ s_3^2 = h_5. \end{cases} \tag{54}$$

If $h_1 = 0$, then $f_1 = f_4 = 0$. This implies that $\varepsilon = 1$ and either $\delta = 1$ or $\delta = -2$, which in each case leads to a contradiction to the assumption $\delta^3\varepsilon^2 + (1-3\delta)\varepsilon + 1 \neq 0$. Hence $h_1 \neq 0$.

By the second-last equation in (54), $s_3 = h_4/h_1$. Hence $h_5 = s_3^2 = h_4^2/h_1^2$, getting

$$f_5 f_1^2 = f_4^2 f_0. \tag{55}$$

Combining the equations of (54), we also have that

$$h_3 = 2s_3 + (s_1 + s_2)^2 - 2s_1 s_2 = 2h_4/h_1 + h_1^2 - 2h_2;$$

equivalently,

$$f_0 f_1 f_3 = f_1^3 - 2f_0 f_1 f_2 + 2f_0^2 f_4. \tag{56}$$

Substituting the coefficients of $f(X,Y)$ (cf. (39)) in (55), we get

$$\delta^2(\varepsilon - 1)\left(\delta^2\varepsilon - 1\right)^3\left(\delta^3\varepsilon^2 - 3\delta\varepsilon + \varepsilon + 1\right) = 0,$$

that implies $\varepsilon = 1$, and combining it with (56)

$$(\delta - 1)^6\delta^2(\delta + 1)^2(\delta + 2)\left(\delta^2 + 3\delta + 1\right) = 0.$$

In our assumptions, the expression above is zero if and only if $\delta^2 + 3\delta + 1 = 0$. Then, if $q = 2^{2m+1}$ for some non-negative integer m or $q \equiv 2, 3 \pmod 5$ odd, the equation

$x^2+3x+1=0$ has no roots in $\mathbb{F}_q$. If $q\equiv 0 \pmod 5$ odd, by hypotheses $\delta\neq 1$ and hence $\delta^2+3\delta+1\neq 0$. Then, in any case, we get a contradiction.

Case 2.2 $s_1=s_2$. By System (50), $t_1=t_2$. Now, since $s_1+t_1=h_1$ and $s_1t_1=h_2$, we obtain that s_1 and t_1 are the solutions of the equation:

$$x^2-\frac{2-\delta^2\varepsilon-\delta\varepsilon}{\delta\varepsilon-1}x+\frac{\delta^2\varepsilon-1}{\delta\varepsilon-1}=0.$$

Let $\xi:=\delta\varepsilon-1$ and $\eta:=\delta^2\varepsilon-1$. Then, the equation can be written as

$$\xi x^2+(\xi+\eta)x+\eta=0;$$

whose solutions are -1 and $-\eta/\xi=\frac{1-\delta^2\varepsilon}{\delta\varepsilon-1}$.

By (49), assuming $s_1=s_2=-1$ and $t_1=t_2=\frac{1-\delta^2\varepsilon}{\delta\varepsilon-1}$ is equivalent to assuming $t_1=t_2=-1$ and $s_1=s_2=\frac{1-\delta^2\varepsilon}{\delta\varepsilon-1}$. Then, without loss of generality, we may choose $s_1=s_2=-1$ and $t_1=t_2=\frac{1-\delta^2\varepsilon}{\delta\varepsilon-1}$.

Let us consider the equations $s_3t_1+s_2t_3=h_4$ and $s_3t_3=h_5$, we have $t_3=s_3\frac{1-\delta^2\varepsilon}{\delta\varepsilon-1}-h_4$, hence s_3 is solution of the following equation

$$\frac{1-\delta^2\varepsilon}{\delta\varepsilon-1}x^2-h_4x-h_5=0,$$

that is equivalent to

$$(1-\delta^2\varepsilon)x^2+(\delta^2\varepsilon+\delta-2)x+(1-\delta)=0.$$

We get that either $s_3=1$ or $s_3=\frac{1-\delta}{1-\delta^2\varepsilon}$. If $s_3=1$, the equation $s_3+s_1t_1+s_2t_2+t_3=h_3$ reads as

$$\delta^3\varepsilon\left(\delta^3\varepsilon^2-3\delta\varepsilon+\varepsilon+1\right)=0$$

against our hypothesis.

Similarly, if $s_3=\frac{1-\delta}{1-\delta^2\varepsilon}$, the equation $s_3+s_1t_1+s_2t_2+t_3=h_3$ reads as

$$\delta^2\varepsilon\left(\delta^2\varepsilon-1\right)=0$$

in contradiction with our hypothesis. This concludes the proof. ☐

Remark 5.6. Note that if $\varepsilon=1$ and either q is an even power of 2 or $q\equiv 1,4 \pmod 5$ odd, by the proof of Lemma 5.5 *Case 2.1*, the conic with equation

$$XY-(\delta+1)X-Y+1=0 \tag{57}$$

is a component of $\mathcal{Q}$ where $\delta\in\mathbb{F}_q$ satisfies $\delta^2+3\delta+1=0$.

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Assume first that $\varepsilon\neq 1$ or that $\varepsilon=1$ and q either an odd power of 2 or $q\equiv 0,2,3 \pmod 5$ odd. Then, from Lemma 5.2, 5.3, 5.5 and Proposition 5.4, the variety $\mathcal{V}$ is an absolutely irreducible curve of genus at most three. The Hasse-Weil Theorem implies that for $q\geqslant 37$ the variety $\mathcal{V}$ has an $\mathbb{F}_q$-rational point. For $q<37$, the result has been directly checked by a GAP script, available at https://pastebin.com/PHQJnAq0.

If $\varepsilon=1$ and either q is an even power of 2 or $q\equiv 1,4 \pmod 5$ odd, we deduce the result from Remark 5.6. In this case, we claim indeed that $\mathcal{Q}$ has an affine point of the form (ℓ,ℓ^{q^s}) and using such point, we are able to show explicitly an affine point of type $(\ell,\ell^{q^s},\ell^{q^{2s}},\ell^{q^{3s}},\ell^{q^{4s}})$ of $\mathcal{C}$ (cf. (38)) which corresponds to an $\mathbb{F}_q$-rational point of $\mathcal{V}$. Putting $Y=X^{q^s}$ in (57), we obtain

$$X^{q^s+1}-(\delta+1)X-X^{q^s}+1=0.$$

This has a solution if and only if $\xi^{q^s+1}-\delta\xi-\delta=0$, where $\xi=X-1$. To show that this equation has a solution in $\mathbb{F}_q$ we apply [21, Theorem 8]. Define

$$M=\begin{pmatrix}0 & \delta\\ 1 & \delta\end{pmatrix}^5.$$

If $q=2^{2m}$ for some positive integer m, then the $(2,2)$-entry of M (that is G_5 in the notation of [21]) is equal to $\delta^3(\delta+1)(\delta+3)$ and lies in $\mathbb{F}_q$. Hence, by [21, Theorem 8], the equation $\xi^{q^s+1}-\delta\xi-\delta=0$ has at least one root.

If $q\equiv 1,4 \pmod 5$ is odd, [21, Theorem 8] shows that the same equation has a solution if and only if

$$\Delta=\mathrm{Tr}(M)^2-4\det(M)$$

is a square in $\mathbb{F}_q$. A direct computation gives

$$\Delta=\delta^5(\delta+4)(\delta^2+3\delta+1)^2=0,$$

so Δ is a square in $\mathbb{F}_q$, and again the equation has a solution. Therefore there exists a point of the form (ℓ,ℓ^{q^s}) on the conic (57), and hence on $\mathcal{Q}$.

By (57), $\ell\neq 1$ and

$$\ell^{q^s}=\frac{(\delta+1)\ell-1}{\ell-1}. \tag{58}$$

By the expression above,

$$\ell^{q^{2s}}=\frac{(\delta+1)\ell^{q^s}-1}{\ell^{q^s}-1}=\frac{(\delta+2)\ell-1}{\ell}. \tag{59}$$

By Lemma 5.2, $\mathcal{Q}$ is birationally equivalent to $\mathcal{C}$ and a point (ℓ,ℓ^{q^s}) of $\mathcal{Q}$ corresponds to a point $(\ell,\ell^{q^s},\bar{c},\bar{d},\bar{e})$ of $\mathcal{C}$ with

$$\bar{c}=\frac{(\ell^{q^s}\delta^2-\ell^{q^s}-\delta+1)}{\ell(\ell^{q^s}\delta-\ell^{q^s}-\delta^2+1)},\quad \bar{d}=\frac{(\ell^{q^s+1}\bar{c}\delta-\ell^{q^s+1}+\ell-\ell^{q^s}\delta^2+\ell^{q^s}-1)}{\ell^{q^s+1}\bar{c}\delta}, \tag{60}$$

and $\bar{e}=-\frac{1}{\ell^{q^s+1}\bar{c}\bar{d}}$ (cf. (43), (42), and (41)). Substituting Equation (58) in the expression of $\bar{c}$, we have that

$$\bar{c}=\frac{(\ell^{q^s}\delta^2-\ell^{q^s}-\delta+1)}{\ell(\ell^{q^s}\delta-\ell^{q^s}-\delta^2+1)}=\frac{(\delta+2)\ell-1}{\ell}, \tag{61}$$

getting $\bar{c}=\ell^{q^{2s}}$. By (58) and (59),

$$\ell^{q^{3s}}=\frac{-\delta+\delta^2\ell+3\delta\ell+\ell-1}{\delta\ell+\ell-1},$$

while by (60), (58) and (61),

$$\bar{d}=\frac{\delta+\delta^2\ell^2+3\delta\ell^2+\ell^2-\delta^2\ell-3\delta\ell-2\ell+1}{(\delta\ell+\ell-1)(\delta\ell+2\ell-1)}.$$

Let us consider

$$\bar{d}-\ell^{q^{3s}}=-\frac{\left(\delta^2+3\delta+1\right)\ell}{\delta\ell+2\ell-1}. \tag{62}$$

Since $\delta^2+3\delta+1=0$, it follows that $\bar{d}=\ell^{q^{3s}}$. An analogous computation shows that $\bar{e}=\ell^{q^{4s}}$, which completes the proof. □

6. On the case $\operatorname{rk} A = \operatorname{rk} B = 4$

6.1. Geometric description. Now assume that $\wp_\Gamma(\Sigma)$ is a MSLS not of pseudoregulus type, that Σ is the set of all points $[x_0,\dots,x_4]$ of $\mathrm{PG}(4,q^5)$ with coordinates rational over $\mathbb{F}_q$, and that Γ has $\operatorname{rk} A = \operatorname{rk} B = 4$ for any choice of σ. Thus $\sigma : (x_0,\dots,x_4) \mapsto (x_0^q,\dots,x_4^q)$ may be assumed. Let $S_A = \langle A, A_1,\dots,A_4\rangle$ and $S_B = \langle B, B_1,\dots,B_4\rangle$.

It may be assumed that S_A and S_B have equations $x_0 = 0$ and $x_4 = 0$, respectively. Since the points A and B have rank four,

$$A = [0, a_1, a_2, a_3, 1] \text{ and } B = [1, b_1, b_2, b_3, 0]. \tag{63}$$

The dependence of A, A_1, B, and B_2 gives

$$\operatorname{rk}\begin{pmatrix} a_1^q - a_1 & a_2^q - a_2 & a_3^q - a_3 \\ b_1^{q^2} - b_1 & b_2^{q^2} - b_2 & b_3^{q^2} - b_3 \end{pmatrix} = 1, \tag{64}$$

and G of coordinates $[0, a_1^q - a_1, a_2^q - a_2, a_3^q - a_3, 0]$ is the intersection of the lines $\langle A, A_1\rangle$, and $\langle B, B_2\rangle$.

The coordinates of the solids joining Γ and the points of Σ are the minors of

$$\begin{pmatrix} 0 & a_1 & a_2 & a_3 & 1 \\ 1 & b_1 & b_2 & b_3 & 0 \\ 0 & a_1^q - a_1 & a_2^q - a_2 & a_3^q - a_3 & 0 \\ u_0 & u_1 & u_2 & u_3 & u_4 \end{pmatrix}, \quad (u_0,\dots,u_4) \in (\mathbb{F}_q^5)^*.$$

Intersecting with the line $x_0 = x_1 = x_4 = 0$ gives the following form for the linear set:

$$\Big\{\langle(\, m_3(x_1 + b_1x_0 + a_1x_4) - m_1(x_3 + b_3x_0 + a_3x_4), m_2(x_1 + b_1x_0 + a_1x_4) \\ -m_1(x_2 + b_2x_0 + a_2x_4))\rangle_{\mathbb{F}_{q^5}} : (x_0,\dots,x_4) \in \mathbb{F}_q^5 \setminus \{(0,\dots,0)\}\Big\},$$

where $(m_1, m_2, m_3) = (a_1^q - a_1, a_2^q - a_2, a_3^q - a_3)$. This can be seen as

$$\begin{pmatrix} m_3 & 0 & -m_1 \\ m_2 & -m_1 & 0 \end{pmatrix}\left[x_0\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + x_4\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right],$$

that is, as the projection from the vertex $M = [m_1, m_2, m_3]$ of the linear set in $\mathrm{PG}(2,q^5)$ defined by the $\mathbb{F}_q$-subspace

$$U_{\mathbf{a},\mathbf{b}} = \langle \mathbf{a}, \mathbf{b}\rangle_{\mathbb{F}_q} + \mathbb{F}_q^3, \tag{65}$$

where $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$.

Summarizing:

Theorem 6.1. *Let* $\mathbb{L} = \wp_\Gamma(\Sigma)$ *be a maximum scattered linear set in* $\mathrm{PG}(1,q^5)$ *not of pseudoregulus type, and assume that* $\operatorname{rk} A = \operatorname{rk} B = 4$. *Then* $\mathbb{L}$ *can be represented as a projection from the vertex* $M = [m_1, m_2, m_3]$ *of the linear set in* $\mathrm{PG}(2,q^5)$ *defined by the* $\mathbb{F}_q$*-subspace* $U_{\mathbf{a},\mathbf{b}}$ *in* (65). *Conversely, if* $M = [a_1^q - a_1, a_2^q - a_2, a_3^q - a_3]$ *has rank three in* $\mathrm{PG}(2,q^5)$ *and* (64) *holds, then the projection from* M *of* $U_{\mathbf{a},\mathbf{b}}$ *is, if maximum scattered, a linear set in* $\mathrm{PG}(1,q^5)$ *not of pseudoregulus type such that* $\operatorname{rk} A = \operatorname{rk} B = 4$.

Note that if M has rank three, then $a_1^q - a_1$, $a_2^q - a_2$, $a_3^q - a_3$ are $\mathbb{F}_q$-linearly dependent, and this implies that $A = [0, a_1, a_2, a_3, 1]$ has rank four.

6.2. Algebraic description. As above, $s=1$ may be assumed. For any two distinct points $P_i=\langle(x_0^{(i)},x_1^{(i)},x_2^{(i)},x_3^{(i)},x_4^{(i)})_{\mathcal{B}}\rangle_{\mathbb{F}_{q^5}}$, $i=1,2$, in Σ, the points A, A_1, B_2, P_1, P_2 are independent; equivalently,

$$\begin{vmatrix} 1 & -\lambda^{q^2} & \mu^{q^2} \\ x_2^{(1)} & x_3^{(1)} & x_4^{(1)} \\ x_2^{(2)} & x_3^{(2)} & x_4^{(2)} \end{vmatrix} \neq 0.$$

This implies that the following linear set is scattered, and projectively equivalent to $\mathbb{L}=\wp_\Gamma(\Sigma)$:

$$L=\{\langle(\mu^{q^2}x_2-x_4,\mu^{q^2}x_3+\lambda^{q^2}x_4)\rangle_{\mathbb{F}_{q^5}}:\langle(x_0,\dots,x_4)_{\mathcal{B}}\rangle_{\mathbb{F}_{q^5}}\in\Sigma\}. \tag{66}$$

By (10) and Proposition 3.1, it may be assumed that $\mu=1$.

From (22) and $e=0$:

$$a=b^{q^4+q^2+1},\ c=b^{-q^3},\ d=-b^{q^2+1},\ \mathrm{N}_{q^5/q}(b)=-1.$$

A $w\in\mathbb{F}_{q^5}^*$ exists such that $b=-w^{1-q^3}$. Hence

$$a=-w^{q^4-q^3},\ b=-w^{1-q^3},\ c=-w^{q-q^3},\ d=-w^{q^2-q^3}. \tag{67}$$

If $\lambda\neq1$, then, by (19), $w=\lambda-1$ may be assumed.

The representing vectors for Σ are obtained from $M_1X^q=X$, where

$$M_1=\begin{pmatrix} 0\ 0\ 0 & -w^{q^4-q^3} & 1 \\ 1\ 0\ 0 & -w^{1-q^3} & -\lambda \\ 0\ 1\ 0 & -w^{q-q^3} & 1 \\ 0\ 0\ 1 & -w^{q^2-q^3} & -\lambda^{q^2} \\ 0\ 0\ 0 & 0 & 1 \end{pmatrix} \tag{68}$$

By solving the equations in term of $x_3=t$ and $x_4=\theta$, one obtains $\theta\in\mathbb{F}_q$, and

$$\begin{aligned} x_0&=-w^{q^4-q^3}t^q+\theta\\ x_1&=-w^{1-q^4}t^{q^2}-w^{1-q^3}t^q+(1-\lambda)\theta\\ x_2&=-w^{q-1}t^{q^3}-w^{q-q^4}t^{q^2}-w^{q-q^3}t^q+(2-\lambda^q)\theta\\ &-w^{q^2}(w^{-q}t^{q^4}+w^{-1}t^{q^3}+w^{-q^4}t^{q^2}+w^{-q^3}t^q+w^{-q^2}t)+2(1-\lambda^{q^2})\theta=0. \end{aligned} \tag{69}$$

The last equation is equivalent to

$$w^{q^2}\,\mathrm{Tr}_{q^5/q}(w^{-q^2}t)=2(1-\lambda^{q^2})\theta. \tag{70}$$

The condition $\operatorname{rk}B=4$ implies that the rank of the following matrix containing the coordinates of v, v_1, v_2, v_3, v_4 with respect the basis $\mathcal{B}$ is four:

$$C=\begin{pmatrix} 1 & -\lambda & 1 & -\lambda^{q^2} & 1 \\ -\lambda^{q^3}a+1 & -\lambda^{q^3}b-\lambda+1 & -\lambda^{q^3}c+1-\lambda^q & 1-\lambda^{q^3}d-\lambda^{q^2} & 1 \\ 0 & 0 & 1 & -\lambda^{q^2} & 1 \\ -\lambda^{q^3}a+1 & -\lambda^{q^3}b-\lambda & -\lambda^{q^3}c+1 & 1-\lambda^{q^3}d-\lambda^{q^2} & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The rank of C is four if and only if

$$\lambda^{q^3+q^2+q+1}a+\lambda^{q^3+q^2+q}b+\lambda^{q^3+q^2}c+\lambda^{q^3}d-1=0. \tag{71}$$

For $\lambda = 1$ this leads to $a+b+c+d-1=0$, equivalent by (67) to $\mathrm{Tr}_{q^5/q}(w)=0$. For $\lambda \neq 1$, since $w=\lambda-1$, the equations (67) are equivalent to

$$a=-\frac{\lambda^{q^4}-1}{\lambda^{q^3}-1},\ b=-\frac{\lambda-1}{\lambda^{q^3}-1},\ c=-\frac{\lambda^{q}-1}{\lambda^{q^3}-1},\ d=-\frac{\lambda^{q^2}-1}{\lambda^{q^3}-1}.$$

Combining with (71), $\mathrm{N}_{q^5/q}(\lambda)=1$ results.

The following straightforward result will be used in the proof of the Theorem 6.3.

Proposition 6.2. *Let* $\rho \in \mathbb{F}_{q^5}$ *such that* $\mathrm{Tr}_{q^5/q}(\rho) \neq 0$*; let* $g : \mathbb{F}_{q^5} \to \mathbb{F}_{q^5}$ *be an* $\mathbb{F}_q$*-linear map such that* $\ker g = \mathbb{F}_q$ *and* $\mathrm{Tr}_{q^5/q} \circ g$ *is the zero map. Also let*

$$S=\{(\theta,y)\colon \theta\in\mathbb{F}_q,\ y\in\mathbb{F}_{q^5},\ \mathrm{Tr}_{q^5/q}(y)=0\},$$
$$T=\{(\theta,y)\colon \theta\in\mathbb{F}_q,\ y\in\mathbb{F}_{q^5},\ \mathrm{Tr}_{q^5/q}(y)+2\theta=0\}.$$

Then the maps

$$\alpha:\mathbb{F}_{q^5}\to S,\ x\mapsto(\mathrm{Tr}_{q^5/q}(\rho x),g(x)), \tag{72}$$

$$\beta:\mathbb{F}_{q^5}\to T,\ x\mapsto(\mathrm{Tr}_{q^5/q}(x),-x-x^{q^2}) \tag{73}$$

are well-defined and bijective.

Theorem 6.3. *If* $\mathrm{rk}\,A=\mathrm{rk}\,B=4$*, then* $\mathbb{L}$ *is equivalent to* $\mathbb{E}$ *or* L_F*, where*

$$\mathbb{E}=\{\langle(\eta(x^q-x)+\mathrm{Tr}_{q^5/q}(\rho x),x^q-x^{q^4})\rangle_{\mathbb{F}_{q^5}} : x\in\mathbb{F}^*_{q^5}\}, \tag{74}$$
$$\eta\neq 0,\ \mathrm{Tr}_{q^5/q}(\eta)=0\neq\mathrm{Tr}_{q^5/q}(\rho),$$

$$F(x)=k(x^q+x^{q^3})+x^{q^2}+x^{q^4},\quad \mathrm{N}_{q^5/q}(k)=1. \tag{75}$$

Proof. Let $\eta=w^{q^2}$ and $y=tw^{-q^2}$. Note that (70) is equivalent to $\mathrm{Tr}_{q^5/q}(y)=0$ for $\lambda=1$, and to

$$\mathrm{Tr}_{q^5/q}(y)+2\theta=0 \tag{76}$$

for $\lambda\neq 1$. By combining (66), (69), the elements of $\mathbb{L}$ are of type

$$\langle(\eta y+\lambda^{q^2}\theta,-\eta^{q^4}(y^q+y^{q^2}+y^{q^3})+(1-\lambda^q)\theta)\rangle_{\mathbb{F}_{q^5}}.$$

For $\lambda=1$, it follows

$$\{\langle(\eta y+\theta,y+y^{q^4})\rangle_{\mathbb{F}_{q^5}} : \theta\in\mathbb{F}_q,\ y\in\mathbb{F}_{q^5},\ \mathrm{Tr}_{q^5/q}(y)=0\}. \tag{77}$$

The form (74) can be obtained by the substitution $(\theta,y)=(\mathrm{Tr}_{q^5/q}(\rho x),x^q-x)$ in (77) according to Proposition 6.2.

For $\lambda\neq 1$, by (76), the pairs are of type

$$(\eta y+(\eta+1)\theta,\eta^{q^4}(y+y^{q^4}+\theta)).$$

Now let us transform the pairs as follows:

$$\begin{pmatrix}0 & \eta^{-q^4}\\ 1 & -\eta^{-q^4+1}-\eta^{-q^4}\end{pmatrix}\begin{pmatrix}\eta y+(\eta+1)\theta\\ \eta^{q^4}(y+y^{q^4}+\theta)\end{pmatrix}=\begin{pmatrix}y+y^{q^4}+\theta\\ -y-ky^{q^4}\end{pmatrix}$$

for $k=1+\eta=\lambda^{q^2}$. By substituting $(\theta,y)=\beta(x)$ we get $(x^{q^3},x+x^{q^2}+k(x^{q^4}+x^q)$, equivalent to L_F.

The norm of k is one as it has been noted as a consequence of (71). □

Proposition 6.4. *If* $\mathrm{Tr}_{q^5/q}(\eta^{-1})\neq 0$*, then the linear set* $\mathbb{E}$ *in* (74) *is projectively equivalent to* L_{F_1} *where*

$$F_1(x)=\mathrm{Tr}_{q^5/q}(\eta^{-1})x^{q^4}-(\eta^{-q^4}+\eta^{-1})\,\mathrm{Tr}_{q^5/q}(x). \tag{78}$$

Proof. Let $\rho = \eta^{-1}$, and

$$g(x) = \rho\left(\mathrm{Tr}_{q^5/q}(\rho)x - \mathrm{Tr}_{q^5/q}(\rho x)\right).$$

Clearly $\mathbb{F}_q \subseteq \ker g(x)$, and $\mathrm{Tr}_{q^5/q}(g(x)) = 0$ for all $x \in \mathbb{F}_{q^5}$. Taking into account (77), for $\theta = \mathrm{Tr}_{q^5/q}(\rho x)$ and $y = g(x)$ we have

$$\eta y + \theta = \mathrm{Tr}_{q^5/q}(\rho)x.$$

In particular, the sum of $\eta g(x)$ with an $\mathbb{F}_q$-linear map of rank one is bijective, and this implies that the rank of $g(x)$ is at least four. Therefore, $\ker g = \mathbb{F}_q$ and the hypotheses of Proposition 6.2 are satisfied. The substitution $y = g(x)$ in $y + y^4$, neglecting the first-degree term, gives

$$\left(\mathrm{Tr}_{q^5/q}(\rho)x, \eta^{-q^4}\,\mathrm{Tr}_{q^5/q}(\eta^{-1})x^{q^4} - (\eta^{-q^4} + \eta^{-1})\,\mathrm{Tr}_{q^5/q}(\eta^{-1}x)\right)$$

and leads to (78). ☐

Remark 6.5. The linear sets of type (74) and (75) have been analyzed by a GAP script for $q \leqslant 25$. None of them is scattered. The script that performed this check can be found at the URL https://pastebin.com/TjrKuu0z.

In particular, for $q \leqslant 9$ and $\mathrm{N}_{q^5/q}(k) = 1 \neq k$, the linear set L_F has points with weight at most 2, i.e., each ratio $F(x)/x$ occurs for at most $q^2 - 1$ distinct non-zero values of $x \in \mathbb{F}_{q^5}^*$. On the other hand, for $k = 1$, L_F is equivalent to the linear set associated with the trace function $\mathrm{Tr}_{q^5/q}(x)$, see [4].

7. Conclusion

In this paper, starting from the results in [22], we proved that the maximum scattered linear sets of $\mathrm{PG}(1, q^5)$ are always of LP type if the rank of A or B is five. The case in which $\mathrm{rk}\,A = \mathrm{rk}\,B = 4$ remains open. In this case, we can describe their form and/or the polynomial that defines them. Any linear set with this shape would be a new type. We have not established their existence and conjecture that they do not exist; in any case, we have ruled out their existence for $q \leqslant 25$. The following theorem summarizes our work. To provide the reader with more insight, we extend the proof to include a description of the steps that led to the result.

Theorem 7.1. *If $\mathbb{L}$ is a maximum scattered linear set in $\mathrm{PG}(1, q^5)$ not of pseudoregulus type, then $\mathbb{L}$ is the projection of a canonical $\mathbb{F}_q$-subgeometry $\Sigma \subseteq \mathrm{PG}(4, q^5)$ from a plane Γ such that $\Gamma \cap \Sigma = \varnothing$. Let σ be a generator of the subgroup $\mathbb{G}$ of $\mathrm{P\Gamma L}(5, q^5)$ fixing Σ pointwise. Define $A = \Gamma \cap \Gamma^{\sigma^4}$ and $B = \Gamma \cap \Gamma^{\sigma^3}$. Then, A and B are points. If $\mathrm{rk}\,A = 5$ or $\mathrm{rk}\,B = 5$, then $\mathbb{L}$ is of LP type. Otherwise $q > 25$, $\mathbb{L}$ is not of LP type and is, up to equivalence, of shape (74) or L_F where $F(x)$ is as in (75).*

Proof and summary of the paper. Let $\mathbb{L}$ be a maximum scattered linear set of $\mathrm{PG}(1, q^5)$, not of pseudoregulus type. By Theorems 1.1 and 1.3, $\mathbb{L}$ is the projection of a canonical $\mathbb{F}_q$-subgeometry Σ of $\mathrm{PG}(4, q^5)$ from a plane Γ. The group $\mathbb{G}$ of collineations fixing Σ pointwise is a cyclic group of order five. By Theorem 1.3, the intersection $\Gamma \cap \Gamma^{\sigma}$ is a point of $\mathrm{PG}(4, q^5)$ for every generator σ of $\mathbb{G}$. This allows us to define two points $A = \Gamma \cap \Gamma^{\sigma^4}$ and $B = \Gamma \cap \Gamma^{\sigma^3}$ and the elements of their orbits $A_i = A^{\sigma^i}$, $B_i = B^{\sigma^i}$, $i = 1, 2, 3, 4$.

The point $A = A_{\sigma,\Gamma}$ depends on the vertex Γ and the generator σ of the collineations group $\mathbb{G}$, and the proven properties hold for every σ. At the beginning of Section 4, we show that if $\mathrm{rk}\,B = 5$, then by replacing σ with another generator τ if necessary, we have $\mathrm{rk}\,A_{\tau,\Gamma} = 5$. For this reason, if $\max\{\mathrm{rk}\,A, \mathrm{rk}\,B\} = 5$, it may be assumed without loss of generality that $\mathrm{rk}\,A = 5$.

Various properties of these points have been studied in Section 2. Overall, the ten points of the orbits of A and B under the action of $\mathbb{G}$ are distinct, and precisely four of them belong to every Γ^{σ^i}. The linear set $\mathbb{L}$ is of type LP if, and only if, the line A_2A_4 intersects Γ (configuration I) or the line B_3B_4 intersects Γ (configuration II), cf. Theorem 2.3. In Section 3, we studied the properties of the vectors representing the above points, linked by equations (6) and (9). We then found properties of $\mathbb{L}$, expressed in terms of vector coordinates of a basis associated with points A, A_1, A_2, A_3, B_4. Using these coordinates, equations (33) allow us to find coordinates of point A_4 in the case $\mathrm{N}_{q^5/q}(\mu)\neq 1$, which leads to a geometric property of fundamental importance in this work: The line A_3B_1 intersects Γ. As a consequence of this, in Proposition 4.8 we have shown that if $\mathbb{L}$ is not of type LP and $\operatorname{rk} A=5$, then, up to projectivities,

$$\mathbb{L}=\{\langle(x-k^{-1}x^{q^{2s}},x^{q^s}-\delta k^{q^{4s}+q^{2s}}x^{q^{2s}})\rangle_{\mathbb{F}_{q^5}}:x\in\mathbb{F}^*_{q^5}\}$$

for some $k\in\mathbb{F}^*_{q^5}$ and $\delta\in\mathbb{F}_q$. In Theorem 4.9, we saw that, for $\varepsilon=\mathrm{N}_{q^5/q}(k)$, it follows that

$$\begin{cases}\delta^2\varepsilon\neq 1\\ \delta^3\varepsilon^2+(1-3\delta)\varepsilon+1\neq 0,\end{cases}\tag{79}$$

and the system of equations

$$\begin{cases}\varepsilon\,\mathrm{N}_{q^5/q}(x)=-1\\ \delta^2\varepsilon x^{q^s}-\delta\varepsilon x^{q^{2s}+q^s+1}(1-x)^{q^{3s}}+(1-x)^{q^s+1}=0\end{cases}\tag{36}$$

have no solution $x\in\mathbb{F}_{q^5}$. However, Theorem 5.1 shows that conditions (79) imply that the system of equations (36) admit a solution. The result is achieved by translating the existence of solutions into the existence of $\mathbb{F}_q$-rational points on an algebraic curve. Therefore, if $\operatorname{rk} A=5$, then $\mathbb{L}$ is of type LP. For the reasons given above, $\mathbb{L}$ is of type LP also if $\operatorname{rk} B=5$.

By virtue of Theorem 6.3, if $\operatorname{rk} A=\operatorname{rk} B=4$, then $\mathbb{L}$ is equivalent to the linear set described in (74), or to L_F where $F(x)$ is the polynomial (75). The linear sets of those two types were analyzed using a GAP script for $q\leqslant 25$ and, as mentioned in Remark 6.5, none of them are scattered. ☐

As a consequence, we have the following result.

Corollary 7.2. *Any maximum scattered linear set (MSLS) in* $\mathrm{PG}(1,q^5)$ *is, up to equivalence in* $\mathrm{P\Gamma L}(2,q^5)$*, one of the following:*

(C1) *a MSLS of pseudoregulus type,* $\{\langle(x,x^q)\rangle_{\mathbb{F}_{q^5}}:x\in\mathbb{F}^*_{q^5}\}$*,*

(C2) *a MSLS of LP type,* $\{\langle(x,x^{q^s}+\delta x^{q^{5-s}})\rangle_{\mathbb{F}_{q^5}}:x\in\mathbb{F}^*_{q^5}\}$*,* $s\in\{1,2\}$*,* $\mathrm{N}_{q^5/q}(\delta)\neq 0,1$*,*

(C3)

$$\begin{gathered}\{\langle(\eta(x^q-x)+\mathrm{Tr}_{q^5/q}(\rho x),x^q-x^{q^4})\rangle_{\mathbb{F}_{q^5}}:x\in\mathbb{F}^*_{q^5}\},\\ \eta\neq 0,\ \mathrm{Tr}_{q^5/q}(\eta)=0\neq\mathrm{Tr}_{q^5/q}(\rho),\end{gathered}$$

(C4)

$$\{\langle(x,k(x^q+x^{q^3})+x^{q^2}+x^{q^4})\rangle_{\mathbb{F}_{q^5}}:x\in\mathbb{F}^*_{q^5}\},\quad \mathrm{N}_{q^5/q}(k)=1.$$

Note that the classes of sets of types (C3) and (C4) might be empty, as they actually are for $q\leqslant 25$.

Acknowledgements. The first author acknowledges the support by the Irish Research Council, grant n. GOIPD/2022/307. The second author is supported by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA - INdAM) and by the European Union under the Italian National Recovery and Resilience Plan (NRRP) of NextGenerationEU, with particular reference to the partnership on "Telecommunications of the Future" (PE00000001 - program "RESTART", CUP: D93C22000910001).

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