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Spencer Backman, Sebastian Manecke & Raman Sanyal

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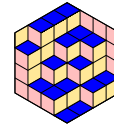


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ABSTRACT Interior and exterior angle vectors of polytopes capture curvature information at faces of all dimensions and can be seen as metric variants of f -vectors. In this context, Gram's relation takes the place of the Euler–Poincaré relation as the unique linear relation among interior angles. We show the existence and uniqueness of Euler–Poincaré-type relations for generalized angle vectors by building a bridge to the algebraic combinatorics of geometric lattices, generalizing work of Klivans–Swartz.

We introduce flag-angles of polytopes as a geometric counterpart to flag- f -vectors. Flag-angles generalize the angle deficiencies of Descartes–Shephard, Grassmann angles, and spherical intrinsic volumes. Using the machinery of incidence algebras, we relate flag-angles of zonotopes to flag- f -vectors of graded posets. This allows us to determine the linear relations satisfied by interior/exterior flag-angle vectors.

1. INTRODUCTION

For a convex polytope $P \subset \mathbb{R}^d$ of dimension d , let $f_i(P)$ be the number of i -dimensional faces of P for $i = 0, 1, \dots, d-1$. The Euler–Poincaré relation states that the face numbers satisfy

$$(1) \quad f_0(P) - f_1(P) + f_2(P) - \dots + (-1)^{d-1} f_{d-1}(P) = 1 - (-1)^d.$$

This simple linear relation among the face numbers is the key to a rich interplay of geometry, combinatorics, and algebra as amply demonstrated, for example, in [22, 39]. In particular, the Euler–Poincaré relation (1) is, up to scaling, the only linear relation among the face numbers of polytopes of fixed dimension d ; cf. [21], [18, Sect. 8.1]. In this paper, we will show the existence and uniqueness of Euler–Poincaré-type relations for certain class of *semi-discrete* invariants and their interplay with algebraic combinatorics. These invariants are generalizations of the well-known *interior* and *exterior* angle vectors of polytopes.

The local geometry of P at a face F is captured by the **tangent cone** (or **inner cone**) $T_F P := \text{cone}(-F + P)$, which is the cone of feasible directions from F . The **interior angle** of P at F is the spherical volume

$$\hat{\nu}(F, P) := \nu(T_F P) = \frac{\text{vol}(T_F P \cap B_d)}{\text{vol}(B_d)},$$

which generalizes the notion of dihedral angle to faces of all dimensions. The i -th interior angle of P is given by $\hat{\nu}_i(P) := \sum_F \hat{\nu}(F, P)$, where F ranges over all faces of

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dimension i . The **interior angle vector** $\widehat{\nu}(P) = (\widehat{\nu}_i(P))_{i=0,\dots,d-1}$ captures curvature information at faces of various dimensions and combines combinatorial as well as metric properties of P . The fundamental relation, called **Gram’s relation**, which is satisfied by interior angle vectors of d -dimensional polytopes is

$$(2) \quad \widehat{\nu}_0(P) - \widehat{\nu}_1(P) + \widehat{\nu}_2(P) - \cdots + (-1)^{d-1}\widehat{\nu}_{d-1}(P) = (-1)^{d+1}.$$

We refer to Grünbaum [18, Sect. 14.4] for a detailed historic account of Gram’s relation and its importance.

Gromov and Milman [16] introduced and studied an *anisotropic* notion of angle of a cone $C \subset \mathbb{R}^d$

$$\nu_K(C) := \frac{\text{vol}(C \cap K)}{\text{vol}(K)},$$

where $K \subset \mathbb{R}^d$ is a fixed centrally-symmetric convex body. These *cone (probability) measures* are related to surface measures of general (star) convex bodies [28, 27, 4]. Using the integral-geometric perspective developed by Perles–Shephard [30] (see also [42]), one easily shows that Gram’s relation is the essentially unique linear relation for the associated interior angles $\widehat{\nu}_K(P)_i = \sum_F \nu_K(\text{T}_F P)$ of d -dimensional polytopes.

In the first part of the paper, we prove the existence and uniqueness of Gram’s relation for a more general class of cone angles by way of an astounding connection to the combinatorics of posets and matroids.

A map $\alpha : \mathcal{C}_d \rightarrow R$ from the collection of convex polyhedral cones \mathcal{C}_d in \mathbb{R}^d into some unital ring R is a **valuation** if $\alpha(\{0\}) = 0$ and

$$\alpha(C \cup C') = \alpha(C) + \alpha(C') - \alpha(C \cap C')$$

for all $C, C' \in \mathcal{C}_d$ such that $C \cup C', C \cap C' \in \mathcal{C}_d$. A valuation α is **simple** if $\alpha(C) = 0$ whenever $\dim C < d$ and we call α a **cone angle** if in addition $\alpha(\mathbb{R}^d) = 1$. Cone valuations play a decisive role in integral geometry [36] and cone angles strictly subsume cone probability measures; see Section 2. Note that we do not require cone angles to be rotationally invariant or to satisfy any positivity conditions. The **interior α -angle** of a polytope P at a face $F \subseteq P$ is then defined as $\widehat{\alpha}(F, P) := \alpha(\text{T}_F P)$. We also define the **exterior α -angle** of P at F as $\check{\alpha}(F, P) := \alpha(\text{N}_F P + \text{aff}_0(F))$, where $\text{aff}_0(F)$ is the linear subspace parallel to F and $\text{N}_F P$ is the **normal cone** of P at F , that is, the cone polar to $\text{T}_F P$. As expected, the interior and exterior **α -angle vector** of P are defined through

$$\widehat{\alpha}_i(P) := \sum_F \widehat{\alpha}(F, P) \quad \text{and} \quad \check{\alpha}_i(P) := \sum_F \check{\alpha}(F, P),$$

where the sums are over all faces $F \subseteq P$ of dimension i , for $i = 0, \dots, d - 1$. Our first main result is this.

THEOREM 1.1. *Let α be a cone angle. Then, up to scaling, the only linear relations satisfied by $\widehat{\alpha}(P)$, respectively $\check{\alpha}(P)$, for any d -dimensional polytope P are*

$$(3) \quad \widehat{\alpha}_0(P) - \widehat{\alpha}_1(P) + \widehat{\alpha}_2(P) - \cdots + (-1)^{d-1}\widehat{\alpha}_{d-1}(P) = (-1)^{d+1},$$

$$(4) \quad \check{\alpha}_0(P) = \sum_v \check{\alpha}(v, P) = 1.$$

Showing the validity of both relations is not difficult. Indeed, in [30] a proof of (2) is sketched that works for general cone angles. For completeness, we give a proof using a conical version of the Brianchon–Gram relation of [1]; see also [24, 37]. The main challenge in proving Theorem 1.1 is uniqueness, as none of the analytic and geometric properties of ν_K carry over to general cone angles. We prove Theorem 1.1

by establishing a powerful connection between the geometry and the combinatorics of zonotopes.

To explain the combinatorial connection, define $\mathbf{L}(F) := \text{aff}_0(F)^\perp$ for any non-empty face $F \subseteq P$ and let $\mathcal{L}(P) := \{\mathbf{L}(F) : \emptyset \neq F \subseteq P\}$ partially ordered by *reverse* inclusion. This is a finite graded poset of rank d . In particular, if P is a zonotope, that is, a Minkowski-sum of segments, then $\mathcal{L}(P)$ is a geometric lattice, called **lattice of flats**. The **Whitney numbers** of the first kind w_i and of the second kind W_i of a graded poset are important enumerative invariants [40, Sect. 3.10], whose precise definition we recall in Section 3. The following result allows us to show the uniqueness in Theorem 1.1 on a purely combinatorial level.

THEOREM 1.2. *Let $Z \subset \mathbb{R}^d$ be a d -dimensional zonotope with lattice of flats $\mathcal{L} = \mathcal{L}(Z)$. For any cone angle α we have*

$$\check{\alpha}_i(Z) = W_i(\mathcal{L}(Z)) \quad \text{and} \quad \hat{\alpha}_i(Z) = (-1)^{d-i} w_{d-i}(\mathcal{L}(Z)^{\text{op}})$$

for all $i = 0, \dots, d - 1$.

For the standard cone angle, the second equation in Theorem 1.2 was proven in Klivans–Swartz [23] using ideas similar to those in [30] involving projections of zonotopes. Theorem 1.2 is then used to show that interior/exterior angle vectors of zonotopes certify the uniqueness of (3) and (4). Theorem 1.2 and Theorem 1.1 are proved in Section 3.

In Section 4, we recast the correspondence between interior/exterior angles and Whitney numbers of the first and second kind in algebraic terms. McMullen [25] showed that $\hat{\alpha}, \check{\alpha}$ can be interpreted as elements in the incidence algebra $\mathcal{I}(\mathcal{F}(P))$ of the face poset $\mathcal{F}(P)$ of P . In Section 4, we show that for a zonotope Z with lattice of flats \mathcal{L} , there is a subalgebra $\mathcal{I}_{\mathcal{L}} \subseteq \mathcal{I}(\mathcal{F}(Z))$ such that the map $F \mapsto \mathbf{L}(F)$ yields a ring map $\mathbf{L}_* : \mathcal{I}_{\mathcal{L}} \rightarrow \mathcal{I}(\mathcal{L})$. We show that $\check{\alpha} \in \mathcal{I}_{\mathcal{L}}$ and $\mathbf{L}_*\check{\alpha} = \zeta_{\mathcal{I}(\mathcal{L})}$. McMullen’s *inverse angles* [26] then allow us to show $\mathbf{L}_*\hat{\alpha}' = \mu_{\mathcal{I}(\mathcal{L})}$, where $\hat{\alpha}'$ is a slight modification of $\hat{\alpha}$. This yields an elegant algebraic proof of Theorem 1.2 and explains the appearance of Whitney numbers of the first and second kind. In addition, we give a simple proof of a beautiful relation due to Klivans and Swartz [23] between spherical intrinsic volumes of a zonotope Z and the characteristic polynomial of $\mathcal{L}(Z)$; see Corollary 4.5.

The second goal of the paper is to introduce *flag-angle vectors* as a unifying geometric concept and to exhibit and exploit parallels to the theory of flag-vectors of posets. To motivate flag-angle vectors, let P be a 3-dimensional polytope. Descartes defined the *angle defect* at a vertex v as $\delta(v, P) = 1 - \sum_F \hat{\nu}(v, F)$, where the sum is over all 2-dimensional faces F containing v and he showed that

$$\sum_v \delta(v, P) = 2.$$

For a d -polytope P and $i = 0, \dots, d - 3$, Shephard [38] defines the *i -th total angle deficiency*

$$\delta_i(P) := f_i(P) - \sum_{G \subset F} \hat{\nu}(G, F),$$

where the sum is over all faces $G \subset F$, with $\dim G = i$ and $\dim F = d - 1$. Generalizing Descartes’ result, Shephard showed that

$$(5) \quad \delta_0(P) - \delta_1(P) + \dots + (-1)^{d-3} \delta_{d-3}(P) = 1 + (-1)^{d-1}.$$

Such generalized Descartes-relations were further studied for manifolds [19] and in relation with stratified curvature [10] and polyhedral Gauss–Bonnet theorems [34].

In contrast to interior angle vectors, total angle deficiencies record the interaction of faces of various dimensions and the generalized Descartes–relation (5) shows that

angle deficiencies are not independent. In a different direction, McMullen [25] showed that exterior angles can be computed from interior angles of flags of faces

$$(6) \quad (-1)^d \check{\nu}_i(P) = \sum_{F_1 \subset F_2 \subset \dots \subset F_k} (-1)^{k+1} \hat{\nu}(F_1, F_2) \hat{\nu}(F_2, F_3) \cdots \hat{\nu}(F_k, P),$$

where the sum is over all flags of faces with $\dim F_1 = i$. Moreover, McMullen showed that various other measures of curvature, such as spherical intrinsic volumes and Grünbaum’s *Grassmann angles* [17] can be computed from chains of interior (or exterior) angles.

DEFINITION 1.3. *Let α be a cone angle. For a d -dimensional polytope P and a non-empty set $S = \{0 \leq s_1 < s_2 < \dots < s_k \leq d - 1\}$, define the **interior flag-angle** by*

$$(7) \quad \hat{\alpha}_S(P) := \sum_{F_1 \subset F_2 \subset \dots \subset F_k} \hat{\alpha}(F_1, F_2) \hat{\alpha}(F_2, F_3) \cdots \hat{\alpha}(F_k, P),$$

where the sum is over all chains of faces of P such that $\dim F_i = s_i$ for $i = 1, \dots, k$. The **exterior flag-angle** $\check{\alpha}_S(P)$ is defined analogously and we set $\check{\alpha}_\emptyset(P) := \hat{\alpha}_\emptyset(P) := 1$.

The vectors $\hat{\alpha}(P) = (\hat{\alpha}_S(P))_S$ and $\check{\alpha}(P) = (\check{\alpha}_S(P))_S$ are called the interior and exterior **flag-angle vectors** of P .

In Sections 5 and 6, we determine the affine spaces spanned by interior and exterior flag-angle vectors, respectively.

THEOREM 1.4. *Let P be a d -dimensional polytope and $S \subseteq [d - 1] := \{1, 2, \dots, d - 1\}$. For any cone angle α , we have*

$$\check{\alpha}_S(P) = \check{\alpha}_{S \cup \{0\}}(P) \quad \text{and} \quad \sum_{i=0}^{t-1} (-1)^i \hat{\alpha}_{S \cup \{i\}}(P) = (-1)^{t+1} \hat{\alpha}_S(P),$$

where $t = \min(S \cup \{d\})$.

Moreover, the affine hull of exterior flag-angles as well as the affine hull of interior flag-angles is of dimension $2^{d-1} - 1$. These spaces are spanned by the flag-angle vectors of zonotopes.

Since $\delta_i(P) = f_i(P) - 2\hat{\nu}_{i,d-1}(P)$, Theorem 1.4 for $S = \{d - 1\}$ together with the Euler–Poincaré relation (1) implies Shephard’s result (5). Theorem 1.4 also determines the spaces of linear relations on all specializations of flag-angles, including spherical intrinsic volumes and Grassmann angles.

Flag angle vectors are *semi-discrete* counterparts to *flag vectors* of polytopes and posets. Let \mathcal{P} be a graded poset of rank $d + 1$ with minimal element $\mathbf{0}$ and maximal element $\mathbf{1}$. For $S \subseteq [d]$, the number of chains of elements

$$\mathbf{0} \prec_{\mathcal{P}} c_1 \prec_{\mathcal{P}} c_2 \prec_{\mathcal{P}} \cdots \prec_{\mathcal{P}} c_k \prec_{\mathcal{P}} \mathbf{1}$$

such that $S = \{\text{rk } c_1, \text{rk } c_2, \dots, \text{rk } c_k\}$ is called the **flag-Whitney number** of the **second kind** $W_S(\mathcal{P})$. The resulting vector $\mathbf{W}(\mathcal{P}) = (W_S(\mathcal{P}))_S$ is commonly known as the **flag-vector** of \mathcal{P} . Flag-vectors of face posets of polytopes or, more generally, of Eulerian posets have received considerable attention, starting with the seminal paper Bayer–Billera [6]. Bayer and Billera determined the linear relations on flag-vectors of Eulerian posets, called the *generalized Dehn–Sommerville relations* and showed that flag-vectors of Eulerian posets of rank $d + 1$ span an affine space of dimension F_d , where F_d is the d -th Fibonacci number. Billera–Ehrenborg–Readdy [8] showed that these spaces are spanned by the flag vectors of d -dimensional zonotopes.

To complete the relation to flag vectors, we introduce the **flag-Whitney numbers** of the **first kind** $w_S(\mathcal{P})$, which extend the ordinary Whitney numbers $w_i(\mathcal{P})$ to flags. They are obtained via inclusion-exclusion from the flag-vector and are complementary to the usual *flag h-vector*. The following result extends Theorem 1.2 to flag-angle vectors. We set $d - S := \{d - s : s \in S\}$.

THEOREM 1.5. *Let α be a cone angle and $Z \subset \mathbb{R}^d$ a full-dimensional zonotope. Then for a nonempty $S \subseteq \{0, 1, \dots, d - 1\}$, we have*

$$\check{\alpha}_S(Z) = W_S(\mathcal{L}(Z)) \quad \text{and} \quad \hat{\alpha}_S(Z) = (-1)^{d-r} w_{d-S}(\mathcal{L}(Z)^{\text{op}}),$$

where $r = \min(S)$.

We prove Theorems 1.4 and 1.5 in Section 5 using the algebraic tools developed in Section 4. As before, Theorem 1.5 is the key to proving the main statement of Theorem 1.4. Corollary 5.4 extends McMullen’s relation (6) to flag-angles, which then shows that it suffices to only consider exterior flag-angle vectors. This amounts to showing that there are no linear relations on flag-vectors of $\mathcal{L}(Z)$ for d -dimensional zonotopes Z , which we do in Theorem 6.1. This strengthens a result of Billera–Hetyei [9]. Since the flag-vector of a zonotope is encoded by the flag-vector of its lattice of flats by results in [5], Theorem 6.1 also strengthens the main result in Billera–Ehrenborg–Readdy [8] in that flag-vectors of Eulerian posets are spanned by the flag vectors of zonotopes. In Section 7 we revisit Grassmann-angle and spherical intrinsic volumes from the perspective of Crofton-type formulas and introduce a generalization for cone angles. This allows us to generalize the main result of Grünbaum’s fundamental paper [17] on Grassmann angles of polytopes.

2. CONE ANGLES AND LINEAR RELATIONS

In this section, we recall the basic geometric constructions and prove the existence of the relations stated in Theorem 1.1.

Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope and $q \in P$. The **tangent cone** of P at q is

$$T_q P := \{u \in \mathbb{R}^d : q + \varepsilon u \in P \text{ for some } \varepsilon > 0\} = \text{cone}(-q + P).$$

It is easy to verify that $T_q P = T_{q'} P$ if and only if $q, q' \in \text{relint}(F)$ for some face $F \subseteq P$ and we recover $T_F P = T_q P = \text{cone}(-F + P)$. Tangent cones capture the local geometry of P at F in that faces of $T_F P$ are in correspondence to faces $G \supseteq F$ under the correspondence $G \mapsto T_F G$. If P is not full-dimensional, we extend the definition to

$$T_F P = \text{cone}(-F + P) + \text{aff}_0(P)^\perp.$$

The **normal cone** of a face $F \subseteq P$ is the polyhedral cone

$$N_F P := \{c \in \mathbb{R}^d : \langle c, x \rangle \geq \langle c, y \rangle \text{ for all } x \in F, y \in P\}.$$

By construction, $N_F P$ is contained in $\text{aff}_0(F)^\perp$. We define the **outer cone** of P at F as the d -dimensional cone

$$O_F P := N_F P + \text{aff}_0(F).$$

As stated in the introduction, a cone angle $\alpha : \mathcal{C}_d \rightarrow R$ is a simple valuation normalized so that $\alpha(\mathbb{R}^d) = 1$. Cone probability measures $\nu_K(C) = \frac{\text{vol}(C \cap K)}{\text{vol}(K)}$ for K a full-dimensional convex body are cone angles, but the notion of cone angles is richer. For example, for any point $q \in \mathbb{R}^d$, let $B_\varepsilon(q)$ be the ball with radius $\varepsilon > 0$ centered at q . Then

$$\omega_q(C) := \lim_{\varepsilon \rightarrow 0} \frac{\text{vol}(B_\varepsilon(q) \cap C)}{\text{vol}(B_\varepsilon(q))}$$

defines a cone angle which, for $q \neq 0$ does not come from a measure. Further instances can be obtained, for example, from the construction of Dehn–Hadwiger functionals [20, Sect. 2.2.2].

The following standard construction yields a *universal* cone valuation. Let $\mathbb{Z}\mathcal{C}_d$ be the free abelian group with generators e_C for $C \in \mathcal{C}_d$. Let $\mathcal{U} \subset \mathbb{Z}\mathcal{C}_d$ be the subgroup generated by

$$e_{C \cup D} + e_{C \cap D} - e_C - e_D$$

for all cones $C, D \in \mathcal{C}_d$ such that $C \cup D, C \cap D \in \mathcal{C}_d$ and let \mathcal{S} be the subgroup generated by all elements e_C for which $\dim C < d$. We call $\mathbb{S} := \mathbb{Z}\mathcal{C}_d / (\mathcal{U} + \mathcal{S})$ the **simple cone group**. If h is a homomorphism from \mathbb{S} to some abelian group G , then $\phi_h : \mathcal{C}_d \rightarrow G$ with $\phi_h(C) := h(e_C)$ is a simple valuation. Volland [41] essentially showed that every valuation lifts to a homomorphism on \mathbb{S} . We record this as follows.

THEOREM 2.1 (Volland). *The map $\mathcal{C}_d \rightarrow \mathbb{S}$ given by $C \mapsto e_C$ is the universal cone valuation: For any simple valuation $\phi : \mathcal{C}_d \rightarrow G$, there is a homomorphism $h : \mathbb{S} \rightarrow G$ such that $\phi = \phi_h$.*

By work of Grömer [15] we can identify elements in $\mathbb{Z}\mathcal{C}_d / \mathcal{U}$ with linear combinations of indicator functions $f = \sum_{i=1}^k a_i [C_i]$ where $C_1, \dots, C_k \in \mathcal{C}_d$ and $a_1, \dots, a_k \in \mathbb{Z}$.

COROLLARY 2.2. *Let $f = \sum_i a_i [C_i]$ and $f' = \sum_i a'_i [C'_i]$. Then $f = f'$ in \mathbb{S} if and only if $f(p) = f'(p)$ for almost all $p \in \mathbb{R}^d$.*

We will make extensive use of this correspondence. In particular, we can define the universal interior and exterior angle vectors. We will describe them a little differently. Let $\mathbb{S}[t]$ be the abelian group of formal polynomials in t with coefficients in \mathbb{S} . For a polytope P , we define

$$\mathbb{T}_P(t) := \sum_F [\mathbb{T}_F P] t^{\dim F} \quad \text{and} \quad \mathbb{O}_P(t) := \sum_F [\mathbb{O}_F P] t^{\dim F},$$

where in both cases the sum is over all nonempty faces $F \subset P$. Thus, if α is a cone angle, then $\hat{\alpha}(P)$ is naturally identified with the coefficients of $\alpha(\mathbb{T}_P)$. Here is the first benefit.

PROPOSITION 2.3. *Let α be a cone angle and $P \subset \mathbb{R}^d$ a full-dimensional polytope. Then $\check{\alpha}_0(P) = 1$.*

Proof. Note that $\mathbb{O}_v P = \mathbb{N}_v P$ for any vertex $v \in P$. Now, for a general $c \in \mathbb{R}^d$, the linear function $x \mapsto \langle c, x \rangle$ will be maximized at a unique vertex of P . Corollary 2.2 implies

$$\sum_v [\mathbb{O}_v P] = [\mathbb{R}^d]$$

as elements in \mathbb{S} . Applying α to both sides of the equation finishes the proof. \square

To complete the first half of Theorem 1.1, recall that the **homogenization** of a polytope $P \subset \mathbb{R}^d$ is the polyhedral cone

$$\text{hom}(P) := \text{cone}(P \times \{1\}) = \{(x, t) \in \mathbb{R}^d \times \mathbb{R} : t \geq 0, x \in tP\}$$

Every face $\{0\} \neq F' \subseteq \text{hom}(P)$ is of the form $\text{hom}(F)$ for some nonempty face $F \subseteq P$. In particular, the definition of tangent cones extends to faces of $\text{hom}(P)$. Moreover,

$$\mathbb{T}_{F'} P \cong \mathbb{T}_{F'} \text{hom}(P) \cap \{(x, t) : t = 0\}.$$

Note that for $F' = \{0\}$, we have $\mathbb{T}_{F'} \text{hom}(P) = \text{hom}(P)$ and hence

$$\mathbb{T}_{F'} \text{hom}(P) \cap \{(x, t) : t = 0\} = \{(0, 0)\}.$$

On the other hand, if $F' = C$, then $\mathbb{T}_C C = \mathbb{R}^{d+1}$. A Brianchon–Gram relation for polyhedral cones was proved in [1].

LEMMA 2.4 ([1, Lem. 4.1]). Let $C \subseteq \mathbb{R}^{d+1}$ be a full-dimensional cone. Then as functions on \mathbb{R}^{d+1}

$$\sum_{F'} (-1)^{\dim F'} [\mathbb{T}_{F'} C] = (-1)^{d+1} [\text{int}(-C)],$$

where the sum is over all nonempty faces $F' \subseteq C$ and $\text{int}(-C)$ denotes the interior of $-C$.

The following proposition proves the first half of Theorem 1.1.

PROPOSITION 2.5. Let α be a cone angle on \mathbb{R}^d and let $P \subset \mathbb{R}^d$ be a full-dimensional polytope. Then

$$\widehat{\alpha}_0(P) - \widehat{\alpha}_1(P) + \widehat{\alpha}_2(P) - \cdots + (-1)^{d-1} \widehat{\alpha}_{d-1}(P) = (-1)^{d+1}.$$

Proof. Let $C = \text{hom}(P) \subset \mathbb{R}^{d+1}$. This is a full-dimensional cone and Lemma 2.4 together with the restriction to $\mathbb{R}^d \times \{0\}$ and the preceding remarks yield the following relation on functions on \mathbb{R}^d

$$(8) \quad [\{0\}] + \sum_F (-1)^{\dim F+1} [\mathbb{T}_F P] + (-1)^{d+1} [\mathbb{R}^d] = (-1)^{d+1} [\emptyset],$$

where the sum is over all nonempty faces $F \subseteq P$ with $F \neq P$. The above equation in \mathbb{S} reads

$$(-1)^{d+1} [\mathbb{R}^d] = \sum_{\emptyset \neq F \subset P} (-1)^{\dim F} [\mathbb{T}_F P] = \mathbb{T}_P(-1)$$

and applying α to both sides, yields the result. □

3. BELT POLYTOPES AND ANGLE VECTORS

A convex polytope $Z \subset \mathbb{R}^d$ is a **zonotope** if there are $z_1, \dots, z_k \in \mathbb{R}^d \setminus \{0\}$ and $t \in \mathbb{R}^d$ such that

$$t + Z = \sum_{i=1}^k [-z_i, z_i] = \{\lambda_1 z_1 + \cdots + \lambda_k z_k : -1 \leq \lambda_1, \dots, \lambda_k \leq 1\}.$$

Zonotopes play an important role in geometric combinatorics [45, Ch.7] as well as convex geometry [13]. Faces of zonotopes are zonotopes and hence all 2-dimensional faces of a zonotope are centrally-symmetric polygons. In fact, this property characterizes zonotopes; see Bolker [11]. A polytope $P \subset \mathbb{R}^d$ is a **belt polytope** (or **generalized zonotope**) if and only if every 2-face $F \subset P$ has an even number of edges and opposite edges are parallel. Belt polytopes were studied by Baladze [3] (see also [11]) and are equivalently characterized by the fact that their normal fans are induced by hyperplane arrangements.

Let \mathcal{H} be a central arrangement of hyperplanes, that is, the collection of some oriented linear hyperplanes $H_i^0 := z_i^\perp$ for $i = 1, \dots, k$. We write $H_i^+ = \{x : \langle z_i, x \rangle > 0\}$ and H_i^- accordingly. For a point $p \in \mathbb{R}^d$, let $\sigma_i = \text{sgn}\langle z_i, p \rangle$ for $i = 1, \dots, k$. Then

$$H_\sigma := H_1^{\sigma_1} \cap H_2^{\sigma_2} \cap \cdots \cap H_k^{\sigma_k}$$

where $\sigma = (\sigma_1, \dots, \sigma_k) \in \{-, 0, +\}^k$ is a relatively open cone containing p . This shows that the collection $\{H_\sigma : \sigma \in \{-, 0, +\}^k\}$ of relatively open cones partitions \mathbb{R}^d . The **lattice of flats** of a hyperplane arrangement \mathcal{H} is the collection of linear subspaces

$$\mathcal{L}(\mathcal{H}) := \{H_{i_1}^0 \cap H_{i_2}^0 \cap \cdots \cap H_{i_r}^0 : 1 \leq i_1 < \cdots < i_r \leq k, r \geq 0\}$$

partially ordered by *reverse* inclusion. This is a graded lattice with minimal element \mathbb{R}^d and maximal element $H_1^0 \cap \cdots \cap H_k^0$. For every relatively open cone H_σ , we have that

$$\text{aff}_0(H_\sigma) = \bigcap_{i:\sigma_i=0} H_i^0$$

is an element of $\mathcal{L}(\mathcal{H})$ and we record the following consequence.

PROPOSITION 3.1. *Let \mathcal{H} be an arrangement of hyperplanes with $\mathcal{L} = \mathcal{L}(\mathcal{H})$. Then any $L \in \mathcal{L}$ is partitioned by the collection of relatively open cones R of \mathcal{H} with $\text{aff}_0(R) \subseteq L$.*

Let $P \subset \mathbb{R}^d$ be a polytope of positive dimension and recall that for a non-empty face $F \subseteq P$, we defined $\mathsf{L}(F) = \text{aff}_0(F)^\perp$. We can associate an arrangement $\mathcal{H}(P)$ with hyperplanes $\mathsf{L}(e)$ for every edge $e \subset P$. Every linear function $\ell(x) = \langle c, x \rangle$ yields a (possibly trivial) orientation on the edges of P and thus determines a relatively open cone of $\mathcal{H}(P)$. It can be shown that for every nonempty face $F \subset P$, the normal cone $N_F P$ is partitioned by some relatively open cones of $\mathcal{H}(P)$. In the language of [45, Sect. 7.1], the fan induced by $\mathcal{H}(P)$ refines the normal fan of P and the refinement is typically strict. It follows that P is a belt polytope if and only if the normal fan of P coincides with the fan induced by $\mathcal{H}(P)$. The name ‘belt polytope’ derives from the following fact: two faces F, F' of a belt polytope P satisfy $\mathsf{L}(F) = \mathsf{L}(F')$ if and only if F and F' are normally equivalent. The collection of faces F with fixed $\mathsf{L}(F)$ are said to be in the same belt.

Thus, if P is a belt polytope, then $\mathcal{L}(P) = \mathcal{L}(\mathcal{H}(P))$ is a lattice graded by dimension with minimum $\mathbf{0} = \mathsf{L}(v) = \mathbb{R}^d$ for every vertex v and maximum $\mathbf{1} = \mathsf{L}(P)$. See [11] for details.

The **Whitney numbers of the second kind** $W_i(\mathcal{L})$ of a graded poset \mathcal{L} count the number of elements $a \in \mathcal{L}$ of rank $\text{rk}_{\mathcal{L}}(a) = i$.

PROPOSITION 3.2. *Let P be a belt polytope of dimension d and let $\mathcal{L} = \mathcal{L}(P)$. Then for any cone angle α*

$$\check{\alpha}_i(P) = W_i(\mathcal{L})$$

for all $i = 0, \dots, d - 1$.

Proof. Let $L \in \mathcal{L}$. From Proposition 3.1 we infer that as elements of \mathbb{S}

$$(9) \quad \sum_F [\mathsf{O}_F P] = \sum_F [L + N_F P] = [\mathbb{R}^d],$$

where the sum is over all faces $F \subseteq P$ with $\mathsf{L}(F) = L$. For $i = 0, 1, \dots, d - 1$ fixed it follows that

$$\sum_{\substack{F \subseteq P \\ \dim F = i}} [\mathsf{O}_F P] = \sum_{\substack{L \in \mathcal{L} \\ \dim L = i}} \sum_{\substack{F \subseteq P \\ \mathsf{L}(F) = L}} [\mathsf{O}_F P] = \sum_{\substack{L \in \mathcal{L} \\ \dim L = i}} [\mathbb{R}^d] = W_i(\mathcal{L})[\mathbb{R}^d]$$

and applying α yields the claim. □

A configuration z_1, \dots, z_n of $n \geq d$ vectors in \mathbb{R}^d is **generic** if any choice of d vectors are linearly independent. The proper faces of the associated zonotope Z are parallelepipeds. This implies that the poset $\mathcal{L}(Z) \setminus \{\mathbf{1}\}$ is isomorphic to the collection of subsets of $[n]$ of cardinality at most $d - 1$ ordered by inclusion and hence depends only on n and d .

COROLLARY 3.3. *For $d \geq 1$ and α a cone angle*

$$\begin{aligned} \text{aff}\{\check{\alpha}(P) : P \subset \mathbb{R}^d \text{ } d\text{-zonotope}\} &= \text{aff}\{\check{\alpha}(P) : P \subset \mathbb{R}^d \text{ } d\text{-polytope}\} \\ &= \{a \in \mathbb{R}^d : a_0 = 1\}. \end{aligned}$$

Proof. Proposition 2.3 implies that \subseteq holds and thus we only need to exhibit d zonotopes whose exterior angle vectors are linearly independent. By Proposition 3.2, it suffices to find d zonotopes $Z_0, \dots, Z_{d-1} \subset \mathbb{R}^d$ such that the d -by- d matrix

$A = (a_{ij})_{i,j=0,\dots,d-1}$ with $a_{ij} = W_i(Z_j)$ has rank d . Let Z_j be the zonotope obtained from a collection of $d + j$ generic vectors. Then

$$a_{ij} = \binom{d+j}{i} \quad \text{for } i, j = 0, 1, \dots, d-1.$$

Row operations together with Pascal's identity then show that A has determinant 1, which proves the claim. \square

The **incidence algebra** $\mathcal{I}(\mathcal{P})$ of a finite poset (\mathcal{P}, \preceq) is the vector space of all functions $h : \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{C}$, such that $h(a, c) = 0$ whenever $a \not\preceq c$ and with multiplication

$$(g * h)(a, c) = \sum_{a \preceq b \preceq c} g(a, b)h(b, c)$$

for $g, h \in \mathcal{I}(\mathcal{P})$; see Stanley [40, Ch. 3] for more on this. The **zeta function** $\zeta_{\mathcal{P}} \in \mathcal{I}(\mathcal{P})$ is given by $\zeta_{\mathcal{P}}(a, c) = 1$ if $a \preceq c$ and $= 0$ otherwise. The zeta function is invertible in $\mathcal{I}(\mathcal{P})$ with inverse given by the **Möbius function** $\mu_{\mathcal{P}} = \zeta_{\mathcal{P}}^{-1}$. More precisely, the Möbius function satisfies $\mu_{\mathcal{P}}(a, a) = 1$ and

$$\mu_{\mathcal{P}}(a, c) = - \sum_{a \prec b \preceq c} \mu_{\mathcal{P}}(b, c)$$

for $a \prec c$. For a graded poset \mathcal{P} of rank d , the **characteristic polynomial** $\chi_{\mathcal{P}}(t) \in \mathbb{Z}[t]$ is defined by

$$\chi_{\mathcal{P}}(t) = \sum_{a \in \mathcal{P}} \mu_{\mathcal{P}}(\mathbf{0}, a) t^{d-\text{rk}(a)} = w_0(\mathcal{P})t^d + w_1(\mathcal{P})t^{d-1} + \dots + w_d(\mathcal{P}).$$

The numbers $w_i(\mathcal{P})$, called the **Whitney numbers of the first kind**, are explicitly given by

$$w_i(\mathcal{P}) = \sum_{a : \text{rk}(a)=i} \mu_{\mathcal{P}}(\mathbf{0}, a).$$

In particular, $w_0(\mathcal{P}) = 1$ and $w_d(\mathcal{P}) = \mu_{\mathcal{P}}(\mathbf{0}, \mathbf{1})$. The characteristic polynomial $\chi_{\mathcal{L}}(t)$ where $\mathcal{L} = \mathcal{L}(\mathcal{H})$ is the lattice of flats of a hyperplane arrangement \mathcal{H} captures a number of important properties. For example, Zaslavsky's celebrated result [44] states that $|\chi_{\mathcal{L}}(-1)|$ is the number of regions of \mathcal{H} ; see also [7, Ch. 3.6]. Here, however, we will be interested in the characteristic polynomial of the opposite poset \mathcal{L}^{op} .

LEMMA 3.4. *Let P be a d -dimensional belt polytope with lattice of flats $\mathcal{L} = \mathcal{L}(P)$ and let α be a cone angle. For any fixed $L \in \mathcal{L}$*

$$\sum_F \hat{\alpha}(F, P) = (-1)^{d-\dim L} \mu_{\mathcal{L}}(L, \mathbf{1}) = (-1)^{d-\dim L} \mu_{\mathcal{L}^{\text{op}}}(\mathbf{0}, L),$$

where the sum is over all faces $F \subseteq P$ with $\mathfrak{L}(F) = L$.

The proof makes use of the fact that tangent and normal cones are related by polarity.

PROPOSITION 3.5. *Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope and $v \in P$ a vertex. Then*

$$(\text{N}_v P)^\vee = \{u \in \mathbb{R}^d : \langle u, x \rangle \leq 0 \text{ for all } x \in \text{N}_v P\} = \text{T}_v P.$$

Proof. Observe that $\mathbf{c} \in \text{N}_v P$ if and only if $\langle \mathbf{c}, v \rangle \geq \langle \mathbf{c}, x \rangle$ for all $x \in P$. That is, if and only if $\langle \mathbf{c}, x - v \rangle \leq 0$ for all $x \in P$ and from $\text{T}_v P = \text{cone}(-v + P)$ we deduce that $\text{T}_v P^\vee = \text{N}_v P$. \square

Proof of Lemma 3.4. We again prove the following more general statement over \mathbb{S}

$$(10) \quad \sum_F [\mathbb{T}_F P] = (-1)^{d-\dim L} \mu_{\mathcal{L}(P)}(L, \mathbf{1}) [\mathbb{R}^d],$$

where the sum is over all faces $F \subseteq P$ with $\mathbb{L}(F) = L$.

Let us assume that $L = \mathbf{0} = \{0\}$. Proposition 3.5 states that $\mathbb{T}_{\mathbf{v}} P$ is precisely the polar cone $N_{\mathbf{v}} P^{\vee}$. That is, if $\mathbf{w} \in \text{int}(\mathbb{T}_{\mathbf{v}} P)$, then the hyperplane \mathbf{w}^{\perp} does not meet $\text{int}(N_{\mathbf{v}} P)$. Note that since P is a belt polytope, the cones $N_{\mathbf{v}} P$ are the regions of $\mathcal{H} = \mathcal{H}(P)$.

Hence, for a generic \mathbf{w} , the left-hand side of (10) is the number of regions of \mathcal{H} that are not intersected by \mathbf{w}^{\perp} . By a classical result of Greene and Zaslavsky [14, Thm. 3.1], this number is independent of \mathbf{w} and is exactly $(-1)^{d-\dim L} \mu_{\mathcal{L}(P)}(\mathbf{0}, \mathbf{1})$.

For $L \neq \mathbf{0}$, let $\pi_L : \mathbb{R}^d \rightarrow L^{\perp}$ be the orthogonal projection along L . Then $\pi_L(P)$ is a belt polytope and $\mathcal{L}(\pi_L(P))$ is isomorphic to the interval $[L, \mathbf{1}] \subseteq \mathcal{L}(P)$. \square

The following shows that the interior angle vectors of zonotopes are determined by the Whitney numbers of the first kind. Together with Proposition 3.2, this proves Theorem 1.2.

Proof of Theorem 1.2. Let P be a belt polytope with lattice of flats $\mathcal{L} = \mathcal{L}(P)$. With the help of Lemma 3.4, we deduce for $L \in \mathcal{L}$ with $\dim L = i$

$$\begin{aligned} \hat{\alpha}_i(P) &= \sum_{\dim F=i} \hat{\alpha}(F, P) = \sum_{\dim L=i} \sum_{\mathbb{L}(F)=L} \hat{\alpha}(F, P) = \sum_{\dim L=i} (-1)^{d-i} \mu_{\mathcal{L}^{\text{op}}}(\mathbf{0}, L) \\ &= (-1)^{d-i} w_{d-i}(\mathcal{L}^{\text{op}}). \end{aligned} \quad \square$$

In [29], Novik, Postnikov, and Sturmfels introduced the **cocharacteristic polynomial** of the lattice of flats: For a zonotope Z of dimension d and lattice of flats $\mathcal{L} = \mathcal{L}(Z)$, its cocharacteristic polynomial is

$$\psi_{\mathcal{L}}(t) = \sum_{L \in \mathcal{L}} |\mu_{\mathcal{L}}(L, \mathbf{1})| t^{d-\dim L} = \sum_{i=0}^d |w_{d-i}(\mathcal{L}^{\text{op}})| t^{d-i} = (-t)^d \chi_{\mathcal{L}^{\text{op}}}(-\frac{1}{t}).$$

In [29] the coefficients of the cocharacteristic polynomial encoded invariants of ideals associated to matroids. Here, cocharacteristic polynomials give us an elegant mean to prove the following theorem, which proves the uniqueness of (3) in Theorem 1.1.

THEOREM 3.6. *For $d \geq 1$, let $\alpha : \mathcal{C}_d \rightarrow \mathbb{R}$ be a cone angle. Then*

$$\text{aff}\{\hat{\alpha}(Z) : Z \subset \mathbb{R}^d \text{ } d\text{-zonotope}\} = \left\{ (a_0, \dots, a_{d-1}) \in \mathbb{R}^d : \sum_{i=0}^{d-1} (-1)^i a_i = (-1)^{d+1} \right\}.$$

Proof. Using Theorem 1.2, it suffices to produce d zonotopes Z_0, \dots, Z_{d-1} whose cocharacteristic polynomials are linearly independent.

For $j \geq 0$, let Z_j be the d -dimensional zonotope of $d + j$ generic vectors and let $\psi_{d,j}(t)$ be its associated cocharacteristic polynomial. From [29, Prop. 4.2] we deduce that these polynomials satisfy the recurrence

$$\psi_{d,j}(t) = \psi_{d-1,j}(t) + \binom{d-1+j}{j} t(t+1)^{d-1}$$

for $d \geq 1$ and $\psi_{0,j}(t) = 1$. We claim that the cocharacteristic polynomials $\psi_{d,j}(t)$ for $0 \leq j \leq d-1$ are linearly independent. Indeed, the recursion and the fact that

$\deg \psi_{d,j}(t) = d$ shows that $\sum_j \lambda_j \psi_{d,j} = 0$ for $\lambda_0, \dots, \lambda_{d-1} \in \mathbb{R}$ if and only if

$$\sum_{j=0}^{d-1} \binom{d-1+j}{j} \lambda_j = 0 \quad \text{and} \quad \sum_{j=0}^{d-1} \psi_{d-1,j}(t) \lambda_j = 0.$$

Iterating this idea, it follows that $\lambda = (\lambda_0, \dots, \lambda_d)$ is in the kernel of the d -by- d matrix A with entries $\binom{i+j}{j}$ for $i, j = 0, \dots, d-1$. Again appealing to Pascal's identity, it is easy to see that $\det A = 1$, which completes the proof. \square

4. CONNECTING ANGLES WITH MÖBIUS INVERSION

In this section we take an algebraic approach to the occurrence of the Whitney numbers of the lattice of flats of a belt polytope in the previous sections. The **face lattice** of a polytope P is the collection $\mathcal{F}(P)$ of faces of P ordered by inclusion. For a given belt polytope P , we define a certain subalgebra of $\mathcal{I}(\mathcal{F}(P))$. As it will turn out the map $F \mapsto \mathbf{L}(F)$ yields a pair of transformations and Theorem 1.2 follows from the fact that the two transformations are adjoint. In particular, we derive a generalization of a result of Klivans and Swartz [23] regarding spherical intrinsic volumes and Whitney numbers.

Let \mathcal{P}, \mathcal{Q} be two posets. A surjective and order preserving map $\phi : \mathcal{P} \rightarrow \mathcal{Q}$ induces a linear transformation $\phi_* : \mathcal{I}(\mathcal{P}) \rightarrow \mathcal{I}(\mathcal{Q})$ by

$$\phi_* h(q, q') := \frac{1}{|\phi^{-1}(q')|} \sum_{\substack{p \in \phi^{-1}(q) \\ p' \in \phi^{-1}(q')}} h(p, p')$$

called the **pushforward** of h . Let $\mathcal{I}_\phi(\mathcal{P}) \subseteq \mathcal{I}(\mathcal{P})$ be the vector subspace of all elements $h \in \mathcal{I}(\mathcal{P})$ such that for all $q, q' \in \mathcal{Q}$

$$(11) \quad \sum_{p \in \phi^{-1}(q)} h(p, p'_1) = \sum_{p \in \phi^{-1}(q)} h(p, p'_2) \quad \text{for all } p'_1, p'_2 \in \phi^{-1}(q').$$

The neutral element $\delta \in \mathcal{I}(\mathcal{P})$ is defined by $\delta(x, y) = 1$ if $x = y$, and $= 0$ otherwise. Clearly, $\delta \in \mathcal{I}_\phi(\mathcal{P})$ and thus $\mathcal{I}_\phi(\mathcal{P}) \neq \emptyset$. For an element $h \in \mathcal{I}_\phi(\mathcal{P})$, the pushforward simplifies to

$$\phi_* h(q, q') = \sum_{p \in \phi^{-1}(q)} h(p, p')$$

for any $p' \in \phi^{-1}(q')$.

PROPOSITION 4.1. $\mathcal{I}_\phi(\mathcal{P})$ is a subalgebra of $\mathcal{I}(\mathcal{P})$ and $\phi_* : \mathcal{I}_\phi(\mathcal{P}) \rightarrow \mathcal{I}(\mathcal{Q})$ is an algebra map.

Proof. Let $g, h \in \mathcal{I}_\phi(\mathcal{P})$. For $q, q' \in \mathcal{Q}$ and $p' \in \phi^{-1}(q')$ arbitrary we compute

$$\begin{aligned} (\phi_* g * \phi_* h)(q, q') &= \sum_{s \in \mathcal{Q}} \phi_* g(q, s) \cdot \phi_* h(s, q') = \sum_{p \in \phi^{-1}(q)} \sum_{s \in \mathcal{Q}} \sum_{r \in \phi^{-1}(s)} g(p, r) \cdot h(r, p') \\ &= \sum_{p \in \phi^{-1}(q)} \sum_{r \in \mathcal{P}} g(p, r) \cdot h(r, p') = \sum_{p \in \phi^{-1}(q)} (g * h)(p, p'). \end{aligned}$$

Since the left-hand side does not depend on the choice of p' , we see that $g * h \in \mathcal{I}_\phi(\mathcal{P})$ and therefore $\phi_* g * \phi_* h = \phi_*(g * h)$. Since $\delta \in \mathcal{I}_\phi(\mathcal{P})$, this shows that $\mathcal{I}_\phi(\mathcal{P})$ is a subalgebra. \square

For a graded poset \mathcal{P} of rank d , we can define a binary operation

$$*_k : \mathcal{I}(\mathcal{P}) \times \mathcal{I}(\mathcal{P}) \rightarrow \mathcal{I}(\mathcal{P})$$

for $k = 0, \dots, d$ by

$$(12) \quad (g *_k h)(a, c) := \sum_{b: \text{rk}(b)=k} g(a, b) h(b, c).$$

By definition $g * h = \sum_k g *_k h$. It is noteworthy that $*_k$ and $*$ are associative operations, i.e., for $0 \leq k \leq l \leq d$ and $g, h, m \in \mathcal{I}(\mathcal{P})$

$$g*(h*_k m) = (g*h)*_k m, \quad g*_k(h*m) = (g*_k h)*m, \quad g*_k(h*_l m) = (g*_k h)*_l m.$$

The proof of Proposition 4.1 carries over verbatim to prove the following corollary.

COROLLARY 4.2. *Let \mathcal{P} and \mathcal{Q} be ranked posets. If $\phi : \mathcal{P} \rightarrow \mathcal{Q}$ is a surjective order preserving map that preserves rank, then*

$$(13) \quad \phi_*(g *_k h) = \phi_* g *_k \phi_* h.$$

Let $C \subset \mathbb{R}^d$ be a polyhedral cone and let $\mathcal{F}_+(C)$ the collection of nonempty faces of C partially ordered by inclusion. For a given cone angle α , we note that the interior and exterior angles

$$\begin{aligned} \widehat{\alpha}(F, G) &= \alpha(\mathbb{T}_F G) = \alpha(\mathbb{T}_F G + \text{aff}_0(G)^\perp) \\ \check{\alpha}(F, G) &= \alpha(\mathbb{O}_F G) = \alpha(\mathbb{N}_F G + \text{aff}_0(F)) \end{aligned}$$

for faces $F \subseteq G \subseteq C$ are naturally elements of the incidence algebra $\mathcal{I}(C) := \mathcal{I}(\mathcal{F}_+(C))$. Furthermore, let us define:

$$\widehat{\alpha}'(F, G) := (-1)^{\dim G - \dim F} \widehat{\alpha}(F, G).$$

We call two cone angles α, β **complementary** if for all polyhedral cones $C \subseteq \mathbb{R}^d$

$$(14) \quad \widehat{\alpha}' * \check{\beta} = \delta_C,$$

where δ_C is the neutral element of $\mathcal{I}(C)$. As a notable example, the standard cone angle ν is self-complementary, i.e. $\widehat{\nu}' * \check{\nu} = \delta_C$ for all polyhedral cones $C \subseteq \mathbb{R}^d$; see [25]. Complementary angles were studied by McMullen [26] under the name of *inverse* angles. In [26], an **angle functional** is a collection of normalized and simple valuations α_L for cones in linear subspaces $L \subseteq \mathbb{R}^d$. The angle of a cone $C \subseteq \mathbb{R}^d$ is then $\alpha_L(C)$ where $L = \text{aff}_0(C)$. In this framework, we define the cone angle α_L on a linear subspace $L \subset \mathbb{R}^d$ by $\alpha_L(C) := \alpha(C + L^\perp)$ for any cone $C \subseteq L$. The next lemma is an adaptation of [26, Lemma 46].

LEMMA 4.3. *For every cone angle α there is a complementary cone angle β .*

Proof. Lemma 46 in [26] guarantees the existence of a complementary angle functional β_L for the angle functional α_L as defined above, such that (14) is satisfied. It can be shown that β_L is of the form $\beta_L(C) = \beta(C + L^\perp)$ for some cone angle β . \square

Let $P \subset \mathbb{R}^d$ be a belt polytope and let $\mathcal{F} = \mathcal{F}(P)$ be the collection of faces of P . As before, we can interpret $\widehat{\alpha}(F, G)$ and $\check{\alpha}(F, G)$ as elements in $\mathcal{I}(\mathcal{F})$, by extending $\widehat{\alpha}(\emptyset, G) = 1$ if $\dim G \leq 0$ and $= 0$ otherwise and $\check{\alpha}(\emptyset, G) = 1$ for all G . In particular, $\widehat{\alpha}' * \check{\beta} = \delta_{\mathcal{F}}$.

Let \mathcal{L}_0 be the set $\mathcal{L}(P) \cup \{\perp\}$ partially ordered by inclusion. Recall that for a non-empty face $F \subseteq P$, $\mathbb{L}(F) = \text{aff}_0(F)^\perp$, where $\text{aff}_0(F)$ is the linear subspace parallel to F . Setting $\mathbb{L}(\emptyset) := \perp$, the map $\mathbb{L} : \mathcal{F}(P) \rightarrow \mathcal{L}_0(P)$ given by $F \mapsto \mathbb{L}(F)$ is a surjective order and rank preserving map.

THEOREM 4.4. *Let P be a belt polytope and $\mathcal{F} = \mathcal{F}(P)$. For every cone angle α , we have $\widehat{\alpha}, \check{\alpha} \in \mathcal{I}_{\mathbb{L}}(\mathcal{F})$ and*

$$\mathbb{L}_* \check{\alpha} = \zeta_{\mathcal{L}_0} \quad \text{and} \quad \mathbb{L}_* \widehat{\alpha}' = \mu_{\mathcal{L}_0}.$$

Proof. Two faces G and G' of P are normally equivalent if $L(G) = L(G')$. Thus equation (11) is satisfied and $\check{\alpha}$ and $\hat{\alpha}$ are elements of $\mathcal{I}_L(\mathcal{F})$. Let $G \subseteq P$ be a face and $U \in \mathcal{L}$ with $U \supseteq L(G)$. Then from (9) in the proof of Proposition 3.2 and $\check{\alpha}(\emptyset, G) = 1$ we infer that

$$\sum_{\substack{F \in \mathcal{F}(P), \\ L(F)=U}} \check{\alpha}(F, G) = 1$$

and hence $(L_*\check{\alpha})(U, U') = 1 = \zeta(U, U')$ for all $U, U' \in \mathcal{L}$ with $U \supseteq U'$.

By Lemma 4.3, there is a cone angle β complementary to α . Using the fact that L_* is an algebra map, we deduce

$$\delta_{\mathcal{L}_0} = L_*(\delta_{\mathcal{F}}) = L_*(\hat{\alpha}' * \check{\beta}) = L_*(\hat{\alpha}') * L_*(\check{\beta}).$$

Replacing $\check{\alpha}$ by $\check{\beta}$ above yields $L_*(\check{\beta}) = \zeta_{\mathcal{L}}$ and thus $L_*(\hat{\alpha}) = \zeta_{\mathcal{L}}^{-1} = \mu_{\mathcal{L}}$. □

Recall that $\nu(C) = \frac{\text{vol}(C \cap B_d)}{\text{vol}(B_d)}$ is the standard cone angle. For a polytope $P \subset \mathbb{R}^d$, the k -th **spherical intrinsic volume** is defined as

$$(15) \quad \bar{\nu}_k(P) := \sum_v \sum_{v \in F} \hat{\nu}(v, F) \check{\nu}(F, P),$$

where the sum is over all vertices $v \in P$ and k -faces $F \subset P$. For a given cone angle α , we denote by $\bar{\alpha}_k(P)$ the generalization of (15) to α .

The machinery developed in this section yields algebraic proofs of Theorem 1.2.

COROLLARY 4.5. *Let α be a cone angle and P a d -dimensional belt polytope. For $k = 0, \dots, d - 1$ the following hold:*

- (i) $\check{\alpha}_k(P) = W_k(\mathcal{L}(P))$;
- (ii) $\hat{\alpha}_k(P) = |w_{d-k}(\mathcal{L}(P)^{\text{op}})|$;
- (iii) $\bar{\alpha}_k(P) = |w_k(\mathcal{L}(P))|$.

Parts (ii) and (iii) were shown by Klivans and Swartz in [23] for the standard cone angle; see also [2, 32]. For the proof, we need the following technical result.

LEMMA 4.6. *Let $\phi : \mathcal{P} \rightarrow \mathcal{Q}$ be a surjective, order and rank preserving map between posets with minimal and maximal elements. If $f \in \mathcal{I}_{\phi}(\mathcal{P})$, then*

$$(\zeta_{\mathcal{P}} *_{k} f)(\mathbf{0}_{\mathcal{P}}, \mathbf{1}_{\mathcal{P}}) = (\zeta_{\mathcal{Q}} *_{k} (\phi_* f))(\mathbf{0}_{\mathcal{Q}}, \mathbf{1}_{\mathcal{Q}}).$$

Proof. Writing out the definition of $\zeta_{\mathcal{P}} *_{k} f$ we obtain

$$\begin{aligned} (\zeta_{\mathcal{P}} *_{k} f)(\mathbf{0}_{\mathcal{P}}, \mathbf{1}_{\mathcal{P}}) &= \sum_{\substack{p \in \mathcal{P} \\ \text{rk}(p)=k}} f(p, \mathbf{1}_{\mathcal{P}}) = \sum_{\substack{q \in \mathcal{Q} \\ \text{rk}(q)=k}} \sum_{p \in \phi^{-1}(q)} f(p, \mathbf{1}_{\mathcal{P}}) \\ &= \sum_{\substack{q \in \mathcal{Q} \\ \text{rk}(q)=k}} (\phi_* f)(q, \mathbf{1}_{\mathcal{Q}}) = (\zeta_{\mathcal{Q}} *_{k} \phi_* f)(\mathbf{0}_{\mathcal{Q}}, \mathbf{1}_{\mathcal{Q}}). \end{aligned} \quad \square$$

Proof of Corollary 4.5. (i) immediately follows from Theorem 4.4 and Lemma 4.6 for $k = i$ and $f = \check{\alpha}$. Relation (ii) follows in the same fashion with $f = \hat{\alpha}$, but note that we obtain the co-Whitney numbers of the first kind. For (iii), we invoke the same lemma for $k = 0$ and $f = \hat{\alpha} *_{i} \check{\alpha}$. □

This algebraic perspective on angles is very helpful and will facilitate proofs and computations in the next sections.

5. FLAG-ANGLE VECTORS

In this and the next section we prove Theorems 1.4 and 1.5. Our strategy of proof is as follows. First, we will show that the interior/exterior flag-angle vectors satisfy the relations stated in Theorem 1.4. This is done in Propositions 5.1 and 5.2. The algebraic machinery developed in Section 4 enables us to prove Theorem 1.5. To complete the proof of Theorem 1.4, we use this combinatorial interpretation of flag-angle vectors for belt polytopes. It suffices to show that there are no linear relations on flag-Whitney numbers of lattices of flats. For the flag-Whitney numbers of the second kind, this is done in Section 6 and, by establishing an algebraic connection (Theorem 5.3) between them, this also addresses the case of flag-Whitney numbers of the first kind.

The following is the analogue of Proposition 2.3.

PROPOSITION 5.1. *Let P be a d -dimensional polytope and $S \subseteq [d - 1]$. Then*

$$\check{\alpha}_S(P) = \check{\alpha}_{S \cup \{0\}}(P).$$

Proof. Let $S = \{s_1, \dots, s_k\}$ and set $s_0 := 0$. Unravelling the definition of exterior flag-angle vectors (see (7)), we compute

$$\begin{aligned} \check{\alpha}_{S \cup \{0\}}(P) &= \sum_{F_0 \subset F_1 \subset F_2 \subset \dots \subset F_k} \check{\alpha}(F_0, F_1) \check{\alpha}(F_1, F_2) \cdots \check{\alpha}(F_k, P) \\ &= \sum_{F_1 \subset F_2 \subset \dots \subset F_k} \check{\alpha}(F_1, F_2) \cdots \check{\alpha}(F_k, P) \sum_{F_0 \subset F_1} \check{\alpha}(F_0, F_1) \\ &= \sum_{F_1 \subset F_2 \subset \dots \subset F_k} \check{\alpha}(F_1, F_2) \cdots \check{\alpha}(F_k, P) \\ &= \check{\alpha}_S(P), \end{aligned}$$

where the sums are over faces F_i with $\dim F_i = s_i$ for $i = 0, \dots, k$ and where the third equality follows from Proposition 2.3. \square

As for the linear relations on interior flag-angle vectors, we take a more algebraic approach. Let P be a d -dimensional polytope with face lattice $\mathcal{F} = \mathcal{F}(P)$ and $S = \{s_1 < s_2 < \dots < s_k\}$ with $S \subseteq [0, d - 1]$. Using (12) together with the fact that $\text{rk}_{\mathcal{F}}(F) = \dim F + 1$, we can give the following expression for the S -entry of the interior flag-angle vector

$$\hat{\alpha}_S(P) = (\zeta_{\mathcal{F}} *_{s_1+1} \hat{\alpha} *_{s_2+1} \cdots *_{s_k+1} \hat{\alpha})(\emptyset, P),$$

where the operation $*_k$ was introduced in (12).

PROPOSITION 5.2. *Let P be a d -polytope. For $S = \{0 \leq s_1 < s_2 < \dots < s_k \leq d - 1\}$ set $t := \min(S \cup \{d\})$. Then*

$$\sum_{i=0}^{t-1} (-1)^i \hat{\alpha}_{S \cup \{i\}}(P) = (-1)^{t+1} \hat{\alpha}_S(P).$$

Proof. Recall that the Möbius function $\mu_{\mathcal{F}} = \zeta_{\mathcal{F}}^{-1}$ is given by

$$\mu_{\mathcal{F}}(F, G) = (-1)^{\dim G - \dim F}$$

for faces $F \subseteq G \subset P$. For a fixed face G , Proposition 2.5 yields

$$(\mu_{\mathcal{F}} * \hat{\alpha})(\emptyset, G) = - \sum_F (-1)^{\dim F} \hat{\alpha}(F, G) = - \sum_{i=0}^{\dim G - 1} (-1)^i \hat{\alpha}_i(G) + (-1)^{\dim G + 1} = 0.$$

The result now follows by evaluating

$$\mu_{\mathcal{F}} * \hat{\alpha} *_{s_1+1} \hat{\alpha} *_{s_2+1} \cdots *_{s_k+1} \hat{\alpha}$$

at (\emptyset, P) . □

Let \mathcal{P} be a graded poset of rank $d + 1$ and let $S = \{s_1 < s_2 < \dots < s_k\} \subseteq [d]$. The **flag-Whitney numbers of the second kind** as defined in the introduction are given by

$$W_S(\mathcal{P}) = (\zeta_{\mathcal{P}} *_{s_1} \zeta_{\mathcal{P}} *_{s_2} \dots *_{s_k} \zeta_{\mathcal{P}})(\mathbf{0}, \mathbf{1}).$$

Similarly, we define the **flag-Whitney numbers of the first kind** by

$$w_S(\mathcal{P}) := (\mu_{\mathcal{P}} *_{s_1} \mu_{\mathcal{P}} *_{s_2} \dots *_{s_k} \zeta_{\mathcal{P}})(\mathbf{0}, \mathbf{1}) = \sum \mu(\mathbf{0}, c_1) \mu(c_1, c_2) \dots \mu(c_{k-1}, c_k),$$

where the sum is over all chains $\mathbf{0} \prec c_1 \prec c_2 \prec \dots \prec c_k$ with $\text{rk } c_i = s_i$ for $i = 1, \dots, k$. Now the same reasoning as in the proof of Corollary 4.5 yields Theorem 1.5:

Proof of Theorem 1.5. Let $\mathcal{F} = \mathcal{F}(P)$ be the face lattice of P and let \mathcal{L}_0 be the poset $\mathcal{L}(P)$ with a new minimal element $\mathbf{0}_{\mathcal{L}_0}$ adjoined. The maximal element of \mathcal{L}_0 is $\mathbf{1}_{\mathcal{L}_0} = \mathbf{L}(P)$. We also set $t_i := s_i + 1$ and $f := \check{\alpha} *_{t_2} \dots *_{t_k} \check{\alpha}$. Note that $f \in \mathcal{I}_{\mathbf{L}}(\mathcal{F})$ and with Lemma 4.6 we compute

$$\begin{aligned} \check{\alpha}_S(P) &= (\zeta_{\mathcal{F}} *_{t_1} \check{\alpha} *_{t_2} \dots *_{t_k} \check{\alpha})(\emptyset, P) \\ &= (\zeta_{\mathcal{F}} *_{t_1} f)(\mathbf{0}_{\mathcal{F}}, \mathbf{1}_{\mathcal{F}}) = (\zeta_{\mathcal{F}} *_{t_1} (\mathbf{L}_* f))(\mathbf{0}_{\mathcal{L}_0}, \mathbf{1}_{\mathcal{L}_0}) \\ &\stackrel{(13)}{=} (\zeta_{\mathcal{L}_0} *_{t_1} \zeta_{\mathcal{L}_0} *_{t_2} \dots *_{t_k} \zeta_{\mathcal{L}_0})(\mathbf{0}_{\mathcal{L}_0}, \mathbf{1}_{\mathcal{L}_0}) \\ &= (\zeta_{\mathcal{L}} *_{s_1} \zeta_{\mathcal{L}} *_{s_2} \dots *_{s_k} \zeta_{\mathcal{L}})(\mathbf{0}_{\mathcal{L}}, \mathbf{1}_{\mathcal{L}}) \\ &= W_S(\mathcal{L}) = W_{d-S}(\mathcal{L}^{\text{op}}). \end{aligned}$$

Similarly for the second statement, for $g := \hat{\alpha} *_{t_2} \dots *_{t_k} \hat{\alpha} \in \mathcal{I}_{\mathbf{L}}(\mathcal{F})$ we obtain:

$$\begin{aligned} \hat{\alpha}_S(P) &= (\zeta_{\mathcal{F}} *_{t_1} \hat{\alpha} *_{t_2} \dots *_{t_k} \hat{\alpha})(\emptyset, P) \\ &= (\zeta_{\mathcal{F}} *_{t_1} g)(\mathbf{0}_{\mathcal{F}}, \mathbf{1}_{\mathcal{F}}) = (\zeta_{\mathcal{F}} *_{t_1} (\mathbf{L}_* g))(\mathbf{0}_{\mathcal{L}_0}, \mathbf{1}_{\mathcal{L}_0}) \\ &\stackrel{(13)}{=} (-1)^{d+1-t_1} \cdot (\zeta_{\mathcal{L}_0} *_{t_1} \mu_{\mathcal{L}_0} *_{t_2} \dots *_{t_k} \mu_{\mathcal{L}_0})(\mathbf{0}_{\mathcal{L}_0}, \mathbf{1}_{\mathcal{L}_0}) \\ &= (-1)^{d-s_1} \cdot (\zeta_{\mathcal{L}} *_{s_1} \mu_{\mathcal{L}} *_{s_2} \dots *_{s_k} \mu_{\mathcal{L}})(\mathbf{0}_{\mathcal{L}}, \mathbf{1}_{\mathcal{L}}) \\ &= (-1)^{d-s_1} \cdot w_{d-S}(\mathcal{L}(P)^{\text{op}}). \end{aligned} \quad \square$$

In order to complete the proof of Theorem 1.4, we observe that the flag-Whitney numbers of the second kind determine the flag-Whitney numbers of the first kind. We show this in more generality. Let \mathcal{P} be a finite poset with $\mathbf{0}$ and $\mathbf{1}$ and let $R := \mathbb{C}[[z_a : a \in \mathcal{P}]]$ be the ring of formal power series with variables indexed by elements of \mathcal{P} . For a unipotent $g \in \mathcal{I}(\mathcal{P})$, i.e., $g(a, a) = 1$ for all $a \in \mathcal{P}$, we define

$$F_g(\mathbf{z}) := \sum g(\mathbf{0}, c_1) z_{c_1} g(c_1, c_2) z_{c_2} \dots z_{c_{k-1}} g(c_{k-1}, c_k) z_{c_k},$$

where the sum is over all multichains $\mathbf{0} \prec c_1 \preceq c_2 \preceq \dots \preceq c_k \prec \mathbf{1}$. Since every multichain comes from a unique chain, we can rewrite $F_g(\mathbf{z})$ to

$$F_g(\mathbf{z}) = \sum_{\mathbf{0} \prec b_1 \prec b_2 \prec \dots \prec b_k \prec \mathbf{1}} g(\mathbf{0}, b_1) \frac{z_{b_1}}{1 - z_{b_1}} g(b_1, b_2) \frac{z_{b_2}}{1 - z_{b_2}} \dots g(b_{k-1}, b_k) \frac{z_{b_k}}{1 - z_{b_k}}.$$

If \mathcal{P} is a graded poset of rank $d + 1$, then for $g = \zeta$, we get

$$G_{\mathcal{P}}(\mathbf{q}) := F_{\zeta}(z_a = q_{\text{rk}(a)} : a \in \mathcal{P}) = \sum_{S \subseteq [d]} W_S(\mathcal{P}) \prod_{i \in S} \frac{q_i}{1 - q_i} \in \mathbb{C}[[q_1, \dots, q_d]].$$

Since the elements $\frac{q_i}{1 - q_i}$ for $i = 1, \dots, d$ are algebraically independent over $\mathbb{C}[[q_1, \dots, q_d]]$, $G_{\mathcal{P}}(\mathbf{q})$ encodes the flag-vector of \mathcal{P} . The relation to the flag-Whitney numbers of the second kind follows from the next theorem.

THEOREM 5.3. *Let $g \in \mathcal{I}(\mathcal{P})$ be unipotent. Then*

$$F_g\left(\frac{1}{\mathbf{z}}\right) = F_{g^{-1}}(\mathbf{z}).$$

Proof. We observe that

$$F_g\left(\frac{1}{\mathbf{z}}\right) = \sum_{\mathbf{0} \prec b_1 \prec b_2 \prec \dots \prec b_k \prec \mathbf{1}} g(\mathbf{0}, b_1) \frac{-1}{1 - z_{b_1}} g(b_1, b_2) \frac{-1}{1 - z_{b_2}} \cdots g(b_{k-1}, b_k) \frac{-1}{1 - z_{b_k}}.$$

The coefficient $g(\mathbf{0}, b_1)g(b_1, b_2) \cdots g(b_{k-1}, b_k)$ now contributes to every multichain supported on a subset of $\{b_1, b_2, \dots, b_k\}$. Rewriting, this is the same as

$$F_g\left(\frac{1}{\mathbf{z}}\right) = \sum_{\mathbf{0} \prec a_1 \prec a_2 \prec \dots \prec a_l \prec \mathbf{1}} h(\mathbf{0}, a_1) \frac{z_{a_1}}{1 - z_{a_1}} h(a_1, a_2) \frac{z_{a_2}}{1 - z_{a_2}} \cdots h(a_{l-1}, a_l) \frac{z_{a_l}}{1 - z_{a_l}},$$

where for $u \prec v$

$$\begin{aligned} h(u, v) &:= \sum_{u \prec b_1 \prec b_2 \prec \dots \prec b_k \prec v} (-1)^k g(u, b_1)g(b_1, b_2) \cdots g(b_k, v) \\ &= \sum_{k \geq 0} (-1)^k (g - \delta)^k(u, v) = g^{-1}(u, v). \end{aligned} \quad \square$$

The above computation is reminiscent of calculation of the antipode applied to the quasisymmetric function associated to a graded poset in Ehrenborg [12]. Applying this statement to a pair α, β of complementary angles allows us to directly relate interior and exterior flag-angles:

COROLLARY 5.4. *Let α be a cone angle with complementary cone angle β . For every d -polytope P , the interior and exterior flag angle vectors are related via*

$$\sum_S (-1)^{d-t} \widehat{\alpha}_S(P) \prod_{i \in S} x_i = \sum_S \check{\beta}_S(P) \prod_{i \in S} -(x_i + 1) \in \mathbb{C}[x_1, \dots, x_d],$$

where the sums are over all $S \subseteq [0, d - 1]$ and $t = \min(S \cup \{d\})$.

Proof. Let $\mathcal{F} = \mathcal{F}(P)$ and $x_i := \frac{q_i}{1 - q_i} \in R$. Then $q_i = \frac{x_i}{x_i + 1}$ and $\frac{-1}{1 - q_i} = -x_i - 1$. Using Theorem 5.3, we compute

$$\begin{aligned} \sum_S (-1)^{d-t} \widehat{\alpha}_S(P) \prod_{i \in S} x_i &= \sum_S \widehat{\alpha}'_S(P) \prod_{i \in S} x_i = \sum_S \widehat{\alpha}'_S(P) \prod_{i \in S} \frac{q_i}{1 - q_i} \\ &= F_{\widehat{\alpha}'}(z_a = q_{\text{rk}(a)} : a \in \mathcal{F}) = F_{\check{\beta}}(z_a = q_{\text{rk}(a)}^{-1} : a \in \mathcal{F}) \\ &= \sum_S \check{\beta}_S(P) \prod_{i \in S} \frac{-1}{1 - q_i} = \sum_S \check{\beta}_S(P) \prod_{i \in S} -(x_i + 1), \end{aligned}$$

where each sum ranges over all $S \subseteq [0, d - 1]$. □

Proof of Theorem 1.4. Propositions 5.1 and 5.2 yield that the linear relations given in Theorem 1.4 hold. In particular, this shows that the dimensions of the affine hulls of interior/exterior flag-angles is at most $2^{d-1} - 1$.

From Theorem 1.5, we infer that

$$\begin{aligned} \text{aff}\{\check{\alpha}(P) : P \text{ } d\text{-polytope}\} &\supseteq \text{aff}\{\check{\alpha}(Z) : Z \text{ } d\text{-zonotope}\} \\ &= \text{aff}\{\mathbf{W}(\mathcal{L}(Z)^{\text{op}}) : Z \text{ } d\text{-zonotope}\}. \end{aligned}$$

Theorem 6.1, that we will prove in the next section, shows that the dimension of the affine hull of flag-vectors of $\mathcal{L}(Z)$ where Z ranges over all d -dimensional zonotopes is of dimension $2^{d-1} - 1$. This proves the claim for exterior flag-angle vectors.

The same reasoning applies to the interior flag-angle vectors and it suffices to determine the affine span of $(w_S(\mathcal{L}(Z)^{\text{op}}))_S$ for d -dimensional zonotopes Z . Analogously to Corollary 5.4, Theorem 5.3 implies that the spaces of flag-Whitney numbers of the

first and of the second kind spanned by posets of rank $d + 1$ are linearly isomorphic, which completes the proof. \square

6. FLAG-WHITNEY NUMBERS AND ZONOTOPES

Let \mathcal{P} be a graded poset with $\mathbf{0}$ and $\mathbf{1}$ of rank $d + 1$. It was shown by Billera and Hetyei [9] that flag-vectors of general graded posets do not satisfy any nontrivial linear relation. That is

$$\dim \text{aff}\{\mathbf{W}(\mathcal{P}) \in \mathbb{R}^{2^d} : \mathcal{P} \text{ graded poset of rank } d + 1\} = 2^d - 1.$$

The only linear relation is given by $W_\emptyset(\mathcal{P}) = 1$.

In light of Theorem 1.5, we can complete the proof of Theorem 1.4 for exterior flag-angles by proving the following refinement of the result of Billera and Hetyei.

THEOREM 6.1. *The flag-vectors of lattices of flats of $(d + 1)$ -dimensional zonotopes span the flag-vectors of rank $d + 1$ posets. That is,*

$$\dim \text{aff}\{\mathbf{W}(\mathcal{L}(Z)) : Z \text{ zonotope of dimension } d + 1\} = 2^d - 1.$$

The result is analogous to that of Billera–Ehrenborg–Readdy [8] where it is shown that the flag-vectors of face lattices of zonotopes span the space of flag-vectors of Eulerian posets. For the proof of Theorem 6.1, we will employ the coalgebra techniques developed in [8].

Let $\mathcal{A} = k\langle a, b \rangle$ be the polynomial ring in noncommuting variables a and b . This is a graded algebra $\mathcal{A} = \bigoplus_{d \geq 0} \mathcal{A}_d$ and a basis for \mathcal{A}_d is given by $\{a - b, b\}^d$. The **ab-Index** of a graded poset \mathcal{P} of rank $d + 1$ is given by

$$\Psi(\mathcal{P}) = \sum_{S \subseteq [d]} W_S(\mathcal{P})x(S)$$

where $x(S) = x_1 x_2 \dots x_d \in \{a - b, b\}^d$ with $x_i = b$ if and only if $i \in S$.

Following [8], we consider two natural operations on zonotopes: If $Z \subset \mathbb{R}^d$ is a zonotope, then $E(Z) := Z \times [0, 1] \subset \mathbb{R}^{d+1}$ is a zonotope of dimension $\dim Z + 1$. This is clearly a combinatorial construction and for the lattice of flats $\mathcal{L} = \mathcal{L}(Z)$, we note that

$$E(\mathcal{L}) := \mathcal{L}(E(Z)) = \mathcal{L} \times C_1,$$

where $C_1 = \{\mathbf{0} \prec \mathbf{1}\}$ is the chain on 2 elements. A vector $u \in \text{aff}_0(Z)$ is in general position with respect to Z if u is not parallel to any face of Z . It can be shown (see [43, Ch. 7]) that the lattice of flats of $M(Z) := Z + [0, u]$ is independent of the choice of u and given by

$$M(\mathcal{L}) := \mathcal{L}(M(Z)) = E(\mathcal{L}) \setminus \{x \in E(\mathcal{L}) : \text{rk}(x) = d + 1\}.$$

Let $\text{Pr}(Z)$ be the orthogonal projection of $M(Z)$ onto the hyperplane u^\perp . This again is a combinatorial operation and $\text{Pr}(\mathcal{L}) := \mathcal{L}(\text{Pr}(Z))$ is obtained from \mathcal{L} by deleting the coatoms, that is, elements of $\text{rk}(\mathcal{L}) - 1$.

In order to determine the effect on the ab-index, we introduce derivations $R, R' : \mathcal{A} \rightarrow \mathcal{A}$ defined on the variables by $R(a) := R(b) := ab$ and $R'(a) := R'(b) := ba$ and linearly extended via

$$R(xy) := R(x)y + xR(y) \quad R'(xy) := R'(x)y + xR'(y)$$

for monomials x, y . Note that both derivations are homogeneous and map \mathcal{A}_d into \mathcal{A}_{d+1} . We also define linear maps $E, M, \text{Pr} : \mathcal{A} \rightarrow \mathcal{A}$ on monomials x by

$$\begin{aligned} \text{Pr}(xa) &:= x & E(x) &:= xa + bx + R(x) \\ \text{Pr}(xb) &:= 0 & M(x) &:= \text{Pr}(E(x)). \end{aligned}$$

In particular, we have

$$M(xa) = xa + bx + R(x) = E(x) \quad \text{and} \quad M(xb) = xb.$$

The following result can be easily obtained by inspecting chains.

LEMMA 6.2. *Let Z be zonotope and $\mathcal{L} = \mathcal{L}(Z)$ its lattice of flats. Then*

$$\begin{aligned} \Psi(E(\mathcal{L})) &= E(\Psi(\mathcal{L})) = \Psi(\mathcal{L})b + a\Psi(\mathcal{L}) + R'(\Psi(\mathcal{L})), \\ \Psi(\text{Pr}(\mathcal{L})) &= \text{Pr}(\Psi(\mathcal{L})), \text{ and} \\ \Psi(M(\mathcal{L})) &= \text{Pr}(E(\Psi(\mathcal{L}))). \end{aligned}$$

Proof of Theorem 6.1. For $d \geq 0$ let

$$\mathcal{Z}_d := \text{span}\{\Psi(\mathcal{L}(Z)) : Z \text{ zonotope of dimension } d + 1\} \subseteq \mathcal{A}_d.$$

We show by induction on d that $\mathcal{Z}_d = \mathcal{A}_d$. For $d = 1$ this is clearly true. Assume that $\mathcal{Z}_d = \mathcal{A}_d$. The key observation is that if x is any monomial in $\mathcal{A}_d = \mathcal{Z}_d$, then also $M(x) \in \mathcal{Z}_d$ and $E(x) \in \mathcal{Z}_{d+1}$.

(i) $xba \in \mathcal{Z}_{d+1}$ for all $x \in \mathcal{Z}_{d-1}$:

$$2E(xb) - M(E(xb)) = xba.$$

(ii) $xab \in \mathcal{Z}_{d+1}$ for all $x \in \mathcal{Z}_{d-1}$:

$$M(E(xa)) - E(xa + bx + R(x)) = xab.$$

(iii) $xba^n \in \mathcal{Z}_{d+1}$ for all $x \in \mathcal{Z}_{d-n}, n = 1, \dots, d - 1$:

For $n = 1$ this is just (i). We may assume that the claim holds for all values $< n + 1$ and compute

$$E(xba^n) = xba^{n+1} + xba^n b + \sum_{i=1}^n x_i ba^i$$

for some $x_i \in \mathcal{A}_{d-i-1}$. Since $x_i ba^i \in \mathcal{Z}_{d+1}$ by induction and $xba^n b \in \mathcal{Z}_{d+1}$ by (ii), we see that $xba^{n+1} \in \mathcal{Z}_{d+1}$.

(iv) $xab^n \in \mathcal{Z}_{d+1}$ for all $x \in \mathcal{Z}_{d-n}, n = 1, \dots, d - 1$:

For $n = 1$ this is just (ii). Assume the statement holds for all values $< n + 1$:

$$E(xab^n) = xab^{n+1} + xab^n a + \sum_{i=1}^n x_i ab^i$$

for some $x_i \in \mathcal{A}_{d-i-1}$. Since $x_i ab^i \in \mathcal{Z}_{d+1}$ by induction and $xab^n a \in \mathcal{Z}_{d+1}$ by (i), we see that $xab^{n+1} \in \mathcal{Z}_{d+1}$.

Since every monomial in \mathcal{A}_{d+1} which contains at least one a and b is of either the form xab^n or xba^n , we see that it remains to show that a^{d+1} and b^{d+1} are in \mathcal{Z}_{d+1} as well. For that we compute

$$\begin{aligned} E(a^d) &= a^{d+1} + ba^d + R(a^d) \\ E(b^d) &= b^d a + b^{d+1} + R(b^d) \end{aligned}$$

Since $ba^d, b^d a, R(a^d), R(b^d) \in \mathcal{Z}_{d+1}$, this finishes the proof. □

In fact we have proven the following statement:

COROLLARY 6.3. *For $d \geq 0$, a vector space basis of \mathcal{A}_d is given by*

$$\{\Phi(\mathcal{L}(\sigma[0, 1])) : \sigma \in \{E, M \circ E\}^d\}.$$

Proof. In the proof of Theorem 6.1 we only needed elements of the form $E(x)$ and $M(E(x))$, $x \in \mathcal{A}_d$, to span \mathcal{A}_{d+1} , thus the assertion follows by induction. □

7. SPHERICAL INTRINSIC VOLUMES AND GRASSMANN ANGLES

We have already seen spherical intrinsic volumes in Section 4, but as a further introduction and motivation to flag-angles, we will give a slightly more general account here. Moreover, we will see how interior and exterior flag-angles are connected. A second goal of this chapter is to show how we can adapt and generalize many results by Grünbaum [17] to a more general version of Grassmann angles.

For convenience, we want to define for a cone $C \subset \mathbb{R}^d$ the **completion** as $\text{cpl } C := C + \text{aff}_0(C)^\perp$. Recall that ν denotes the spherical volume. For $0 \leq r \leq d$, the **r -th spherical intrinsic volume** ν^r of a cone $C \in \mathcal{C}^d$ is defined as:

$$\nu^r(C) := \sum_{\substack{F \subseteq C \text{ face} \\ \dim F=r}} \nu(\text{cpl } F) \cdot \nu(\text{O}_F C),$$

where $\text{O}_F C = \text{cpl } N_F C = N_F C + \text{aff}_0(F)$. Note that $\nu^d(C) = \nu(C)$ and $\nu^i(C) = \nu^{d-i}(C^\vee)$ for all i . The name *spherical* intrinsic volume stems from the similarity to a formula for the usual intrinsic volumes V^r of a polytope P

$$V^r(P) = \sum_{\dim F=r} \text{vol}_r(F) \cdot \nu(\text{O}_F P),$$

where vol_r denotes the usual r -dimensional volume; see [33, Section 4.2]. The intersection of a face $F \subseteq C$ with the unit sphere \mathbb{S}^d is a spherical polytope and $\nu(\text{cpl } F)$ is the normalized spherical volume.

We would like to replace ν in the definition of ν^r with more general valuations. In fact, we could replace both occurrences of ν with different valuations. Let α, β be cone angles. For $0 \leq r \leq d$, we define the **generalized spherical intrinsic volume** $\xi^r = \xi^r(\alpha, \beta) : \mathcal{C}^d \rightarrow \mathbb{R}$ by

$$(16) \quad \xi^r(C) := \sum_{\substack{F \subseteq C \text{ face} \\ \dim F=r}} \alpha(\text{cpl } F) \cdot \beta(\text{O}_F C),$$

It is well known that ν^r is a valuation for all r ; see [35, Lemma 2.3.2] for an elementary proof. This remains true for the generalized spherical intrinsic volumes:

THEOREM 7.1. *Let $\alpha, \beta : \mathcal{C}^d \rightarrow \mathbb{R}$ be cone angles. Then $\xi^r(\alpha, \beta)$ is a valuation for $0 \leq r \leq d$.*

Note that $\xi(\alpha, \beta)$ is not a simple valuation: if C is a linear subspace of dimension $r < d$, then $\xi^r(\alpha, \beta)(C) = \alpha(\mathbb{R}^d)\beta(\mathbb{R}^d) = 1$. Furthermore, recall that we can view the associated interior and exterior cone angles $\hat{\alpha}$ and $\check{\beta}$ as elements in the incidence algebra of the face lattice $\mathcal{F}(C)$. This allows the interpretation

$$\xi^r(C) = (\hat{\alpha} *_r \check{\beta})(F, C),$$

where F is the smallest non-empty face of C , that is, the largest linear subspace contained in C . Theorem 7.1 suggests that higher products such as $(\hat{\alpha}_1 *_r \check{\alpha}_2) *_s \check{\alpha}_3$ are valuations as well. This, unfortunately, is not the case as the simplicity of α and β is essential in the proof of Theorem 7.1.

Proof of Theorem 7.1. By [31] it suffices to show that $\xi = \xi^r(\alpha, \beta)$ is a *weak* valuation: For every cone $C \subset \mathbb{R}^d$ and H a linear hyperplane we need to show that

$$(17) \quad \xi(C) = \xi(C \cap H^{\leq}) + \xi(C \cap H^{\geq}) - \xi(C \cap H).$$

It is sufficient to assume that $C \not\subseteq H$ and that H meets the relative interior of C . Then the cones $C^{\leq} := C \cap H^{\leq}$ and $C^{\geq} := C \cap H^{\geq}$ are of the same dimension as C and $C^= := C \cap H$ is of dimension $\dim(C) - 1$.

To show (17), we need to consider all r -faces of C^{\leq} , C^{\geq} , and $C^=$. These faces are either faces of C or are obtained by intersecting faces of C with H^{\leq} , H^{\geq} or H in the following ways. Let F be an r -face of C .

- Case 1. If $\text{relint}(F) \cap H = \emptyset$, then F is contained in H^{\leq} or H^{\geq} and F is an r -face of C^{\leq} or C^{\geq} , respectively.
- Case 2. If $F \subseteq H$, then F is an r -face of C^{\leq} , C^{\geq} , and $C^=$.
- Case 3. If H intersects $\text{relint}(F)$ in a proper subset, then $F \cap H^{\leq}$ is an r -face of C^{\leq} and $F \cap H^{\geq}$ is an r -face of C^{\geq} . Further more $F \cap H$ is an $(r - 1)$ -face of C^{\leq} , C^{\geq} and $C^=$.

Furthermore, let $G \subseteq C$ be an $(r + 1)$ -face.

- Case 4. If $\text{relint}(G) \cap H \neq \emptyset$, then $G \cap H$ is an r -face of C^{\leq} , C^{\geq} , and $C^=$.

We consider the contributions of each case to (17) separately:

Case 1. Without loss of generality we can assume that $F \subseteq H^{\leq}$ so that F is a face of C^{\leq} as well. In this case $O_F C = O_F C^{\leq}$ and thus F gives the same contribution to $\xi(C)$ and $\xi(C^{\leq})$ and none to $\xi(C^{\geq})$ and $\xi(C^=)$.

Case 2. As F is a face of all four cones, we have $O_F C = O_F C^{\leq} \cup O_F C^{\geq}$ with $O_F C^{\leq} \cap O_F C^{\geq} = O_F C^=$. The contribution on the right-hand side is then

$$\alpha(\text{cpl } F)(\beta(O_F C^{\leq}) + \beta(O_F C^{\geq}) - \beta(O_F C^=)),$$

which equals to $\alpha(\text{cpl } F)\beta(O_F C)$ as β is a valuation.

Case 3. Set $F^{\leq} = F \cap H^{\leq}$ and $F^{\geq} = F \cap H^{\geq}$, which are faces of C^{\leq} and C^{\geq} , respectively. Since the normal cone is polar to the tangent cone and the tangent cone of a face F is determined by any neighborhood of a point $q \in \text{relint}(F)$, we have $O_F C = O_{F^{\leq}} C^{\leq} = O_{F^{\geq}} C^{\geq}$. The contribution on the right-hand side is therefore

$$(\alpha(\text{cpl } F^{\leq}) + \alpha(\text{cpl } F^{\geq}))\beta(O_F C)$$

which is precisely $\alpha(\text{cpl } F)\beta(O_F C)$ since α is a simple valuation.

Case 4. Here $F^= = F \cap H$ is a common face of C^{\leq} , C^{\geq} , and $C^=$. Since $O_{F^=} C^= = O_{F^=} C^{\leq} \cup O_{F^=} C^{\geq}$ and β is a simple valuation, the contribution to the right-hand side is 0. □

Since $\xi^r = \xi^r(\alpha, \beta)$ is a valuation, we immediately obtain the following from the Brianchon-Gram relation. For a d -polytope P and a face $F \subseteq P$ define $\widehat{\xi}_i^r(P) := \sum_F \xi^r(T_F P)$ where the sum is over all i -faces F of P . Applying ξ^r to the general form of the Brianchon–Gram relation (8), we obtain

COROLLARY 7.2. *Let $\alpha, \beta : \mathcal{C}^d \rightarrow \mathbb{R}$ be cone angles and $\xi^r = \xi^r(\alpha, \beta)$ for some $0 \leq r \leq d$. Then*

$$\widehat{\xi}_0^r(P) - \widehat{\xi}_1^r(P) + \widehat{\xi}_2^r(P) - \dots + (-1)^{\dim d} \cdot \widehat{\xi}_d^r(P) = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{if } r > 0 \end{cases}.$$

If β is the (unique) cone angle complementary to α , we will simplify the notation and write $\xi^r(\alpha) := \xi^r(\alpha, \beta)$. When unraveling the definition of complementary angles, we obtain an equation sometimes called the Gauss–Bonnet Theorem for polyhedral cones [2]. Let χ be the Euler characteristic on cones with $\chi(D) = 0$ if D is not a linear subspace and $\chi(D) = (-1)^{\dim D}$ otherwise.

LEMMA 7.3. *Let α be a cone angle and $C \in \mathcal{C}^d$. Then*

$$\sum_{r=0}^d (-1)^r \xi^r(\alpha)(C) = \chi(C).$$

REMARK 7.4. The classical spherical intrinsic volumes furthermore sum to 1. This is not necessarily true for generalized spherical intrinsic volumes, but one can show using some calculation involving Lemma 2.4 that this holds for $\xi(\alpha)$ when we additionally assume that the cone angle α is *even*, that is, that $\alpha(C) = \alpha(-C)$ for all polyhedral cones $C \subseteq \mathbb{R}^d$.

We will now express the spherical intrinsic volumes in a different basis. Using integral geometry, we can relate the usual spherical intrinsic volumes ν^r to certain integrals over the **Grassmannian** $\text{Gr}^{r,d}$ of r -dimensional linear subspaces in \mathbb{R}^d . The **Haar measure** μ is the unique $O(d)$ -invariant measure on $\text{Gr}^{r,d}$ such that $\mu(\text{Gr}^{r,d}) = 1$. The r -th **Grassmann-angle** of a cone $C \subset \mathbb{R}^d$

$$\kappa^r(C) := \mu(\{L \in \text{Gr}^{r,d} : L \cap C = \{0\}\})$$

was introduced by Grünbaum in [17] as a generalization of interior and exterior angles. Indeed $2\nu(C) = 1 - \kappa^1(C)$ and $2\nu(C^\vee) = \kappa^{d-1}(C)$. To generalize the Grassmann angles to arbitrary valuations we need to shift our point of view. For a fixed pointed cone $C \in \mathcal{C}^d$, define the μ -measurable function $\varepsilon_C : \text{Gr}^{r,d} \rightarrow \{0, 1\}$ with $\varepsilon_C(L) := 1$ if $C \cap L = \{0\}$ and 0 otherwise. The Grassmann-angle can now be expressed as the integral over ε_C

$$\kappa^r(C) = \int_{\text{Gr}^{r,d}} \varepsilon_C(L) d\mu(L)$$

Recall the kinematic formulas for cones.

THEOREM 7.5 ([2, Theorem 5.1]). *Let $C \subseteq \mathbb{R}^d$ be a polyhedral cone. Then for $0 \leq r \leq d$ and $1 \leq k \leq d$:*

$$\int_{\text{Gr}^{r,d}} \nu^k(C \cap L) d\mu(L) = \nu^{k+d-r}(C), \quad \int_{\text{Gr}^{r,d}} \nu^0(C \cap L) d\mu(L) = \sum_{j=0}^{d-r} \nu^j(C).$$

For a fixed cone, we have almost surely $\varepsilon_C(L) = \chi(C \cap L)$. From Lemma 7.3 we get

$$\chi(C) = \sum_{i=0}^d (-1)^i \nu^i(C),$$

and we compute

$$\begin{aligned} \kappa^r(C) &= \int_{\text{Gr}^{r,d}} \varepsilon_C(L) d\mu(L) = \int_{\text{Gr}^{r,d}} \chi(C \cap L) d\mu(L) \\ &= \sum_{i=0}^d (-1)^i \int_{\text{Gr}^{r,d}} \nu^i(C \cap L) d\mu(L) = \sum_{j=0}^{d-r} \nu^j(C) + \sum_{i=1}^d (-1)^i \nu^{i+d-r}(C) \\ &= \sum_{j=0}^{d-r} \nu^j(C) + \sum_{i=d-r+1}^d (-1)^{i+d-r} \nu^i(C). \end{aligned}$$

This is a slight variation of the usual Crofton-formulas, which better serves our purposes. We refer to [2] for further details. It is not hard to see that spherical intrinsic volumes and Grassmann-angles encode the same quantities in a different basis, and conversely we obtain the intrinsic volumes from the Grassmann-angles as follows:

$$\nu^r = \frac{1}{2}(\kappa^{d-r-1} - \kappa^{d-r+1})$$

for $r = 1, \dots, d - 1$ as well as $\nu^0 = \frac{1}{2}(\kappa^d + \kappa^{d-1})$ and $\nu^d = \frac{1}{2}(\kappa^0 - \kappa^1)$.

Using the generalized spherical volumes allows us to give a generalization of the Grassmann angles, too, by taking the Crofton-formulas as a definition. Thus we define for any two cone angles $\alpha, \beta : \mathcal{C}^d \rightarrow \mathbb{R}$ the **generalized r -th Grassmann-angle**

$$\kappa^r(\alpha, \beta) := \sum_{j=0}^{d-r} \xi^j(\alpha, \beta) + \sum_{i=d-r+1}^d (-1)^{i-d+r} \xi^i(\alpha, \beta).$$

we will simply write $\xi^r(\alpha) = \xi^r(\alpha, \beta)$ and $\kappa^r(\alpha) = \kappa^r(\alpha, \beta)$ if β is the (unique) complementary angle to α .

As a corollary of Theorem 7.1, we have:

COROLLARY 7.6. *Every generalized Grassmann-angle is a valuation.*

From this observation we can draw short proofs for most of the results in Grünbaum’s original paper on Grassmann angles [17] where at the same time we replace the usual Grassmann-angle with our generalized notion $\kappa^r = \kappa^r(\alpha)$. Let us denote by $\widehat{\kappa}_i^r(P)$ the sums of all r -th generalized Grassmann angles of the i -faces of a d -polytope $P \subseteq \mathbb{R}^d$, that is

$$\widehat{\kappa}_i^r(P) := \sum_F \kappa^r(T_F P).$$

We have:

COROLLARY 7.7 (Generalization of Grünbaum [17, Theorem 3.3]). *Let $P \subset \mathbb{R}^d$ be a d -polytope and $\kappa^r = \kappa^r(\alpha)$ for a cone angle α and $0 \leq r \leq d$. Then*

$$\sum_{i=0}^{d-r} (-1)^i \cdot \widehat{\kappa}_i^r(P) = 1.$$

Proof. Since $\xi^0(\{0\}) = 1$ and $\xi^r(\{0\}) = 0$ for all $1 \leq r \leq d$, we have $\kappa^r(\{0\}) = 1$ for all $0 \leq r \leq d$. Applying κ^r to both sides of (8) yields

$$(18) \quad \kappa^r(\{0\}) + \sum_F (-1)^{\dim F+1} \kappa^r(T_F P) + (-1)^{d+1} \kappa^r(\mathbb{R}^d) = 0.$$

If $F \subseteq P$ is a face with $\dim F > d - r$, then $\xi^j(T_F P) = 0$ for all $j \leq d - r$, as the smallest face of $T_F P$ has dimension $\dim F$ and thus the sum in (16) is empty. Thus, by Lemma 7.3

$$\begin{aligned} \kappa^r(T_F P) &= \sum_{j=0}^{d-r} \xi^j(T_F P) + \sum_{i=d-r+1}^d (-1)^{i-d+r} \xi^i(T_F P) \\ &= (-1)^{d-r} \sum_{i=0}^d (-1)^i \xi^i(T_F P) = 0. \end{aligned}$$

With that, we obtain the claim by rearranging (18). □

In a similar fashion, most of the results in [17] can be shown for generalized Grassmann angles. For example, [17, Theorem 3.5] follows from an application κ^r to Lemma 2.4.

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SPENCER BACKMAN, Department of Mathematics and Statistics, University of Vermont, Burlington, Vermont, USA
E-mail : Spencer.Backman@uvm.edu

SEBASTIAN MANECKE, Institut für Mathematik, Goethe-Universität Frankfurt, Germany
E-mail : manecke@math.uni-frankfurt.de

RAMAN SANYAL, Institut für Mathematik, Goethe-Universität Frankfurt, Germany
E-mail : sanyal@math.uni-frankfurt.de