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On the association scheme of perfect matchings and their designs

John Bamberg & Lukas Klawuhn

Dedicated to the memory of Kai-Uwe Schmidt,
who started us on the journey of this research.

Abstract We investigate generalisations of 1-factorisations and hyperfactorisations of the complete graph K_{2n}. We show that they are special subsets of the association scheme obtained from the Gelfand pair $(S_{2n}, S_2 \wr S_n)$. This unifies and extends results by Cameron (1976) and gives rise to new existence and non-existence results. Our methods involve working in the group algebra $\mathbb{C}[S_{2n}]$ and using the representation theory of S_{2n}.

1. Introduction

It is well-known that the complete graph K_{2n} on an even number of vertices can always be decomposed into perfect matchings, called a *1-factorisation*. Equivalently, we can view a 1-factorisation as a set of perfect matchings of K_{2n} such that every edge of K_{2n} is contained in exactly one perfect matching. From this point of view, one can generalise 1-factorisations to *hyperfactorisations*, studied by Jungnickel and Vanstone [13]. A *hyperfactorisation* is a set of perfect matchings of K_{2n} such that every pair of disjoint edges is contained in exactly c perfect matchings, where c is a constant called the *index* of the hyperfactorisation. Boros, Jungnickel, and Vanstone [3] showed that K_{2n} has a non-trivial hyperfactorisation for every $n \geqslant 5$.

Cameron [5] generalised hyperfactorisations (or 'parallelisms of $\binom{X}{t}$') to partition systems. He defines an s-(t, n) *partition system* to be a collection S of partitions of an n-set X into t-subsets having the property that given any s disjoint t-subsets of X, there is a unique partition in S which has all of the given t-subsets as parts. This is a natural way to generalise 1-factorisations and hyperfactorisations of complete graphs. Indeed, a 1-factorisation of K_n is simply a 1-$(2, n)$ partition system. Independently, Stinson [23] defined *hyperresolutions* of a resolvable block design, which turn out to be 2-(t, n) partition systems. Cameron constructed infinitely many examples of 2-$(2, n)$ partition systems from hyperovals of projective planes of even order.

We generalise the notion of s-$(2, n)$ partition systems to λ-*factorisations* by replacing the parameter s with a partition λ and use the theory of association schemes to investigate them. In an s-$(2, n)$ partition system, any set of $s - 1$ subsets of size 2 appears a constant number of times. The counting involved does not give rise to restrictions on the parameters s and n (see [5, Theorem 7.2]). We show that similar

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counting arguments work for λ-factorisations and give rise to non-trivial divisibility conditions by comparing different partitions λ.

This work can be considered a follow-up to the work of Rands [20] on the association scheme of perfect matchings of the complete graph K_{2n}. Rands calculated the eigenvalues of the association scheme for $n \in \{4, 5, 6\}$, and he also observed that we can view abstract hyperovals and 1-factorisations as cliques for this association scheme. We take this viewpoint further and consider λ-factorisations as designs in the association scheme, and use Delsarte theory to produce results on the existence of such designs. Our main result is a characterisation of λ-factorisations as designs:

Theorem 1.1 (Paraphrase of Theorem 4.4). *A non-empty set of perfect matchings is a λ-factorisation if and only if its dual degree set contains no partition $\mu \vdash n$ with $\lambda \trianglelefteq \mu \neq (n)$.*

In Section 4, we give several interesting corollaries of this result. One of which (Corollary 4.12), is that for every $n \geqslant 4$, an $(n-2, 2)$-factorisation⁽¹⁾ of index coprime to 3 (in particular index 1) can only exist if $n \equiv 0 \pmod 3$. Also, in Corollary 4.17 we show that there is no $(2, 1, \ldots, 1)$-factorisation for all $n \geqslant 4$, generalising a result of Cameron [5, Theorem 7.6] on partition systems.

2. Preliminaries

In this section, we fix some notation and introduce the basic definitions of partitions and irreducible characters.

2.1. Partitions.

Many combinatorial objects can be indexed by partitions. Here, we give a brief overview of the main definitions. We mostly follow [17, Ch. I].

An (integer) *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers that sum to a finite number and satisfy $\lambda_i \geqslant \lambda_{i+1}$ for every $i \in \mathbb{N}$. The *size* of λ is the number $|\lambda| = \sum_{i \geqslant 1} \lambda_i$ and if $|\lambda| = n$, we say that λ is a partition of n, denoted by $\lambda \vdash n$. We often write partitions as finite sequences $(\lambda_1, \ldots, \lambda_k)$ only writing down the non-zero parts. The number of non-zero parts of λ is the *length* of λ, denoted by $l(\lambda)$. We use the notation $\lambda!$ for the product of the factorials $\lambda_i!$.

For a partition $\lambda \vdash n$, we denote by 2λ the partition of $2n$ with parts $(2\lambda_1, 2\lambda_2, \ldots)$. If $\lambda \vdash n$, then the *Young diagram* of λ is an array of n boxes with left-justified rows and top-justified columns such that row i contains exactly λ_i boxes. We use the dominance order $\trianglelefteq$ to compare partitions. Let $\lambda, \mu \vdash n$ be two partitions. We write $\lambda \trianglelefteq \mu$ and say that λ is *dominated* by μ or μ *dominates* λ if

$$\sum_{i=1}^{k} \lambda_i \leqslant \sum_{i=1}^{k} \mu_i \quad \text{for all } k \geqslant 1.$$

A helpful fact is that we have $\lambda \trianglelefteq \mu$ if and only if the Young diagram of λ can be transformed into the Young diagram of μ by moving boxes from the end of a row of λ to a higher row one at a time. When interpreting a partition as a sequence, this corresponds to adding vectors of the form $(0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots)$ to the partition.

We will also make heavy use of *set partitions.*

Definition 2.1. *Let $n \in \mathbb{N}$. A* set partition *is a set $\{P_1, P_2, \ldots\}$ of non-empty disjoint subsets $P_i \subseteq \{1, \ldots, n\}$ such that $P_1 \cup P_2 \cup \ldots = \{1, \ldots, n\}$. The partition λ whose parts are the sizes $|P_i|$ (ordered non-increasingly) is called the* shape *of the*

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(1) This is a set D of perfect matchings of K_{2n} such that for every 4-subset F of the vertices, there are precisely c elements of D which have two edges contained in F (and c is the *index*).

set partition. An ordered set partition *is a tuple* $(P_1, P_2, \ldots)$ *of non-empty disjoint subsets* $P_i \subseteq \{1, \ldots, n\}$ *such that* $\{P_1, P_2, \ldots\}$ *is a set partition. The shape of an ordered set partition is the tuple* $(|P_1|, |P_2|, \ldots)$ *(which is not necessarily ordered non-increasingly).*

A *perfect matching* of the complete graph K_{2n} is simply a set partition of the set $\{1, \ldots, 2n\}$ into n subsets of size 2. Its shape is $(2, \ldots, 2) \vdash 2n$. It is well-known that K_{2n} has exactly

$$(2n-1)!! = (2n-1) \cdot (2n-3) \cdot \cdots \cdot 5 \cdot 3 \cdot 1$$

perfect matchings.

2.2. Characters. Since many formulas in association schemes defined on groups involve irreducible characters, we now turn to representation and character theory. We refer the reader to [22] for a more extensive introduction to the representation theory of finite groups. For a finite group G, we will refer to ordinary representations and characters, and the *degree* of a character will be the dimension of the vector space for the associated representation.

We remind the reader that the class functions of G form an inner product space under the inner product given by

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}$$

and the irreducible characters form an orthonormal basis of the space of class functions with respect to $\langle \cdot, \cdot \rangle$.

Furthermore, if φ is an arbitrary character and χ is an irreducible character, then $\langle \varphi, \chi \rangle$ is the *multiplicity* of χ in the decomposition of φ into irreducible characters. If $\langle \varphi, \chi \rangle \neq 0$, then χ is called an *irreducible constituent* of φ. If the multiplicity of every irreducible constituent of φ is equal to 1, then φ is called *multiplicity-free.*

Let $H \leqslant G$. If χ is a character of G, then restricting χ to H gives a character of H. This character is called the *restriction* of χ to H and denoted by $\chi \downarrow_H^G$. There is a way to extend a character of a subgroup H to a character of G called *induction.* If χ is a character of H, then the induced character is denoted by $\chi \uparrow_H^G$. The function 1_G with $1_G(g) = 1$ for all $g \in G$ is a character of G, called the *trivial character* of G.

2.3. Character theory of the symmetric group. It is well-known that the irreducible characters of the symmetric group S_n are indexed by the partitions of n. If $\lambda \vdash n$, then χ^λ denotes the irreducible character of the symmetric group S_n indexed by the partition λ. Notice that the symmetric group S_n naturally acts on ordered set partitions. The stabiliser of an ordered set partition is of the form

$$S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_{l(\lambda)}}.$$

The subgroup S_λ is called a *Young subgroup* of shape λ. All Young subgroups of the same shape are conjugate to each other. Hence, all characters obtained by inducing a character from a Young subgroup of a given shape to the whole group are equal. For two partitions λ, μ of n, we have "Young's rule" (cf. [21, Theorem 2.11.2]):

$$\langle 1 \uparrow_{S_\lambda}^{S_n}, \chi^\mu \rangle \neq 0 \text{ if and only if } \lambda \trianglelefteq \mu.$$

3. Association schemes and Gelfand pairs

3.1. Association schemes. Association schemes are a helpful structure to investigate interesting combinatorial structures and they provide powerful techniques to analyse them. We refer the reader to [2] for an extensive introduction to association schemes. An *association scheme* (with d classes) is a pair $(X, \mathcal{R})$ of a finite set X with $|X| \geqslant 2$ and a family of relations $\mathcal{R} = (R_i)_{i=0,\dots,d}$ subject to the following conditions:

(A0) $\mathcal{R}$ forms a set partition of $X \times X$; that is, $R_i \cap R_j = \varnothing$ for $i \neq j$ and $\bigcup\limits_{i=0}^{d} R_i = X \times X$.
(A1) R_0 is the 'equality' relation; that is, $R_0 = \{(x, x) \in X \times X \mid x \in X\}$.
(A2) $\mathcal{R}$ is closed under converses; that is, $R_i^\top = \{(y, x) \in X \times X \mid (x, y) \in R_i\} = R_j$ for some $j \in \{0, \dots, d\}$.
(A3) Let $i, j, k \in \{0, \dots, d\}$. If $(x, y) \in R_k$, then the number p_{ij}^k of elements $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ only depends on i, j and k but not on the choice of (x, y).
(A4) For every $i, j, k \in \{0, \dots, d\}$ we have $p_{ij}^k = p_{ji}^k$.

For finite and non-empty sets X and Y, let $\mathbb{C}(X, Y)$ denote the set of all complex $|X| \times |Y|$-matrices with rows indexed by X and columns indexed by Y. For a matrix $A \in \mathbb{C}(X, Y)$ and $x \in X$, $y \in Y$, the (x, y)-entry of A is denoted by $A(x, y)$. If $|Y| = 1$, then we omit Y, so $\mathbb{C}(X)$ is the set of complex column vectors indexed by X.

Let (X, R) be an association scheme. For $i \in \{0, \dots, d\}$, let $A_i \in \mathbb{C}(X, X)$ be the i-th *adjacency matrix* given by

$$A_i(x, y) = \begin{cases} 1, & \text{if } (x, y) \in R_i \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Let $\mathbb{A} = \operatorname{sp}(A_0, \dots, A_d)$ be the vector space generated by $A_0, \dots, A_d$ over the complex numbers. Then $\mathbb{A}$ is a commutative matrix algebra with basis $A_0, \dots, A_d$ that contains the identity matrix and is closed under conjugate transposition. Thus, the zero-one-matrices A_i define an association scheme. The algebra $\mathbb{A}$ is called the *Bose–Mesner algebra* of the association scheme.

We can rewrite the definition of an association scheme in terms of matrices. Let X be a finite set with $|X| \geqslant 2$ and $(A_i)_{i=0,\dots,d}$ be a family of matrices in $\mathbb{C}(X, X)$ with entries 0 or 1 subject to the following conditions:

(A0') $\sum_{i=0}^{d} A_i = J$ where J denotes the all-ones matrix;
(A1') $A_0 = I$ where I denotes the identity matrix;
(A2') $A_i^\top = A_j$ for some $j \in \{0, \dots, d\}$;
(A3') For every $i, j \in \{0, \dots, d\}$ we have $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$;
(A4') For every $i, j \in \{0, \dots, d\}$ we have $A_i A_j = A_j A_i$.

The matrices A_i of an association scheme fulfil these conditions and if a family of matrices fulfils these conditions, it generates the Bose–Mesner algebra of an association scheme.

Since the Bose–Mesner algebra $\mathbb{A}$ of an association scheme is commutative and closed under conjugate transposition, all of its elements are normal and can be simultaneously diagonalised via a unitary matrix. Therefore, there exists a basis $E_0, \dots, E_d$ of $\mathbb{A}$ consisting of Hermitian matrices with the property

$$E_k E_l = \delta_{kl} E_k. \tag{2}$$

The matrices E_k are the minimal idempotents of $\mathbb{A}$.

Let V_k be the column space of E_k. The spaces V_k are called the *eigenspaces* of the association scheme because they are the common eigenspaces of the matrices A_i. From (2), the V_k are pairwise orthogonal and $\mathbb{C}(X) = \bigoplus_{k=0}^{d} V_k$.

Let $Y \subseteq X$ be a non-empty subset. We can associate two sequences of numbers with Y: the *inner distribution* a and the *dual distribution* a'. The inner distribution of Y is the tuple $(a_0, \dots, a_d)$, where

$$a_i = \frac{1}{|Y|} \sum_{x,y \in Y} A_i(x,y), \tag{3}$$

and the dual distribution of Y is the tuple $(a'_0, \dots, a'_d)$, where

$$a'_k = \frac{|X|}{|Y|} \sum_{x,y \in Y} E_k(x,y). \tag{4}$$

It is immediate that the inner distribution is non-negative. The same holds for the dual distribution. Let $\mathbb{1}_Y \in \mathbb{C}(X)$ be the characteristic vector of Y, so $\mathbb{1}_Y(x) = 1$ if $x \in Y$ and $\mathbb{1}_Y(x) = 0$ otherwise. Since the matrix E_k is Hermitian, it follows that

$$\frac{|Y|}{|X|} a'_k = \mathbb{1}_Y^\top E_k \mathbb{1}_Y = \mathbb{1}_Y^* E_k^* E_k \mathbb{1}_Y = ||E_k \mathbb{1}_Y||^2. \tag{5}$$

Thus, the dual distribution is real and non-negative. Furthermore, the case $a'_k = 0$ occurs if and only if $\mathbb{1}_Y$ is orthogonal to V_k.

As both $A_0, \dots, A_d$ and $E_0, \dots, E_d$ are bases of $\mathbb{A}$, there are numbers $P_i(k), Q_k(i) \in \mathbb{C}$ such that

$$A_i = \sum_{k=0}^{d} P_i(k) E_k \quad \text{and} \quad E_k = \frac{1}{|X|} \sum_{i=0}^{d} Q_k(i) A_i.$$

Since the matrices E_k are pairwise orthogonal idempotents, the first equation shows that $P_i(k)$ is an eigenvalue of A_i for every $i \in \{0, \dots, d\}$. The matrix $P = (P_j(i))_{i,j=0,\dots,d}$ is called the *matrix of eigenvalues* of the association scheme. Writing $Q = (Q_j(i))_{i,j=0,\dots,d}$, we find that $PQ = |X|I$. The matrix Q is called the *matrix of dual eigenvalues* of the association scheme.

Using the dual eigenvalues, we can give another formula for the dual distribution of a subset Y. It is the Q-transform of the inner distribution, that is

$$a'_k = \sum_{i=0}^{n} Q_k(i) a_i. \tag{6}$$

Thus, knowledge of the dual eigenvalues is crucial for computing the dual distribution.

The main reason why the theory of association schemes is such a powerful tool in combinatorics, is the observation that interesting combinatorial structures can often be characterised using the inner or dual distribution. Delsarte [8] calls these objects *cliques* and *designs*, respectively.

Definition 3.1. *Let Y be a non-empty subset of X with inner distribution $(a_0, \dots, a_d)$ and dual distribution $(a'_0, \dots, a'_d)$.*

(a) *Let $D \subseteq \{1, \dots, d\}$. We call Y a D-*clique *or a D-*code *if $a_d = 0$ for every $d \notin D$.*

(b) *Let $T \subseteq \{1, \dots, d\}$. We call Y a T-*design *if $a'_t = 0$ for every $t \in T$.*

To match the definition of T-design with the expression we used to describe our main theorem, the reader will need the customary notion of the *dual degree set* of a set. The dual degree set of Y is the set of indices $j \in \{1, \dots, d\}$ such that $E_j \mathbb{1}_Y \neq 0$. Therefore, Y is a T-design if and only if its dual degree set is disjoint from T.

3.2. Group algebras. The theory of complex group algebras will prove useful to derive results about association schemes. Denote by $\mathbb{C}[G]$ the complex group algebra of G. It is the vector space $\mathbb{C}^G$ of complex functions on G together with convolution as multiplication. The group algebra $\mathbb{C}[G]$ is generated by the basis $(e_\sigma)_{\sigma\in G}$. We can think of a function $f : G \to \mathbb{C}$ as the formal sum

$$f = \sum_{\sigma\in G} f(\sigma)e_\sigma.$$

Going the other way, we can think of an element $f \in \mathbb{C}[G]$ as a function from G to $\mathbb{C}$ where $f(\sigma)$ is the coefficient of e_σ when f is written in terms of the basis $(e_\sigma)_{\sigma\in G}$. Since every character χ of G is a function from G to $\mathbb{C}$, every character is an element of the group algebra $\mathbb{C}[G]$. The multiplication of two elements $f, g \in \mathbb{C}[G]$ is defined as

$$(f * g)(\sigma) = \sum_{\tau\in G} f(\sigma\tau^{-1})g(\tau).$$

Let H be a subgroup of G. Denote by e_H the element

$$e_H = \frac{1}{|H|}\sum_{h\in H} e_h.$$

Then $e_H * e_H = e_H$ and e_H is an idempotent of the group algebra. As a function from G to $\mathbb{C}$, we have $e_H(\sigma) = |H|^{-1}$ if $\sigma \in H$ and $e(\sigma) = 0$ otherwise. Inside $\mathbb{C}[G]$, we can investigate the subalgebra $e_H * \mathbb{C}[G] * e_H$. As convolution of a function $f : G \to \mathbb{C}$ with e_H roughly corresponds to averaging f over H (from the left or from the right), we find that $e_H * \mathbb{C}[G] * e_H$ is the algebra of functions $f : G \to \mathbb{C}$ such that

$$f(h\sigma h') = f(\sigma) \quad \text{for all } h, h' \in H,$$

that is, the functions that are both left- and right-invariant under multiplication by elements of H. Thus, $e_H * \mathbb{C}[G] * e_H$ is the algebra of functions from G to $\mathbb{C}$ that are constant on the double cosets $H\sigma H$ of G. This algebra will be denoted by $\mathbb{C}[H\backslash G/H]$. It inherits the inner product $\langle\cdot,\cdot\rangle$ from $\mathbb{C}[G]$.

3.3. Gelfand pairs. We refer the reader to [17, Ch. VII.1] for an introduction to Gelfand pairs. A (finite) *Gelfand pair* is a pair of a group G and a subgroup $H \leqslant G$ such that the group algebra $\mathbb{C}[H\backslash G/H]$ is commutative. Equivalently, (G, H) is a Gelfand pair if and only if the induced character $1\uparrow_H^G$ of H is multiplicity-free (by Schur's Lemma). We will only work with finite Gelfand pairs, hence the following theory assumes that G is finite.

Every Gelfand pair (G, H) gives rise to an association scheme by considering the transitive action of G on the (left) cosets of H, and taking the orbits of G on pairs for the relations. Letting $G = S_{2n}$ and $H = S_2 \wr S_n$, we have that

$$1\uparrow_{S_2\wr S_n}^{S_{2n}} = \sum_{\lambda\vdash n} \chi^{2\lambda},$$

hence $(S_{2n}, S_2 \wr S_n)$ is a Gelfand pair. We refer the reader to [17, Ch. VII.2] for an excellent treatment of this Gelfand pair. In spirit with the theory of Coxeter groups, we denote the subgroup $S_2\wr S_n$ by B_n. We have that $|B_n| = 2^n n!$. The subgroup B_n is not a normal subgroup of S_{2n} so S_{2n}/B_n is not a group and hence we cannot work directly with characters and permutation representations. However, every Gelfand pair has so called *zonal spherical functions* that behave very similarly to irreducible characters, and many calculations with irreducible characters in a group can be mimicked with zonal spherical functions in a Gelfand pair.

The zonal spherical functions of the Gelfand pair (S_{2n}, B_n) are functions from S_{2n} to the complex numbers. They are indexed by the partitions of n. In the group algebra $\mathbb{C}[S_{2n}]$, the zonal spherical functions ω^λ are given by

$$\omega^\lambda = \chi^{2\lambda} * e_{B_n} = e_{B_n} * \chi^{2\lambda}$$

where $\chi^{2\lambda}$ is the irreducible character of S_{2n} indexed by the partition 2λ. From the definition, it is immediate that the zonal spherical functions are constant on the double cosets $B_n \sigma B_n$ so we find that $\omega^\lambda \in \mathbb{C}[B_n \backslash S_{2n}/B_n]$. The double cosets are indexed by partitions of n and we denote the value of the zonal spherical function ω^λ on the double coset of type ρ by ω^λ_ρ. Explicitly, the value of ω^λ on an element $\sigma \in S_{2n}$ is given by

$$\omega^\lambda(\sigma) = \frac{1}{|B_n|} \sum_{b \in B_n} \chi^{2\lambda}(\sigma b).$$

Following Macdonald [17, Ch. VII.2], we use the inner product

$$\langle f, g \rangle = \sum_{\sigma \in S_{2n}} f(\sigma)\overline{g(\sigma)}$$

for $f, g \in \mathbb{C}[B_n \backslash S_{2n}/B_n]$, omitting the scaling by $|S_{2n}|$. The zonal spherical functions have the following properties.

Theorem 3.2 (cf. [17, Ch. VII (1.4)]). *Let $n \in \mathbb{N}$ and $\lambda, \mu \vdash n$. Then,*

(i) $\omega^{(n)} = 1$;
(ii) $\omega^\lambda(\sigma^{-1}) = \omega^\lambda(\sigma)$ *for all* $\sigma \in S_{2n}$;
(iii) $\langle \omega^\lambda, \omega^\mu \rangle = \delta_{\lambda\mu}(2n)!\chi^{2\lambda}(1)^{-1}$;
(iv) $\omega^\lambda * \omega^\mu = \delta_{\lambda\mu} c_\lambda \omega^\lambda$ *for some* $c_\lambda \neq 0$;
(v) *The zonal spherical functions* $(\omega^\lambda)_{\lambda \vdash n}$ *form an orthogonal basis of the algebra* $\mathbb{C}[B_n \backslash S_{2n}/B_n]$.

So the zonal spherical functions can be thought of as analogues of the irreducible characters of a group.

3.4. The perfect matching association scheme. We now describe the association scheme obtained from the Gelfand pair (S_{2n}, B_n) in more detail. (Some of what we cover below can be found in [16, §4].) The base set of the association scheme is the coset space S_{2n}/B_n. As B_n is the stabiliser of a perfect matching of $2n$ points, we find that the cosets of B_n are in bijection with the perfect matchings of the complete graph on the vertices $1, \dots, 2n$. We can thus think of cosets of B_n as perfect matchings and vice versa.

Denote by m^* the perfect matching of the vertices $2i-1$ and $2i$ for $i = 1, \dots, n$. This can be thought of as a *base matching* or *identity matching*. It corresponds to the coset of the identity, so $m^* = B_n$. The relations of the scheme are indexed by partitions of n. Two perfect matchings $m, \widetilde{m}$ are in relation R_λ (denoted by $d(m, \widetilde{m}) = \lambda$) if the (disjoint) union of m and $\widetilde{m}$ consists of even cycles of lengths $2\lambda_1, 2\lambda_2, \dots$. Figure 1 gives an example.

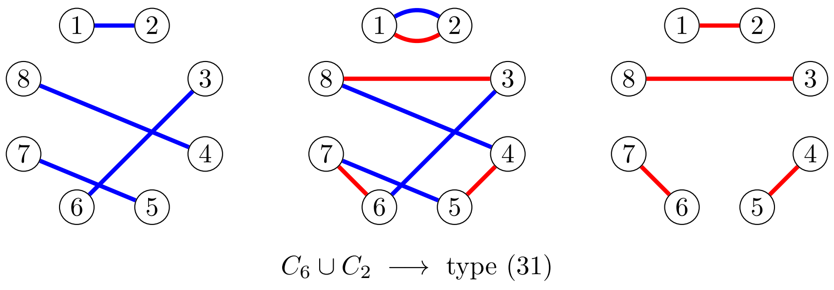

Figure 1. Two perfect matchings and their union.

The set $\Omega_\lambda = \{\sigma B_n \mid d(m^*, \sigma B_n) = \lambda\}$ is called the λ-*sphere*. We have that

$$d(\sigma B_n, \tau B_n) = \lambda \iff d(B_n, \sigma^{-1}\tau B_n) = \lambda \iff \sigma^{-1}\tau B_n \in \Omega_\lambda.$$

If $\sigma \in S_{2n}$ such that $\sigma B_n \in \Omega_\lambda$, we say that σ has *coset type* λ. This is justified by the fact that σ and τ have the same coset type if and only if $\tau \in B_n\sigma B_n$ (see [17, Ch. VII (2.1)]). The double cosets are indexed by partitions of n. The reader should note that the zonal spherical functions are constant on the λ-spheres just like the irreducible characters of a group are constant on the conjugacy classes.

3.5. Eigenvalues of the perfect matching association scheme. Knowledge of the (dual) eigenvalues of an association scheme is of crucial importance when investigating the dual distribution and designs. Godsil and Meagher [11] have computed the eigenvalues of an association scheme obtained from a finite Gelfand pair. We repeat the result here.

Theorem 3.3 ([11, Cor. 13.8.2]). *Let (G, H) be a finite Gelfand pair and $\mathbb{A}$ be the association scheme obtained from the action of G on the left cosets of H. The matrix idempotents of $\mathbb{A}$ are E_ϕ where the entries are given by*

$$E_\phi(x_iH, x_jH) = \frac{\phi(1)}{|G|}\sum_{\sigma\in H}\phi(x_i\sigma x_j^{-1})$$

and ϕ is an irreducible constituent of $1\uparrow_H^G$.

Corollary 3.4. *The idempotents of the perfect matching association scheme are given by*

$$E_\mu(x_iB_n, x_jB_n) = \frac{\chi^{2\mu}(1)}{(2n)!}\sum_{\sigma\in B_n}\chi^{2\mu}(x_i\sigma x_j^{-1}).$$

The (x_iB_n, x_jB_n)-entry of E_μ is equal to $Q_\mu(\rho)/(2n-1)!!$ where $d(x_iB_n, x_jB_n) = \rho$, equivalently $x_i^{-1}x_j$ has coset type ρ. We obtain

$$Q_\mu(\rho) = \chi^{2\mu}(1)\omega_\rho^\mu.$$

Proof. Notice that

$$\begin{aligned}\frac{\chi^{2\mu}(1)}{(2n)!}\sum_{\sigma\in B_n}\chi^{2\mu}(x_i\sigma x_j^{-1}) &= \frac{\chi^{2\mu}(1)}{(2n)!}\sum_{\sigma\in B_n}\chi^{2\mu}(x_j^{-1}x_i\sigma)\\ &= \frac{\chi^{2\mu}(1)|B_n|}{(2n)!}\omega^\mu(x_j^{-1}x_i)\\ &= \frac{\chi^{2\mu}(1)}{(2n-1)!!}\omega^\mu(x_j^{-1}x_i).\end{aligned}$$

The value ω^μ only depends on the coset type of $x_j^{-1}x_i$ which has the same coset type as $(x_j^{-1}x_i)^{-1} = x_i^{-1}x_j$. ☐

Using this corollary and Equation (6), we obtain a formula for the dual distribution. The dual distribution of a non-empty set $Z \subseteq S_{2n}/B_n$ is given by

$$a'_\mu(Z) = \sum_{\rho\,\vdash\, n} Q_\mu(\rho)a_\rho(Z) = \chi^{2\mu}(1)\sum_{\rho\,\vdash\, n}\omega_\rho^\mu a_\rho(Z).$$

Hence,

$$a'_\mu(Z) = 0 \iff \sum_{\rho\,\vdash\, n}\omega_\rho^\mu a_\rho(Z) = 0.$$

4. Designs

In this section, we introduce a generalisation of hyperfactorisations of K_{2n}.

Definition 4.1. *Let $\lambda \vdash n$ and consider all set partitions of shape 2λ. A set $D \neq \varnothing$ of perfect matchings is called a* λ-factorisation *if there exists a constant $c > 0$ such that for every set partition P of shape 2λ there are exactly c matchings $m \in D$ such that m refines P. The constant c is called the* index *of D.*

Example 4.2. For every n, an $(n-1,1)$-factorisation with index 1 is just a 1-factorisation, and an $(n-2,1,1)$-factorisation is a hyperfactorisation. Cameron's examples [5] arising from hyperovals of projective planes are $(n-2,1,1)$-factorisations of index 1 where $2n = 2^a + 2$ for some integer $a \geqslant 3$. Indeed, Cameron's s-$(2,2n)$ partition systems are precisely the $(n-s,1,1,\dots,1)$-factorisations with index 1.

For every n and $t \leqslant n/2$, an $(n-t,t)$-factorisation is a set $D \neq \varnothing$ of perfect matchings such that for every subset $S \subseteq \{1,\dots,2n\}$ of size $2t$, there are a constant number of perfect matchings in D that have t edges that all lie inside of S. Hence, every subset of size $2t$ is covered constantly often by the perfect matchings of D.

A hyperoval $\mathcal{O}$ of a projective plane Π of order q is a set of $q+2$ points such that every line of Π intersects $\mathcal{O}$ in 0 or 2 points. Each point P not in $\mathcal{O}$ defines a fixed-point-free permutation of $\mathcal{O}$, taking an element X to the unique element X^P of $\mathcal{O}\backslash\{X\}$ lying on the line spanned by P and X. The set of q^2-1 permutations of $\mathcal{O}$ arising this way gives rise to an *abstract hyperoval.* Indeed, each permutation is a complete product of 2-cycles, and so the set of perfect matchings that arise is identical to Cameron's construction. However, not all abstract hyperovals arise from a hyperoval.

Example 4.3. Abstract hyperovals were introduced by J. G. Thompson in [24]. An *abstract hyperoval* $A(X)$ on a finite set X is a set of fixed-point-free involutions on X with the following property: for any four elements $a,b,c,d \in X$, there is a unique $u \in A(X)$ such that $a^u = b$ and $c^u = d$. If $|X| = n+2$, then we say that $A(X)$ has *order* n, and we see from the definition that n is even and $|A(X)| = n^2 - 1$. De Clerck [7] showed that an abstract hyperoval of order $2s$ is equivalent to a partial geometry with parameters $\mathrm{pg}(s, 2s-2, s-1)$. We refer to [6] and [19] for more on this connection.

Now a perfect matching defines a fixed-point-free involution, and vice versa, because a fixed-point-free involution is a complete product of disjoint 2-cycles. So if n is even, a 2-$(2,n+2)$ partition system is a set of n^2-1 perfect matchings of K_{n+2} such that every pair of disjoint edges $\{a,b\}$ and $\{c,d\}$ lies in exactly one perfect matching. So we see readily that abstract hyperovals of order n and 2-$(2,n+2)$ partition systems are equivalent objects. This observation generalises [5, Theorem 7.3], and can also be deduced from [6, Chapter 7] and [10, Lemma 1].

For $n = 2,4$ the only abstract hyperovals are obtained by taking the set of all fixed-point-free involutions. There is no example for $n = 6$ (see [18]) and there are precisely two examples for $n = 8$ (see also [18]). So there are only two 2-$(2,10)$ partition systems up to equivalence, and they have stabilisers $\mathrm{P\Gamma L}(2,8)$ and $M_9 : C_3$ respectively. In our language, there are two $(3,1,1)$-factorisations of index 1. We will look at related examples in Section 5. We remark that Thompson [24] explored the use of the representation theory of S_{2n} to derive restrictions on the existence of abstract hyperovals, and we take this viewpoint further and apply this theory to λ-factorisations.

Our goal is to show the following theorem that characterises λ-factorisations as T-designs in an association scheme.

Theorem 4.4. *Let $D \subseteq S_{2n}/B_n$ be a non-empty set of perfect matchings and $(a'_\mu)_{\mu \vdash n}$ be its dual distribution. Then*

$$D \text{ is a } \lambda\text{-factorisation} \iff a'_\mu = 0 \text{ for all } \mu \vdash n \text{ with } \lambda \trianglelefteq \mu \neq (n).$$

4.1. Antidesigns. We will prove our main theorem (Theorem 4.4) using the notion of *antidesigns*.

Definition 4.5. *Let (X, R) be an association scheme and $A, D \subseteq X$ be non-empty subsets of X with dual distributions $(a'^{A}_k)_{k=0,\dots,d}$ and $(a'^{D}_k)_{k=0,\dots,d}$, respectively.*

1. *The pair (A, D) is called* design-orthogonal, *if $a'^{A}_k \cdot a'^{D}_k = 0$ for every $k > 0$.*
2. *Let $T = \{k \in \{1, \dots, d\} \mid a'^{A}_k \neq 0\}$. Then we call A a T-*antidesign.

It is clear from the definition that if the pair (A, D) is design-orthogonal and A is a T-antidesign, then D must be a T-design. Hence, we can characterise T-designs using T-antidesigns. It turns out (see [9]) that if an association scheme admits an automorphism group that acts transitively on each relation R_i, then a pair (A, D) is design-orthogonal if and only if the intersection $|D \cap \varphi(A)|$ is constant when φ varies over all automorphisms of (X, R).

Applied to the association scheme of perfect matchings, we find that S_{2n} acts transitively on each of the relations R_μ. Let $\lambda \vdash n$ and let P_λ be a set partition of shape 2λ. Consider the set A of all perfect matchings that refine P_λ. A λ-factorisation D of index c intersects A in precisely c elements. Note that S_{2n} acts transitively on the set partitions of shape 2λ and that, from the definition of a λ-factorisation, $|D \cap \varphi(A)|$ is constant for all $\varphi \in S_{2n}$. Thus, the pair (A, D) is design-orthogonal. If A is a T-antidesign, then D is a T-design. Hence, it suffices to compute the set T such that A is a T-antidesign.

We find that A is the set of left cosets of B_n with representatives in $S_{2\lambda}B_n$, where $S_{2\lambda}$ is a Young subgroup. The size of A in the association scheme S_{2n}/B_n is

$$|A| = \frac{|S_{2\lambda}B_n|}{|B_n|} = \frac{|S_{2\lambda}|}{|S_{2\lambda} \cap B_n|} = \frac{|S_{2\lambda}|}{|B_\lambda|} = \frac{(2\lambda)!}{2^n \lambda!}$$

where $B_\lambda = B_{\lambda_1} \times B_{\lambda_2} \times \cdots \times B_{\lambda_{l(\lambda)}}$. We have that $|B_\lambda| = 2^n \lambda!$.

First, we compute the inner distribution of A.

Lemma 4.6. *The inner distribution $a_\rho(A)$ is given by*

$$a_\rho(A) = \frac{1}{2^n \lambda!} |S_{2\lambda} \cap B_n x_\rho B_n|$$

where $x_\rho \in S_{2n}$ is any element of coset type ρ.

Proof. First, recall that $d(\sigma B_n, \tau B_n) = \rho$ if and only if the coset type of $\sigma^{-1}\tau$ is ρ. Since every matching in A corresponds to a left coset σB_n and all elements of σB_n have the same coset type, we get the number of pairs $(\sigma B_n, \tau B_n)$ with $d(\sigma B_n, \tau B_n) = \rho$ by counting the number of pairs $(\sigma, \tau) \in (S_{2\lambda}B_n)^2$ such that $\sigma^{-1}\tau$ is of coset type ρ and dividing the result by $|B_n|^2$. Write

$$S_{2\lambda}B_n = \bigcup_{r \in R} S_{2\lambda} r$$

where $R \subseteq B_n$ is a complete set of representatives of the decomposition of $S_{2\lambda}B_n$ into right cosets of $S_{2\lambda}$. Then every element of $z \in A$ can be uniquely written as $z = \sigma r$ with $\sigma \in S_{2\lambda}$ and $r \in R$. Note that $|R| = |B_n : S_{2\lambda} \cap B_n| = |B_n : B_\lambda| = |A||B_n|/|S_{2\lambda}|$.

Counting the pairs in the symmetric group gives

$$\begin{aligned}
&|\{(\sigma,\tau) \in (S_{2\lambda}B_n)^2 : \text{coset type}(\sigma^{-1}\tau) = \rho\}|\\
&= |\{(\sigma,r,\tau,s) \in (S_{2\lambda}\times R)^2 : (\sigma r)^{-1}\tau s \in B_n x_\rho B_n\}|\\
&= |\{(\sigma,r,\tau,s) \in (S_{2\lambda}\times R)^2 : r^{-1}\sigma^{-1}\tau s \in B_n x_\rho B_n\}|\\
&= |R|^2|\,\{(\sigma,\tau) \in S_{2\lambda}^2 : \sigma^{-1}\tau \in B_n x_\rho B_n\}|\\
&= |S_{2\lambda}|\,|R|^2|\,\{\sigma \in S_{2\lambda} : \sigma \in B_n x_\rho B_n\}|\\
&= |S_{2\lambda}||R|^2|S_{2\lambda}\cap B_n x_\rho B_n|\\
&= \frac{|A|^2|B_n|^2}{|S_{2\lambda}|}|S_{2\lambda}\cap B_n x_\rho B_n|\\
&= \frac{|A||B_n|^2}{|B_\lambda|}|S_{2\lambda}\cap B_n x_\rho B_n|.
\end{aligned}$$

Dividing by $|B_n|^2$ gives the number of pairs in the scheme S_{2n}/B_n and dividing by $|A|$ gives the inner distribution. ☐

Now let $\theta^\lambda = e_{B_n} * 1_{2\lambda} * e_{B_n} \in \mathbb{C}[B_n\backslash S_{2n}/B_n]$ where $1_{2\lambda}$ is the trivial character of $S_{2\lambda}$. In $\mathbb{C}[S_{2n}]$, $1_{2\lambda}$ is the characteristic vector of the subgroup $S_{2\lambda}$. Explicitly, we have

$$\theta^\lambda(\sigma) = (e_{B_n} * 1_{2\lambda} * e_{B_n})(\sigma) = \frac{1}{|B_n|^2}\sum_{a,b\in B_n} 1_{2\lambda}(a\sigma b).$$

We can decompose θ^λ in terms of the orthogonal basis of $\mathbb{C}[B_n\backslash S_{2n}/B_n]$ given by the zonal spherical functions ω^μ. Let $x_\rho \in S_{2n}$ be any element of coset type ρ, recall that ω^μ and θ^λ are constant on the double cosets and calculate

$$\begin{aligned}
\langle \omega^\mu, \theta^\lambda\rangle &= \sum_{\sigma\in S_{2n}} \omega^\mu(\sigma)\overline{\theta^\lambda(\sigma)} = \sum_{\rho\vdash n} |B_n x_\rho B_n|\omega^\mu_\rho \theta^\lambda(x_\rho)\\
&= \frac{1}{|B_n|^2}\sum_{\rho\vdash n}|B_n x_\rho B_n|\,\omega^\mu_\rho \sum_{a,b\in B_n} 1_{2\lambda}(a x_\rho b).
\end{aligned}$$

So it remains to calculate the inner sum. We find

$$\sum_{a,b\in B_n} 1_{2\lambda}(a x_\rho b) = \sum_{a\in B_n}\sum_{b\in B_n} 1_{2\lambda}(a x_\rho b) = \sum_{a\in B_n} |S_{2\lambda}\cap a x_\rho B_n|.$$

We have $a x_\rho B_n = b x_\rho B_n \iff a^{-1}b \in x_\rho B_n x_\rho^{-1}$. Since $a, b \in B_n$, we find that $a x_\rho$ and $b x_\rho$ are representatives of the same coset if and only if $a^{-1}b \in B_n \cap x_\rho B_n x_\rho^{-1}$, equivalently $a(B_n \cap x_\rho B_n x_\rho^{-1}) = b(B_n \cap x_\rho B_n x_\rho^{-1})$. Let $R \subseteq B_n$ be a system of representatives of the left cosets of $B_n\,/\,(B_n \cap x_\rho B_n x_\rho^{-1})$. Then we find

$$\begin{aligned}
\sum_{a\in B_n} |S_{2\lambda}\cap a x_\rho B_n| &= \sum_{r\in R}\sum_{\sigma\in B_n\cap x_\rho B_n x_\rho^{-1}} |S_{2\lambda}\cap r\sigma x_\rho B_n|\\
&= \sum_{r\in R}\sum_{\sigma\in B_n\cap x_\rho B_n x_\rho^{-1}} |S_{2\lambda}\cap r x_\rho B_n|\\
&= \sum_{r\in R} |B_n\cap x_\rho B_n x_\rho^{-1}|\,|S_{2\lambda}\cap r x_\rho B_n|\\
&= |B_n\cap x_\rho B_n x_\rho^{-1}|\sum_{r\in R}|S_{2\lambda}\cap r x_\rho B_n|\\
&= |B_n\cap x_\rho B_n x_\rho^{-1}|\,|S_{2\lambda}\cap B_n x_\rho B_n|.
\end{aligned}$$

We have that (cf. [17, Ch. VII (2.3)])

$$|B_n x_\rho B_n| = \frac{|B_n|^2}{|B_n \cap x_\rho B_n x_\rho^{-1}|}.$$

Putting it all together, we obtain

$$\begin{aligned}\langle \omega^\mu, \theta^\lambda \rangle &= \frac{1}{|B_n|^2} \sum_{\rho \vdash n} |B_n x_\rho B_n| \, \omega_\rho^\mu \sum_{a \in B_n} |S_{2\lambda} \cap a x_\rho B_n| \\ &= \frac{1}{|B_n|^2} \sum_{\rho \vdash n} |B_n x_\rho B_n| \, \omega_\rho^\mu \, |B_n \cap x_\rho B_n x_\rho^{-1}| \, |S_{2\lambda} \cap B_n x_\rho B_n| \\ &= \sum_{\rho \vdash n} \omega_\rho^\mu \, |S_{2\lambda} \cap B_n x_\rho B_n| = 2^n \lambda! \sum_{\rho \vdash n} \omega_\rho^\mu a_\rho(A) \quad \text{(Lemma 4.6)} \\ &= \frac{2^n \lambda!}{\chi^{2\mu}(1)} \chi^{2\mu}(1) \sum_{\rho \vdash n} \omega_\rho^\mu a_\rho(A) = \frac{2^n \lambda!}{\chi^{2\mu}(1)} a'_\mu(A).\end{aligned}$$

Hence, $\langle \omega^\mu, \theta^\lambda \rangle$ gives the dual distribution of A up to a non-zero constant.

Indeed, $\langle \omega^\mu, \theta^\lambda \rangle$ is a positive multiple of the coefficient $u_{\mu,\lambda}$ of the monomial symmetric function m_λ in the expansion of the zonal polynomial Z_μ (see the proof of [17, Ch. VII (2.22)]). This coefficient is 0 unless $\mu \trianglerighteq \lambda$. Now by [17, Ch. VII (2.23)], Z_μ is the *Jack symmetric function* with parameter $\alpha = 2$. By [17, Ch. VI (10.13),(10.15)], $u_{\mu,\lambda} > 0$ whenever $\mu \trianglerighteq \lambda$. It follows that

$$\langle \omega^\mu, \theta^\lambda \rangle \neq 0 \iff \mu \trianglerighteq \lambda.$$

We can now prove Theorem 4.4.

Proof of Theorem 4.4. A λ-factorisation D is a T-design where T is the set of partitions such that the set A of left cosets of B_n with representatives in $S_{2\lambda}B_n$ is a T-antidesign. By the calculation above, the dual distribution of A is $a'_\mu(A) = C\langle \omega^\mu, \theta^\lambda \rangle$ for some non-zero constant C. Since $\langle \omega^\mu, \theta^\lambda \rangle \neq 0$ if and only if $\mu \trianglerighteq \lambda$, the result follows. ☐

By characterising λ-factorisations in terms of the dual distribution, we can compare λ-factorisations of different types.

Theorem 4.7. *Let $\lambda, \mu \vdash n$ and $\varnothing \neq D \subseteq S_{2n}/B_n$ be a λ-factorisation. If $\mu \trianglerighteq \lambda$, then D is also a μ-factorisation.*

Proof. Follows immediately from Theorem 4.4 and the fact that the dominance order $\trianglelefteq$ is transitive. ☐

Remark 4.8. Notice that the converse of Theorem 4.7 does not necessarily hold if λ is not 'small' enough. Consider $n = 6$ and suppose D is a $(3,2,1)$-factorisation. Then D is also a μ-factorisation for $\mu \in \{(3,3),(4,1,1),(4,2),(5,1)\}$. Now D is a λ-factorisation where $\lambda = (4,1,1)$, but we also have that D is a μ-factorisation where $\mu = (3,3)$; but λ and μ are incomparable in the dominance order.

Theorem 4.7 gives divisibility constraints on the indices reminiscent of the arithmetic conditions on the parameters of block designs.

Theorem 4.9. *Let $\lambda, \mu \vdash n$ with $\mu \trianglerighteq \lambda$ and $\varnothing \neq D \subseteq S_{2n}/B_n$ be a λ-factorisation of index c_λ. Then D is a μ-factorisation of index c_μ where*

$$c_\mu = c_\lambda \frac{(2\mu_1 - 1)!!(2\mu_2 - 1)!! \ldots}{(2\lambda_1 - 1)!!(2\lambda_2 - 1)!! \ldots}.$$

Hence $(2\lambda_1 - 1)!!(2\lambda_2 - 1)!! \ldots$ divides $c_\lambda(2\mu_1 - 1)!!(2\mu_2 - 1)!! \ldots$ for all $\mu \trianglerighteq \lambda$.

Proof. We count the pairs (m, P) of matchings $m \in D$ that refine a set partition P of shape 2λ in two ways. First, we count the number of set partitions of $\{1, \dots, 2n\}$ of shape 2λ, and then we count the number of set partitions that one matching m refines.

Denote by m_i the number of parts of λ of size i. Then the number of set partitions of shape 2λ is

$$\frac{(2n)!}{\prod\limits_{i \geqslant 1, m_i > 0} (2i)!^{m_i} m_i!}$$

because the parts of a set partition of shape 2λ have even size and the parts of size $2i$ are stabilised by the wreath product $S_{2i} \wr S_{m_i}$.

Now consider any perfect matching m. Counting the number of set partitions that m refines comes down to counting in how many ways the n edges of m can be grouped into parts according to λ. We find that there are

$$\frac{n!}{\prod\limits_{i \geqslant 1, m_i > 0} i!^{m_i} m_i!}$$

set partitions of shape 2λ that m refines. Hence, double counting the pairs (m, P) of matchings $m \in D$ that refine a set partition P of shape 2λ, we obtain

$$c_\lambda \frac{(2n)!}{\prod\limits_{i \geqslant 1, m_i > 0} (2i)!^{m_i} m_i!} = |D| \frac{n!}{\prod\limits_{i \geqslant 1, m_i > 0} i!^{m_i} m_i!}$$

and so

$$|D| = c_\lambda \frac{(2n)!}{n!} \prod_{i \geqslant 1, m_i > 0} \left(\frac{i!}{(2i)!} \right)^{m_i}.$$

Now $(2n)!/n! = 2^n (2n-1)!!$, the powers of 2 cancel and we can rewrite the product as

$$|D| = c_\lambda (2n-1)!! \prod_{i=1}^{l(\lambda)} \frac{1}{(2\lambda_i - 1)!!}.$$

By Theorem 4.7, we find that D is also a μ-factorisation, so we can also compute the size of D in terms of μ. Equating both counts gives

$$c_\lambda (2n-1)!! \prod_{i=1}^{l(\lambda)} \frac{1}{(2\lambda_i - 1)!!} = c_\mu (2n-1)!! \prod_{i=1}^{l(\mu)} \frac{1}{(2\mu_i - 1)!!}$$

and so

$$c_\mu = c_\lambda \frac{(2\mu_1 - 1)!!(2\mu_2 - 1)!! \dots}{(2\lambda_1 - 1)!!(2\lambda_2 - 1)!! \dots}.$$

Since c_μ is an integer, the result follows. ☐

The formula in Theorem 4.9 also reveals a simple way to obtain the size of a λ-factorisation of index c_λ (i.e. use $\mu = (n)$):

$$c_\lambda \frac{(2n-1)!!}{(2\lambda_1 - 1)!!(2\lambda_2 - 1)!! \dots}.$$

The following is a simple yet powerful special case of Theorem 4.9.

Corollary 4.10. *Let $\lambda \vdash n$, $\lambda \neq (n)$, and let $D \neq \varnothing$ be a λ-factorisation of index c_λ. If k, l with $k \leqslant l$ are distinct parts of λ (which may be the same if λ has multiple parts of the same size), then $2k - 1$ divides $(2l + 1)c_\lambda$. In particular, if D is of index 1, then $2k - 1$ divides $2l + 1$.*

Proof. Write $\lambda = \widetilde{\lambda} \cup (l,k)$ and let $\mu = \widetilde{\lambda} \cup (l+1,k-1)$. Then we have $\lambda \trianglelefteq \mu$. Now use Theorem 4.9 to find that

$$c_\mu = c_\lambda \frac{(2(k-1)-1)!!(2(l+1)-1)!!}{(2k-1)!!(2l-1)!!} = c_\lambda \frac{2l+1}{2k-1}. \qquad \square$$

This severely restricts the possible partitions λ such that a λ-factorisation of index 1 exists. Applying Theorem 4.9 to partitions with two parts, we get the following:

Corollary 4.11. *Let $t \leqslant n/2$ and $\varnothing \neq D \subseteq S_{2n}/B_n$ be an $(n-t,t)$-factorisation of index c_t. Then D is an $(n-t+1,t-1)$-factorisation of index c_{t-1} where*

$$c_{t-1} = c_t \frac{2n-2t+1}{2t-1} = c_t \frac{2n}{2t-1} - c_t.$$

Proof. Use Theorem 4.9 with $\lambda = (n-t,t)$ and $\mu = (n-t+1,t-1)$, which satisfy $\mu \trianglerighteq \lambda$. $\square$

Notice that c_{t-1} is an integer if and only if $2t-1 \mid c_t \cdot 2n$. This gives the following non-existence result.

Corollary 4.12. *For every $n \geqslant 4$, an $(n-2,2)$-factorisation of index coprime to 3 (in particular index 1) can only exist if $n \equiv 0 \pmod 3$.*

Proof. Use Corollary 4.11 for $t = 2$. $\square$

Note that Corollary 4.11 can be applied repeatedly, i.e. an $(n-t,t)$-factorisation is an $(n-s,s)$-factorisation for every $s = t-1, t-2, \dots, 1$. For example, an $(n-3,3)$-factorisation of index 1 (where $n \geqslant 6$) is an $(n-2,2)$-factorisation of index $\frac{2n-5}{5}$, so $n \equiv 0 \pmod 5$. Now an $(n-2,2)$-factorisation of index $\frac{2n-5}{5}$ is an $(n-1,1)$-factorisation of index $\frac{2n-5}{5} \cdot \frac{2n-3}{3}$, so $(2n-5)(2n-3) \equiv 0 \pmod 3$, hence $n \equiv 0,1 \pmod 3$. So an $(n-3,3)$-factorisation of index 1 can only exist if $n \equiv 0, 10 \pmod{15}$. In general, we get the following divisibility criteria.

Theorem 4.13. *Let $t \leqslant n/2$. If an $(n-t,t)$-factorisation of index 1 exists, then*

$$(2t-1)(2t-3)\dots(2k+1) \text{ divides } (2n-(2t-1))(2n-(2t-3))\dots(2n-(2k+1))$$

for every $k = t-1, t-2, \dots, 1$. Equivalently,

$$\frac{(2(n-k)-1)!!(2k-1)!!}{(2(n-t)-1)!!(2t-1)!!}$$

is an integer for every $k = t-1, t-2, \dots, 1$.

Proof. Apply Theorem 4.9 or Corollary 4.11 to the chain $(n-t,t) \trianglelefteq (n-t+1,t-1) \trianglelefteq \dots \trianglelefteq (n-2,2) \trianglelefteq (n-1,1)$. $\square$

Note that Theorem 4.13 also holds in the case $k = 0$ but the resulting divisibility condition is a weaker version of the divisibility condition for $k = 1$.

For many combinatorial objects there exists a process called *derivation* that builds smaller objects from bigger ones. This is also the case for λ-factorisations.

Definition 4.14. *Let $\lambda \vdash n$, $\lambda \neq (n)$, and D be a λ-factorisation. Let k be a part of λ and $S \subseteq \{1, \dots, 2n\}$ be a set of size $2k$. For a perfect matching $m \in D$, we write $m \smallsetminus S$ for the set of edges of m that are disjoint from S. Then the set*

$$D_S = \{m \smallsetminus S \mid m \text{ refines } \{S, [2n] \smallsetminus S\}\}$$

is called the derivation *of D at S.*

The following theorem is immediate.

Theorem 4.15. *Let $\lambda \vdash n$, $\lambda \neq (n)$, and D be a λ-factorisation of index c. Let k be a part of λ, $S \subseteq \{1, \dots, 2n\}$ be a set of size $2k$ and write $\lambda = \widetilde{\lambda} \cup (k)$ for a partition $\widetilde{\lambda} \vdash n-k$. Then the derivation D_S is a $\widetilde{\lambda}$-factorisation of index c.*

Proof. Without loss of generality, we can assume that $S = \{2(n-k)+1, \dots, 2n\}$ (otherwise, permute the elements of $\{1, \dots, 2n\}$ suitably). Consider the set P_S of all set partitions of $\{1, \dots, 2n\}$ of shape 2λ that contain S. We find

$$P_S = \{S \cup \widetilde{P} \mid \widetilde{P} \text{ is a set partition of } \{1, \dots, 2(n-k)\} \text{ of shape } 2\widetilde{\lambda}\}.$$

Now let $\widetilde{P}$ be any set partition of $\{1, \dots, 2(n-k)\}$ of shape $2\widetilde{\lambda}$. Then $S \cup \widetilde{P} \in P_S$ and $S \cup \widetilde{P}$ has shape 2λ. Since D is a λ-factorisation, we have that there are exactly c perfect matchings $m \in D$ that refine $S \cup \widetilde{P}$. All matchings $m \in D$ that refine a set partition of P_S also refine the set partition $\{S, [2n] \setminus S\}$ and thus give rise to elements $m \setminus S$ of D_S. Hence, there are exactly c perfect matchings $\widetilde{m} \in D_S$ that refine $\widetilde{P}$ and D_S is a $\widetilde{\lambda}$-factorisation of index c. ☐

For two partitions λ, μ, we write $\mu \preccurlyeq \lambda$ if every part of μ is a part of λ. Then Theorem 4.15 implies the following.

Corollary 4.16. *Let $\lambda \vdash n$ and D be a λ-factorisation of index c. Then there exist μ-factorisations of index c for every partition $\mu \preccurlyeq \lambda$.*

Proof. Use derivation multiple times. ☐

Corollary 4.16 implies that if a λ-factorisation of index c does not exist, then there is no μ-factorisation of index c where all parts of λ turn up as parts of μ. Thus, knowledge of the existence or non-existence of λ-factorisations where $\lambda \vdash n$ with small n gives insight into the existence for bigger values of n. For example, we can generalise a result of Cameron [5, Theorem 7.6] and Mathon [18] that there is no 2-$(2,8)$ partition system.

Corollary 4.17. *Let $n \geqslant 4$. There is no $(2,1,\dots,1)$-factorisation of index 1. In particular, there is no $(n-2)$-$(2,2n)$ partition system.*

Proof. By Cameron [5, Theorem 7.6] and Mathon [18], there is no $(2,1,1)$-factorisation of index 1. Now the result follows from Corollary 4.16. ☐

Alternatively, the theory of symmetric functions can be used for a longer proof of Corollary 4.17. We present it here, because it also shows that there is a Krein parameter of the association scheme on perfect matchings that is guaranteed to be 0, and this is of independent interest.

Example 4.18. Let S be a putative $(2,1,\dots,1)$-factorisation of index 1. By Theorem 4.4, the dual distribution of S is of the form $(1, x, 0, \dots, 0)$ where the positive rational number x appears in the coordinate indexed by the partition $\mu := (1,1,\dots,1)$. We now calculate the Krein parameter $q^{\mu}_{\mu\mu}$, via the formula given in [2, Theorem 3.6 and 3.5(i)] that uses the eigenvalues and the valencies of the association scheme.

The valency k_ρ of the perfect matching scheme is the size of the ρ-sphere Ω_ρ. This is the size of the double coset $B_n x_\rho B_n$ of type ρ. By [17, Ch. VII (2.3)], it is given by the formula $|B_n|^2 z_{2\rho}^{-1}$ where $z_{2\rho}$ is a non-zero number whose exact value need not concern us. The eigenvalue $P_\mu(\rho)$ of the perfect matching scheme is given by the formula $P_\mu(\rho) = k_\rho \omega^\mu_\rho$ (cf. [11, Lemma 13.8.3]).

We will use the relation $\equiv$ to denote 'up to a nonzero scalar'. Using the formula for the value of the zonal spherical function $\omega_\rho^{(1,\dots,1)}$ given in [17, Ch. VII Ex. 2(b)], we find

$$\begin{aligned} q^{\mu}_{\mu\mu} &\equiv \sum_{\rho \vdash n} \frac{1}{k_\rho^2} \left(P_\mu(\rho)\right)^3 = \sum_{\rho \vdash n} k_\rho \left(\omega^\mu_\rho\right)^3 \equiv \sum_{\rho \vdash n} z_{2\rho}^{-1} \omega_\rho^{(1,\dots,1)} \cdot \left(\omega_\rho^{(1,\dots,1)}\right)^2 \\ &= \sum_{\rho \vdash n} z_{2\rho}^{-1} \omega_\rho^{(1,\dots,1)} \cdot \left(\frac{(-1)^{n-l(\rho)}}{2^{n-l(\rho)}}\right)^2 = \sum_{\rho \vdash n} z_{2\rho}^{-1} \omega_\rho^{(1,\dots,1)} \cdot \frac{1}{2^{2n-2l(\rho)}} \\ &= 4^{-n} \sum_{\rho \vdash n} z_{2\rho}^{-1} \omega_\rho^{(1,\dots,1)} \cdot 4^{l(\rho)} \\ &\equiv c_{(1,\dots,1)}(4). \end{aligned}$$

The last step follows from [17, Ch. VII Ex. 2(c)]. Here,

$$c_\lambda(X) = \prod_{(i,j) \in \lambda} (X - i + 2j - 1)$$

is the *content polynomial* (see [17, Ch. VII (2.24)]). For the partition $\mu = (1, \dots, 1)$, we find

$$c_{(1,\dots,1)}(X) = \prod_{i=1}^{n} (X - i + 1).$$

Clearly, we have $c_{(1,\dots,1)}(4) = 0$ for all $n \geqslant 5$. Therefore, by [1, Corollary 1.5], S consists of half of the perfect matchings. This is a contradiction, because there is an odd number of perfect matchings of K_{2n}.

Table 1 gives explicit constrains that follow from Theorem 4.13 and Corollary 4.16 for λ-factorisations of index 1 where λ has few and small parts.

λ	Constraints
$(n-t,1,\dots,1)$	$t \neq n-2$
$(n-1,1)$	none, always exists
$(n-2,2)$	$n \equiv 0 \pmod{3}$
$(n-3,3)$	$n \equiv 0, 10 \pmod{15}$
$(n-3,2,1)$	$n \equiv 1 \pmod{3}$
$(n-4,4)$	$n \equiv 0, 21 \pmod{35}$
$(n-4,3,1)$	$n \equiv 1, 11 \pmod{15}$
$(n-4,2,2)$	does not exist
$(n-4,2,1,1)$	does not exist
$(n-5,5)$	$n \equiv 0, 36, 126, 162, 225, 252 \pmod{315}$

Table 1. Criteria for the existence of λ-factorisations of index 1 for some λ.

Remark 4.19. A near-perfect matching is a matching in a graph where all but one vertex are matched, and the graph must have an odd number of vertices. We can define λ-factorisations for a complete graph on an odd number of vertices, say $2n-1$, by considering only the partitions $\lambda \vdash 2n-1$ that have exactly one odd part. Every such partition extends uniquely to a partition $\widehat{\lambda} \vdash 2n$ that only has even parts. Likewise, every near-perfect matching of K_{2n-1} extends uniquely to a perfect matching of K_{2n}. Conversely, deleting the element $2n$ of a set partition of $\{1, \dots, 2n\}$ with only even parts (resp. a perfect matching) leaves a set partition with exactly one odd part (resp.

a near-perfect matching). Thus, there is a bijection between near-perfect matchings of K_{2n-1} and perfect matchings of K_{2n} and the corresponding λ-factorisations are essentially the same structures.

We could use similar techniques from the theory of association schemes to study near-perfect matchings of K_{2n-1}. Indeed, we obtain a natural association scheme from the Gelfand pair (S_{2n-1}, B_{n-1}) where B_{n-1} is the centraliser of the permutation $(1\,2)(3\,4)\cdots(2n-3\;2n-2)$. See [16, §11] for more.

5. EXAMPLES FOR SMALL n

We give an overview of the existence of λ-factorisations of index 1 for small n. The first case is $n = 3$, with the only nontrivial partition being $\lambda = (2, 1)$, and a $(n-1, 1)$-factorisation of index 1 is a 1-factorisation (see Example 4.2), and so exists. For $n = 4$, we have $\lambda \in \{(2, 1, 1), (2, 2), (3, 1)\}$, where again, the $\lambda = (3, 1)$ case is not interesting. Corollary 4.10 shows that there is no example for $\lambda = (2, 2)$. It was shown by Cameron [5, Theorem 7.6] and Mathon [18] that there is no example for $\lambda = (2, 1, 1)$.

For $n = 5$, the partitions are totally ordered in the dominance order: $(1, 1, 1, 1, 1) \trianglelefteq (2, 1, 1, 1) \trianglelefteq (2, 2, 1) \trianglelefteq (3, 1, 1) \trianglelefteq (3, 2) \trianglelefteq (4, 1) \trianglelefteq (5)$. By Corollary 4.10, a λ-factorisation of index 1 does not exist for $\lambda \in \{(2, 2, 1), (3, 2)\}$. By Corollary 4.16, a $(2, 1, 1, 1)$-factorisation of index 1 does not exist. Again, a $(4, 1)$-factorisation of index 1 is a 1-factorisation (and so exists), and a $(3, 1, 1)$-factorisation is a hyperfactorisation, which exists in this case (see Example 4.3).

Now suppose $n = 6$. We can immediately rule out λ-factorisations of index 1 for $\lambda \in \{(2, 2, 1, 1),\ (2, 2, 2),\ (3, 2, 1),\ (3, 3)\}$ with a simple application of Corollary 4.10. Corollary 4.17 rules out $\lambda = (2, 1, 1, 1, 1)$. So the only nontrivial partitions of 6 that survive are $(3, 1, 1, 1)$, $(4, 1, 1)$, and $(4, 2)$. A $(4, 1, 1)$-factorisation of index 1 is equivalent to an abstract hyperoval of order 10. It was shown in [14] that such an object does not exist, via an exhaustive computer search. There is an example of a $(4, 2)$-factorisation of index 1 – which is explored in Example 5.1. To show that $(3, 1, 1, 1)$-factorisations of index 1 do not exist, we use the fact (Theorem 4.15) that the derivation of a putative example would yield a hyperfactorisation of K_{10}, and we know there are only two examples (up to isomorphism). So we construct the two examples, extend them to sets of perfect matchings of K_{12} (by adjoining $\{11, 12\}$ to each matching), and then we use a constraint satisfaction solver such as Gurobi [12] to show that there is no $(3, 1, 1, 1)$-factorisation whose derivation is the given hyperfactorisation. The GAP code is attached as an appendix, where we make use of the package Gurobify [15].

EXAMPLE 5.1. Here we give an example of a $(4, 2)$-factorisation. First we let $\mathrm{AGL}(1, 11)$ be the stabiliser of ∞ in the action of $\mathrm{PGL}(2, 11)$ on the projective line $\mathrm{PG}(1, 11)$. Let $\mathbb{F}_{11}^{\square}$ be the set of nonzero squares in the finite field $\mathbb{F}_{11}$ of order 11. Consider the following two perfect matchings on the 12 points of the projective line $\mathrm{PG}(1, 11) = \mathbb{F}_{11} \cup \{\infty\}$:

$$\{\{0, \infty\}\} \cup \{\{x, -x\} : x \in \mathbb{F}_{11}^{\square}\}, \quad \{\{0, \infty\}\} \cup \{\{x, 7x\} : x \in \mathbb{F}_{11}^{\square}\}.$$

Then the orbits of $\mathrm{AGL}(1, 11)$ on these perfect matchings are orbits of size 11 and 22, respectively. Their union forms a $(4, 2)$-factorisation of index 1. Note that this example is very much related to the Steiner system $S(5, 6, 12)$ – the *small Witt design*. If we take the subset $\{\infty\} \cup \mathbb{F}_{11}^{\square}$ of the projective line $\mathrm{PG}(1, 11)$ and its orbit under $\mathrm{PSL}(2, 11)$, then we have 132 subsets of size 6 such that any 5 points lie in a unique block.

λ	Exists?
(2,1,1,1,1)	No, Corollary 4.17
(2,2,1,1)	No, Corollary 4.10
(2,2,2)	No, Corollary 4.10
(3,1,1,1)	No (by computer)
(3,2,1)	No, Corollary 4.10
(3,3)	No, Corollary 4.10, Theorem 4.13
(4,1,1)	No, [14]
(4,2)	Yes
(5,1)	Yes

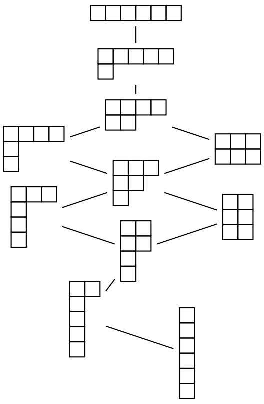

Table 2. The case $n = 6$. On the right is the dominance Hasse diagram of the partitions of 6. On the left is the existence data for λ-factorisations of index 1.

The authors failed to generalise this example to more than this one example, despite the large amount of symmetry that this example possesses. It would be interesting to find λ-factorisations admitting $\mathrm{AGL}(1, q)$ (as automorphisms), where $2n = q + 1$.

For $n = 7$, we can again apply Corollary 4.10 and Corollary 4.17 to show that most of the λ-factorisations (of index 1) do not exist. The only nontrivial partitions to consider are $(4, 2, 1)$, and $(5, 1, 1)$. Now a $(5, 1, 1)$-factorisation of index 1 is a hyperfactorisation of K_{14} of index 1. To the authors' knowledge, it is unknown whether such a hyperfactorisation exists. If it were to exist, then it would also give rise to a 5-design on 14 points with block size 6 and index 15 (see [3, p. 10]). We emphasise that this 5-design would have repeated blocks as the complete design on 14 points with block size 6 is a 5-design of index 9. There is a 5-design on 14 points with block size 6 and index 3 due to Brouwer [4] that does not have repeated blocks. The 5-design obtained from a hypothetical $(5, 1, 1)$-factorisation of index 1 could be a 5-fold copy of the design by Brouwer. Finally, the authors could not determine by computer or otherwise whether a $(4, 2, 1)$-factorisation of index 1 exists or not.

For $n = 8$, after applying Corollary 4.10 and Corollary 4.17, we are left with nontrivial λ-factorisations (of index 1) with $\lambda \in \{(5, 1, 1, 1), (6, 1, 1)\}$. Again, we do not know if these partitions have examples or not. One can see a pattern emerging here where the possible valid partitions have relatively small length. Finally, for $n = 9$, we have an example of a hyperfactorisation of index 1 of K_{18} from Cameron's construction, and so the case $\lambda = (7, 1, 1)$ turns out to have an example. So the remaining partitions where we do not have a resolution to the existence question are $(5, 1, 1, 1, 1), (6, 1, 1, 1), (7, 2)$.

6. Open questions

We list some open questions for further research that the reader might be interested to work on. We expect that they are roughly ordered increasing in difficulty.

1. Can we construct more $(n - 2, 2)$-factorisations that are not also hyperfactorisations like Example 5.1? Is there an infinite family?
2. Can we construct any non-trivial $(n - 3, 3)$-factorisation? Since this would also yield an $(n - 2, 2)$-factorisation by Theorem 4.7, one should investigate $(n - 2, 2)$-factorisations first.
3. Can we construct any non-trivial $(n - 3, 1, 1, 1)$-factorisation?

(4) Can we investigate similar questions for the coset algebra $S_{kn}/(S_k \wr S_n)$ of uniform set partitions? Since this is not a Gelfand pair for $k \geqslant 3$, a different approach is needed.

Appendix: GAP code

```
# First, we construct the two hyperfactorisations of K_{10}
n := 5;
s2n := SymmetricGroup(2*n);
matching := List([1..n], i -> [2*i-1,2*i]);
matchings := Set(Orbit(s2n, matching, OnSetsSets));;
perm := Action(s2n, matchings, OnSetsSets);;
maxs := MaximalSubgroupClassReps(perm);;
s9 := First(maxs, t -> Size(t)=Factorial(9));
IsSymmetricGroup(s9);
maxs := MaximalSubgroupClassReps(s9);;
m := First(maxs, t -> Size(t)=432);
subgroup216 := MaximalNormalSubgroups(m)[1];
orbits := Filtered(Orbits(subgroup216), t -> Size(t)<=63);;
Sort(orbits, {x,y} -> Size(x) > Size(y));
# Both examples below are equivalent to Mathon's example
o1 := Union(orbits{[1,3]});;
o2 := Union(orbits{[2,3]});;

psl := PSL(2,8);
iso := IsomorphismGroups(SymmetricGroup(9),s9);;
psl_on_matchings := Image(iso, psl);
# This example arises from a hyperoval of PG(2,8) ...
# Cameron's example
o3 := First(Orbits(psl_on_matchings), t -> Size(t)=63);;

# Now extend each one with [11,12] to get a partial configuration
extensions := List([matchings{o1},matchings{o3}], mm ->
   List(mm, t -> Concatenation(t,[[11,12]])));;

# now change n and set up the Gurobi program
LoadPackage("gurobify");
n := 6;
s2n := SymmetricGroup(2*n);
matching := List([1..n], i -> [2*i-1, 2*i]);
matchings := Set(Orbit(s2n, matching, OnSetsSets));;
perm := Action(s2n, matchings, OnSetsSets);;
lambda := [3,1,1,1];
# one set partition of shape 2*lambda
p := []; max := 0;
for s in 2*lambda do
   Add(p, max + [1..s]);
   max := Maximum(Union(p));
od;
refines := {x,y} -> ForAll(x, t -> ForAny(y, u -> IsSubset(u,t)));
cp := Filtered(matchings, t -> refines(t,p));;
antidesign1 := SubsetToCharacteristicVector(cp, matchings);;
# These are the antidesigns, as 0,1-vectors.
# We seek a set that intersects each antidesign in precisely
# 1 element.
```

```
antidesigns := Orbit(perm, antidesign1, Permuted);;

for i in [1,2] do
   Print("Doing example ", i, "\n");
   ext := extensions[i];
   varnames := List([1..Size(matchings)], i -> Concatenation("x",
       String(i)));;
   vartypes := ListWithIdenticalEntries( Size(varnames), "Binary" );;
   model := GurobiNewModel(vartypes, varnames);;

   # constraint: meets each antidesign in 1 element
   GurobiAddMultipleConstraints(model,antidesigns,
      ListWithIdenticalEntries(Size(antidesigns),"="),
      ListWithIdenticalEntries(Size(antidesigns),1));

   # constraint: derives to known hyperfactorisation
   GurobiAddConstraint(model,
            SubsetToCharacteristicVector(ext,matchings), "=", 63);
   status := GurobiOptimiseModel(model);

   if status = 3 then
      Print("no example found\n");
   else
      Print("something else has happened\n");
   fi;
od;
```

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