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Color rules for cyclic wreath products and semigroup algebras from projective toric varieties

Fabián Levicán-Santibáñez & Marino Romero

Abstract We introduce the notion of "color rules" for computing class functions of $Z_k \wr S_n$, where Z_k is the cyclic group of order k and S_n is the symmetric group on n letters. Using a general sign-reversing involution and a map of order k, we give a combinatorial proof that the irreducible decomposition of these class functions is given by a weighted sum over semistandard tableaux in the colors. Since using two colors at once is also a color rule, we are consequently able to decompose arbitrary tensor products of representations whose characters can be computed via color rules. This method extends to class functions of $G \wr S_n$ where G is a finite abelian group. We give a number of applications, including decomposing tensor powers of the defining representation, along with a combinatorial proof of the Murnaghan–Nakayama rule for $Z_k \wr S_n$.

Our main application is to the study of the linear action of $Z_k \wr S_n$ on bigraded affine semigroup algebras arising from the product of projective toric varieties. In the case of the product of projective spaces, our methods give the decomposition of these bigraded characters into irreducible characters, thus deriving equivariant generalizations of Euler-Mahonian identities.

1. Introduction

We will present a general method for computing irreducible decompositions of class functions of $G \wr S_n$ for when G is an abelian group and S_n is the symmetric group on n letters (though we will focus on the case when $G = Z_k$ is the cyclic group of order k, from which the more general case follows). Our proof is purely combinatorial and noteworthy for a number of reasons.

First, the family of class functions we will be interested in are attained by "coloring" cycles according to some "color rule" (see Section 3), and any character can be written as a sum of these types of class functions (see Subsection 4.2.3). Given such a class function, we will show that the multiplicities of irreducible characters can be enumerated by weighted sums over certain semistandard tableaux in the colors. This will be our main result, Theorem 3.8.

There are a number of consequences from this perspective. For one, tensor products of representations whose characters are computed via color rules can be immediately decomposed into irreducible representations, since, conveniently, having two simultaneous coloring rules is itself a single coloring rule! This is explained by Proposition 3.4. As a consequence, we get a large number of applications, including the decomposition of tensor powers of the defining representation of $Z_k \wr S_n$. Furthermore, the class functions are not necessarily characters of a representation and can include weights.

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For example, in Subsection 4.2.2, we get a combinatorial proof for the Murnaghan-Nakayama rule for $Z_k \wr S_n$ written in terms of multi-Schur functions. These methods also restrict to consequences for the symmetric group, as we will show in Subsection 4.1.

Second, our proof uses a remarkably general weight-preserving, sign-reversing involution method. The core idea behind this method stems from a special case of our setting, found in [16], where a sign-reversing involution is used to decompose tensor powers of the defining representation of S_n. This idea was also later used in [14] to give a combinatorial proof of the Murnaghan-Nakayama rule for S_n. As previously mentioned, our framework specializes to these cases. Most importantly, our point of view gives a unified combinatorial framework for understanding arbitrary products of these types of class functions.

A key application of our methods is to the study of certain bigraded group characters arising from actions on affine semigroup algebras, $K[P^{\times n}]$, coming from the product of projective toric varieties. We will study a bigrading introduced by Chapoton in [7], which is similar to (but distinct from) the one introduced by Reiner and Rhoades in [20]. Then, we will endow $K[P^{\times n}]$ with the structure of a bigraded (or multigraded) $Z_k \wr S_n$-module. In particular, we will compute the character of $K[P^{\times n}]$ and give a combinatorial rule for computing its decomposition into irreducible representations (see Theorem 5.11). For $k = 1$, we will also describe a link to Stapledon's *equivariant Ehrhart theory* [23] by introducing a new refinement using this bigrading.

An interesting case is when $P = \Delta_1 = [0, 1]$ is the unit interval. A regular sequence in $K[(\Delta_1)^{\times n}]$ is given in [1] (for $k = 1$) and [5] (for any k). In [5], they interpret the Hilbert series of quotients of $K[(\Delta_1)^{\times n}]$ by these regular sequences as *Euler-Mahonian identities*, which have appeared in various contexts within algebraic combinatorics, as discussed after their Theorem 1.1.

As a special case of our work, we will study the products of unit m-simplices $P = \Delta_m$, corresponding geometrically to the product of projective spaces with the Segre embedding. We will give a regular sequence in $K[(\Delta_m)^{\times n}]$ for $k = 1$, conjecture a regular sequence for every k, and therefore give an expression for the bigraded character of the quotients of $K[(\Delta_m)^{\times n}]$ by these regular sequences. In particular, for $m = 1$ and any k and n, our methods give an equivariant generalization of these aforementioned Euler-Mahonian identities (see Table 2).

Our work is organized in the following way. Section 2 introduces the main combinatorial structures, along with the representation theory of $Z_k \wr S_n$. Section 3 will introduce the definition of a "color rule" and state the main combinatorial result. In Section 4, we give a list of examples and applications from this point of view. This will be followed by Section 5, where we introduce affine semigroup algebras and projective toric varieties, describe the action of $Z_k \wr S_n$, and state our results about the product of projective spaces. We also discuss links to equivariant Ehrhart theory and to Euler-Mahonian identities. Lastly, Section 6 gives a proof of our main theorem regarding color rules.

2. Preliminaries

2.1. Partitions. A partition $\lambda = (\lambda_1, \cdots, \lambda_\ell) \vdash n$ is a sequence of non-increasing positive integers whose total sum is $|\lambda| = n$ and length is $\ell(\lambda) = \ell$. We represent partitions by its diagram in the French convention, see Figure 1.

We will write $\vec{\lambda} = (\lambda^0, \lambda^1, \ldots, \lambda^{k-1}) \vdash_k n$ if $\vec{\lambda}$ is a sequence of k partitions such that $|\lambda^0|+|\lambda^1|+\cdots+|\lambda^{k-1}| = n$. Note that some of these partitions may be empty. We give this sequence a Young diagram interpretation by drawing successive partitions corner to corner. More specifically, the south-east corner of λ^i is connected to the north-west corner of λ^{i+1}. If λ^i is an empty partition, we let λ^i be a cell with $\varnothing$ written within, and we proceed with the concatenation. Similarly to ordinary integer partitions, we denote its size by $|\vec{\lambda}| = n$. For example, the sequence $\vec{\lambda} = ((3,1),(\varnothing),(2,2))$ depicted on the right-hand side of Figure 1 has size 8. It is important that we leave the empty square empty, and any interactions with cells of $\vec{\lambda}$ should avoid this square.

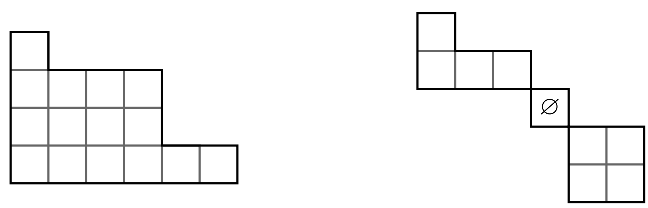


FIGURE 1. The partition $\lambda = (6,4,4,1)$ is drawn on the left; and on the right is the sequence of partitions $\vec{\lambda} = ((3,1),\varnothing,(2,2)) \vdash_3 8$.

Suppose we are given a multiset $F = [f_1^{m_1}, f_2^{m_2}, \dots]$, and suppose that we fix an order on the elements of F, say $f_1 < f_2 < \cdots$ (the choice generally does not matter). A semistandard tableau $T \in \mathrm{SSYT}(\vec{\gamma}, F)$ of shape $\vec{\gamma}$ and content F is a filling of the cells of $\vec{\gamma}$ so that every cell gets one element from F; f_i appears at most m_i times; the rows are weakly increasing when read from left to right; and columns are strictly increasing from bottom to top. See Figure 2.

The set of standard Young tableaux of shape $\vec{\gamma}$, $\mathrm{SYT}(\vec{\gamma}) \coloneqq \mathrm{SSYT}(\vec{\gamma}, [1,2,\dots,|\vec{\gamma}|])$, consists of semistandard Young tableaux with entries in $\{1,\dots,|\vec{\gamma}|\}$ so that each label appears exactly once. Every element $T \in \mathrm{SSYT}(\vec{\gamma}, F)$ can be "standardized" to an element $\mathrm{stand}(T) \in \mathrm{SYT}(\vec{\gamma})$ where we replace the entries by their index in the reading order. That is, if there are a_i cells labeled f_i, then replace the a_1 cells containing f_1 by $1,\dots,a_1$, increasing from left to right, replace the a_2 cells containing f_2 with $a_1+1,\dots,a_1+a_2$, increasing from left to right, and so on. For an example, see Figure 2.

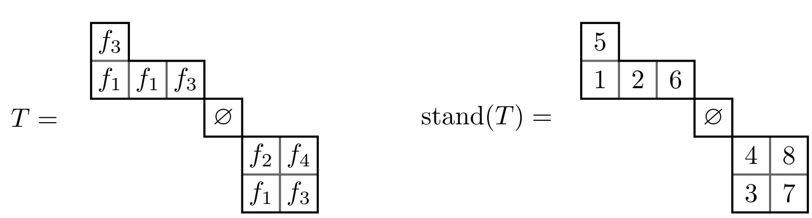


FIGURE 2. On the left, we have a semistandard tableau $T \in \mathrm{SSYT}(((3,1),\varnothing,(2,2)),[f_1^3, f_2, f_3^6, f_4])$. On the right, we have its standardization $\mathrm{stand}(T) \in \mathrm{SYT}(((3,1),\varnothing,(2,2)))$.

2.2. WREATH PRODUCTS. For references regarding wreath products with the symmetric group and connections to symmetric functions, see [15] and [12].

For the remainder of the paper, let n and k be fixed positive integers. Let Z_k be the cyclic group consisting of the k^{th} roots of unity $\{u_0, u_1, \dots, u_{k-1}\}$, generated by u_1 (so that $u_r = u_1^r$); and let S_n be the symmetric group on the letters $\{1,2,\dots,n\}$.

Let $Z_k \wr S_n$ denote the group formed by the set of n by n matrices whose rows and columns have only one nonzero entry coming from Z_k. Since this group is introduced in matrix form, the elements σ of $Z_k \wr S_n$ are given a natural "defining" representation, which is the matrix $\Pi_{\mathrm{def}}(\sigma)$ from their definition. For instance, one such matrix in $Z_4 \wr S_6$ is given by

$$\Pi_{\mathrm{def}}(\tau) = \begin{pmatrix} u_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_0 & 0 \\ 0 & 0 & 0 & u_1 & 0 & 0 \\ 0 & u_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_3 \end{pmatrix}.$$

We may also consider the elements in $Z_k \wr S_n$ as weighted permutations. That is, $\sigma \in Z_k \wr S_n$ can be written in two-line notation as

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ u_{a_1}\sigma_1 & u_{a_2}\sigma_2 & \cdots & u_{a_n}\sigma_n \end{pmatrix},$$

where $u(\sigma, \sigma_i) \coloneqq u_{a_i}$ is some k^{th} root of unity and $\sigma_1 \cdots \sigma_n$ is a permutation in S_n. We will then say that i goes to $u_{a_i}\sigma_i$. We will often refer to the numbers $1, \ldots, n$ appearing in $\sigma_1 \cdots \sigma_n$ as indices, so that the index i appears with the root of unity $u(\sigma, i)$. In terms of the representation $\Pi_{\mathrm{def}}(\sigma)$, $u(\sigma, i)$ is the root of unity in row i.

We will use a particular cycle notation. Given $\sigma \in Z_k \wr S_n$, we can write σ uniquely as a product of disjoint cycles in the following way: $u(\sigma, i)\ i$ precedes $u(\sigma, \sigma_i)\ \sigma_i$ in a cycle; every cycle begins with the smallest index i; and arrange the cycles so that they are decreasing from left to right according to the smallest index in each cycle. For example, the element τ above can be written as

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ u_0 1 & u_2 5 & u_3 2 & u_1 4 & u_0 3 & u_3 6 \end{pmatrix} = (u_3 6)(u_1 4)(u_3 2, u_2 5, u_0 3)(u_0 1) \in Z_4 \wr S_6.$$

Because we are writing the first indices of each cycle in decreasing order, we will call this unique arrangement the *decreasing cycle notation* of σ.

Remark 2.1. Given σ in decreasing cycle notation, one can erase the parentheses and get a bijection from $Z_k \wr S_n$ to itself, where the above permutation in cycle notation would be sent to $u_3 6\ u_1 4\ u_3 2\ u_2 5\ u_0 3\ u_0 1$, written in one-line notation. One gets the original cycle notation by taking a left-to-right minimum as the location of a new cycle.

Given a cycle $c = (u_{a_1} i_1, \ldots, u_{a_m} i_m)$, we say that c is a C_j-cycle if $u_{a_1} \cdots u_{a_m} = u_j$. This u_j will also be called the C-type of the cycle c. We may also say that c is an m-cycle, in reference to its length; so 1-cycles are cycles of length 1, whereas C_0-cycles are cycles of C-type $u_0 = 1$. The conjugacy classes are indexed by sequences of partitions $\vec{\lambda} \vdash_k n$:

$$C_{\vec{\lambda}} = \{\sigma \in C_k \wr S_n : \text{the } C_j\text{-cycles in } \sigma \text{ are of lengths } \lambda^j_1, \ldots, \lambda^j_{\ell(\lambda^j)} \text{ for } j = 0, \ldots, k-1\}.$$

From the previous example $\tau \in Z_4 \wr S_6$, since $u_3 = u_3$, $u_1 = u_1$, $u_3 u_2 u_0 = u_1$, and $u_0 = u_0$,

$$\begin{array}{rl} (u_0 1) & \text{is a } C_0\text{-cycle,} \\ (u_3 2, u_2 5, u_0 3) \text{ and } (u_1 4) & \text{are } C_1\text{-cycles, and} \\ (u_3 6) & \text{is a } C_3\text{-cycle.} \end{array}$$

Therefore, τ is in the conjugacy class described by the sequence of partitions

$$((1), (3,1), (\varnothing), (1)) \vdash_4 6.$$

2.3. Irreducible characters. The irreducible characters of Z_k are given by the functions $\{\Psi^j : 0 \leqslant j < k\}$ defined by setting $\Psi^j(u) = u^j$. In particular, these characters are themselves representations and multiplicative: $\Psi^j(uv) = \Psi^j(u)\Psi^j(v)$. This will be the property required to prove Theorem 3.8, which is why we are able to generalize the main result to wreath products with abelian groups.

For each $\vec{\gamma} \vdash_k n$ there is a corresponding irreducible character of $Z_k \wr S_n$, $\chi^{\vec{\gamma}}$, whose value at σ can be computed by a Murnaghan-Nakayama rule, as shown in [15]. This is what we now describe.

For a partition λ, a *rim hook* ζ is a contiguous path along the cells of the north-east boundary of λ consisting of East and South steps whose removal leaves a partition $\lambda \setminus \zeta$. We insert the rim hook by drawing a line segment through the sequence of cells. And for a rim hook ζ, we define

$$\operatorname{sign}(\zeta) \coloneqq (-1)^{\text{number of South steps in } \zeta}.$$

For example, Figure 3 depicts a rim hook on the left; and the second image is not a rim hook since the removal of the sequence of cells does not leave a partition. Since the first shape has a rim hook that traverses 2 horizontal lines, its sign is given by $(-1)(-1) = +1$.

Figure 3. The path depicted on the left is a rim hook with positive sign, while the one on the right-hand side is not a rim hook.

Let $c = (c_1, \ldots, c_m)$ be a composition of n, and let $\vec{\lambda} \vdash_k n$. Also, let $a = (a_1, \ldots, a_m)$ be a sequence of elements in Z_k. A *rim hook tableau* $T \in \mathrm{RHT}_{\vec{\lambda},(c,a)}$ of shape $\vec{\lambda}$ and type (c, a) is created by placing a rim hook ζ_1 in $\vec{\lambda}$ of length c_1, placing a rim hook ζ_2 in $\vec{\lambda} \setminus \zeta_1$ of length c_2, placing a rim hook ζ_3 in $\vec{\lambda} \setminus \zeta_1 \setminus \zeta_2$ of length c_3, and so on. We then assign each rim hook a weight w given by

$$w(\zeta_i) \coloneqq \Psi^j(a_i)\operatorname{sign}(\zeta_i),$$

where ζ_i was inserted in partition λ^j, and set

$$w(T) \coloneqq \prod_{i=1}^{\ell(c)} w(\zeta_i).$$

As an example for when $k = 2$ and $n = 8$, Figure 4 lists all rim hook tableaux of shape $((3,1),(2,2))$ and type $(1,3,1,3),(1,-1,-1,1)$, where the rim hooks are numbered to describe the order in which they were placed. The figure is displayed so that the tableaux of weight $+1$ are in the first row, and the tableaux with weight -1 are in the second row.

Proposition 2.2 (The wreath Murnaghan-Nakayama rule [24, 15]). *Let $c(\sigma)$ be the cycle structure of $\sigma \in Z_k \wr S_n$ written in some fixed order, and $a(\sigma)$ be the C-types of the respective cycles. Then, the irreducible character indexed by $\vec{\lambda}$ is given by*

$$\chi^{\vec{\lambda}}(\sigma) = \sum_{T \in \mathrm{RHT}_{\vec{\lambda},(c(\sigma),a(\sigma))}} w(T).$$

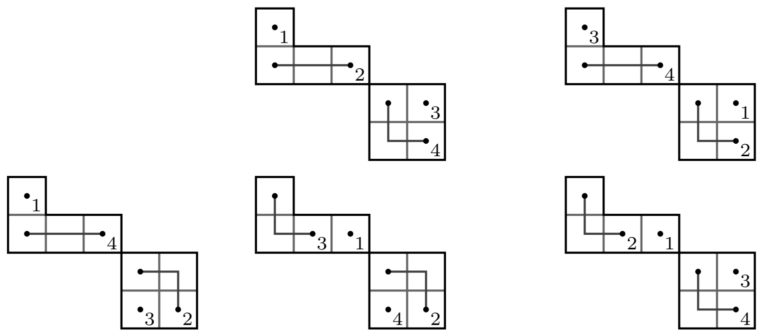


Figure 4. All rim hook tableaux of shape $((3,1),(2,2))$ and type $(1,3,1,3),(1,-1,-1,1)$.

This means that, from the previous example,

$$\text{if } \sigma \in C_{((3,1),(3,1))}, \text{ then } \chi^{((3,1),(2,2))}(\sigma) = -1.$$

3. Color rules and the main result

We assume that we are working with some ring, say $R = \mathbb{C}((t_1, t_2, \dots))$, with indeterminates $t_1, t_2, \dots$; and we are interested in computing a class function $\chi : Z_k \wr S_n \to R$. It is also important to mention that we will use brackets to denote multisets, so that $[1^2, 2] = [1, 1, 2] = [1, 2, 1]$ and $[1, \varnothing] \neq [1]$.

3.1. Definitions.

Definition 3.1 (Color rules). *A* color rule *is a multiset of colors $F = [f_1^{m_1}, f_2^{m_2}, \dots]$. A coloring of an element $\sigma \in Z_k \wr S_n$ under the color rule is a pair (f, σ) where $f = (f_{r_1}, \dots, f_{r_n})$ gives a multiset $[f_{r_1}, \dots, f_{r_n}]$ contained (with multiplicities) in F and satisfies the following condition:*

If i and j are indices in the same cycle of σ, then $f_{r_i} = f_{r_j}$.

The set of all colorings by the coloring rule F will be denoted by $F(Z_k \wr S_n)$.

Remark 3.2. A coloring $(f, \sigma) \in F(Z_k \wr S_n)$ can also be described as a composition of two functions: the first function maps $i \in \{1, \dots, n\}$ to its cycle in σ, and the second sends each cycle to a color in $\{f_1, f_2, \dots\}$. In this way, each element of the same cycle gets the same color. The condition on multiplicities is that the preimage of f_i has cardinality at most m_i.

Given two color rules F and G, the set of colorings $F \times G(Z_k \wr S_n)$ is defined as the set of pairs (h, σ) where $h = ((f_{r_1}, g_{s_1}), \dots, (f_{r_n}, g_{s_n}))$ is composed of two colorings $f = (f_{r_1}, \dots, f_{r_n})$ and $g = (g_{s_1}, \dots, g_{s_n})$. I.e., $((f, \sigma), (g, \sigma)) \in F(Z_k \wr S_n) \times G(Z_k \wr S_n)$. In particular, this means that if i and j are in the same cycle of σ, then $f_{r_i} = f_{r_j}$ and $g_{s_i} = g_{s_j}$.

Definition 3.3. *A class function $\chi : Z_k \wr S_n \to R$ is given by the coloring rule F if there is*

1. *a "value" function $p_F : \{f_1, f_2, \dots\} \to \mathbb{Z}$ and*
2. *a "weight" function $\rho_F : \{f_1, f_2, \dots\} \to R$*

so that $\chi(\sigma)$ can be computed as

$$\chi(\sigma) = \sum_{(g,\sigma) \in F(Z_k \wr S_n)} \text{weight}(g, \sigma),$$

where

$$\mathrm{weight}(g,\sigma) \coloneqq \prod_{i=1}^{n} \mathrm{weight}(g_i, u(\sigma,i)) \qquad \textit{and} \qquad \mathrm{weight}(f_i, u_j) \coloneqq u_j^{p(f_i)} \rho(f_i).$$

When the color rule, weight, and values are clear, we will write $p = p_F$ *and* $\rho = \rho_F$*.*

The following proposition follows from the definition and states that tensor products of representations whose characters are given by color rules have characters which are also given by a color rule.

Proposition 3.4. *Suppose we have two class functions,* χ^M *and* $\chi^{M'}$*, given by two color rules* F *and* G*, respectively, with values* p_F, p_G *and weights* ρ_F, ρ_G*. Then the class function* $\chi^M \chi^{M'}$ *is given by the color rule* $F \times G$ *with value* $p(f,g) = p_F(f) + p_G(g)$ *and weight* $\rho(f,g) = \rho_F(f)\rho_G(g)$*. In particular, if* χ^M *and* $\chi^{M'}$ *are graded characters for some modules* M *and* M'*, then the graded character of* $M \otimes M'$ *is given by the color rule* $F \times G$*.*

3.2. Examples.

Example 3.5. The most basic case is already remarkably interesting, as we will see in Subsection 4.2.1. Let $F = [1, \varnothing^{n-1}]$, meaning we must color a 1-cycle with the color 1. Let $p(1) = \rho(1) = \rho(\varnothing) = 1$ and $p(\varnothing) = 0$. The reason this is interesting is that then

$$\sum_{(g,\sigma)\in F(Z_k \wr S_n)} \mathrm{weight}(g,\sigma) = \sum_{\sigma_i = i} u(\sigma,i) \; = \mathrm{tr}(\Pi_{\mathrm{def}}(\sigma))$$

gives the trace of the matrix which defines the element σ. It is therefore the character of the defining representation.

Note that taking the character $\mathrm{tr}^m \Pi_{\mathrm{def}}$ corresponds to the m-fold tensor product of the defining representation. The character $\mathrm{tr}^m \Pi_{\mathrm{def}}(\sigma)$ can be computed by coloring the cycles of σ in the following way: over each cycle of length 1 place a subset of $\{1,\dots,m\}$ so that no number is ever repeated. The value is then given by

$$\mathrm{tr}^m \Pi_{\mathrm{def}}(\sigma) = \sum_{S} \prod_{\sigma_i = i} u(\sigma,i)^{|S_i|},$$

where the sum ranges over all $S = (S_1,\dots,S_n)$ forming a set partition $\cup S_i = \{1,\dots,m\}$ ($S_i \cap S_j = \varnothing$ for $i \neq j$), and $S_i = \varnothing$ if i is not in a 1-cycle ($\sigma_i \neq i$).

For instance, for $n = 8$, $k = 3$, and $m = 3$, the following is an example of a coloring:

$$\begin{array}{cccccc} \{2\} & \varnothing & \varnothing\,\varnothing\,\varnothing & \{1,3\} & \varnothing & \varnothing \\ (1) & (u_2 2) & (3\ u_1 6\ 4) & (u_1 5) & (u_2 8\ u_1 7) \end{array}.$$

We see that the weight of this coloring is $u_0^{|\{2\}|} \times u_1^{|\{1,3\}|} = u_2$.

Example 3.6. One of the main examples we will see is given by the colors

$$\mathsf{N}_{n,d} \coloneqq [0^n, 1^n, \dots, d^n]\,.$$

Define

$$p(i) = i \qquad\qquad \text{and} \qquad\qquad \rho(i) = q^i.$$

We will see that the sum

$$\sum_{d \geqslant 0} t^d \sum_{(g,\sigma)\in \mathsf{N}_{n,d}(Z_k \wr S_n)} \mathrm{weight}(g,\sigma)$$

is the character of a certain semigroup algebra, $K[(\Delta_1)^{\times n}]$ (see Theorem 5.25).

3.3. The main result. We will now state the main result regarding the decomposition of class functions given by color rules into irreducible characters, and we defer its proof to Section 6. Even though the proof is noteworthy on its own, it will be more useful to cover how this result can be applied, especially to the objects we are most interested in.

Definition 3.7. *Let $F = [f_1^{m_1}, f_2^{m_2}, \dots]$ be a color rule with value function p and weight function ρ. Let $F_r = [f_i^{m_i} \in F : p(f_i) = r \mod k]$ be the submultiset of r-valued colors. For $\vec{\gamma} \vdash_k n$, let*

$$\mathrm{SSYT}_k(\vec{\gamma}, F) := \mathrm{SSYT}(\gamma^0, F_0) \times \cdots \times \mathrm{SSYT}(\gamma^{k-1}, F_{k-1})$$

denote the set of all semistandard tableaux of shape $\vec{\gamma}$ whose entries in γ^r form a submultiset of F_r. Furthermore, define for $T \in \mathrm{SSYT}_k(\vec{\gamma}, F)$ the weight

$$\rho(T) = \prod_{f_i} \rho(f_i)^{a_i}$$

where a_i is the number of times f_i appears in T.

Then we have the following decomposition:

Theorem 3.8. *Let χ be given by a color rule F with value p and weight ρ. For any $\vec{\gamma} \vdash_k n$,*

$$\langle \chi, \chi^{\vec{\gamma}} \rangle = \sum_{T \in \mathrm{SSYT}_k(\vec{\gamma}, F)} \rho(T).$$

4. Some examples

We now list some applications. It is important to remark that from Proposition 3.4, we can immediately take any arbitrary tensor products of the representations which are subsequently listed here and immediately give a combinatorial description of their decomposition into irreducible representations.

4.1. The symmetric group.

When $k = 1$, we are looking at characters of the symmetric group. Since, $\vec{\gamma} = \lambda$ will consist of only one partition, we will write $\mathrm{SSYT}(\lambda, F)$ instead of $\mathrm{SSYT}_1(\vec{\gamma}, F)$.

4.1.1. *Symmetric functions and the Frobenius map.* It will be useful to recall the following connection between class functions of the symmetric group and symmetric functions. For us, class functions are maps $\chi : S_n \to R = \mathbb{C}((t_1, t_2, \dots))$, which are constant on conjugacy classes. And our symmetric functions will have coefficients in R.

Let C_λ be the conjugacy class of S_n with permutations of cycle type λ, which is also viewed as the class function which indicates if a permutation is in C_λ. The Frobenius map, from class functions to symmetric functions, is defined by

$$\mathrm{Frob}(C_\lambda) = p_\lambda / z_\lambda,$$

where $z_\lambda = n!/|C_\lambda|$ and $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_{\ell(\lambda)}}$ is the power sum symmetric function generated by $p_r = x_1^r + x_2^r + \cdots$. Then the Frobenius map sends a class function $\chi = \sum_\lambda \chi(\lambda) C_\lambda$ to

$$\mathrm{Frob}(\chi) = \sum_{\lambda \vdash n} \chi(\lambda) p_\lambda / z_\lambda,$$

where $\chi(\lambda)$ is the value of χ on C_λ. A well-known property of the Frobenius map is that the irreducible character χ^λ is sent to

$$\mathrm{Frob}(\chi^\lambda) = s_\lambda := \sum_{T \in \mathrm{SSYT}(\lambda, [1^n, 2^n, \dots])} \rho(T),$$

where the color rule from $[1^n, 2^n, \dots]$ has weight $\rho(i) = x_i$. This is, in fact, also a consequence of Theorem 3.8. The (complete) homogeneous symmetric function

$$h_n = \sum_{a_1 \leqslant \cdots \leqslant a_n} x_{a_1} \cdots x_{a_n}$$

then corresponds to the trivial representation. Theorem 3.8 also would imply that if a module M is isomorphic to $1 \uparrow_{S_{\mu_1} \times \cdots \times S_{\mu_\ell}}^{S_n}$, that is M is generated by a single element whose stabilizer is isomorphic to $S_{\mu_1} \times \cdots \times S_{\mu_\ell}$, then

$$\mathrm{Frob}\left(1 \uparrow_{S_{\mu_1} \times \cdots \times S_{\mu_\ell}}^{S_n}\right) = h_\mu,$$

which is further addressed in Section 4.1.4. Before looking at our examples, we would like to warn readers that we will be making light use of plethystic substitution. The important part to remember is that if $y_1, y_2, \dots$ are positive monomials in R, then $F[y_1 + y_2 + \cdots]$ corresponds to variables substitution $F(y_1, y_2, \dots)$ in the symmetric function. More generally, when not all the terms are positive monomials, one must first expand $F = \sum_\lambda c_\lambda p_\lambda$ in terms of power sums, then set, for any expression E,

$$p_r[E(t_1, t_2, \dots)] = E(t_1^r, t_2^r, \dots).$$

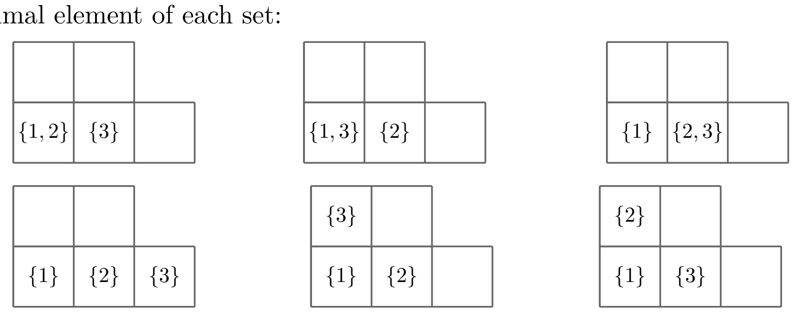
When $X = x_1 + x_2 + \cdots$, we get $p_r[X] = x_1^r + x_2^r + \cdots = p_r$, so that for any symmetric function F, we have $F[X] = F$.

4.1.2. *The defining representation.* The character for the defining representation of S_n is given by $\sigma \mapsto \mathrm{fxd}(\sigma)$, the number of fixed points of σ. Example 3.5 describes the color rule: fxd^m is computed by choosing a set partition $\{S_1, \dots, S_r\}$ of $\{1, \dots, m\}$, then placing each S_i over a unique fixed point. This means $\mathrm{SSYT}(\lambda, [1, \varnothing^{n-1}]^{\times m})$ can be interpreted as the set of semistandard tableaux whose entries (which are possibly empty) form a set partition of $\{1, \dots, m\}$. Theorem 3.8 specializes to the combinatorial proof of the following result given in [16].

Proposition 4.1. *For $\lambda \vdash n$,*

$$\langle \chi^\lambda, \mathrm{fxd}^m \rangle = |\,\mathrm{SSYT}(\lambda, [1, \varnothing^{n-1}]^{\times m})|.$$

For instance, suppose $n = 5$, $m = 3$, and $\lambda = (3, 2)$. Then $\mathrm{SSYT}(\lambda, [1, \varnothing^4]^{\times 3})$ would contain the following semistandard tableaux, where the order on colors is given by the minimal element of each set:



This means that

$$\langle \chi^{(3,2)}, \mathrm{fxd}^3 \rangle = 6.$$

4.1.3. *Actions on words.* Let $F = [1^n, \dots, N^n]$. This means every cycle can get any color. Given a word $a_1 \cdots a_n \in \{1, \dots, N\}^n$ from N letters, we have the action of S_n on the positions, given by

$$\sigma(a_1 \cdots a_n) = a_{\sigma_1} \cdots a_{\sigma_n}.$$

Let χ be the graded character of this action, given by

$$\chi(\sigma) = \sum_{\sigma(a_1\cdots a_n)=a_1\cdots a_n} t_{a_1}\cdots t_{a_n}.$$

Then χ is given by the color rule F with $\rho(f_i) = t_{f_i}$. As a consequence of Theorem 3.8, we have

$$\langle \chi^\lambda, \chi\rangle = \sum_{T\in \mathrm{SSYT}(\lambda,[1^n,\ldots,N^n])} \rho(T) = s_\lambda[t_1+\cdots+t_N].$$

This gives

$$s_\lambda[t_1+\cdots+t_N] = \langle s_\lambda, \mathrm{Frob}(\chi)\rangle\,,$$

where Frob is the Frobenius characteristic map in Subsection 4.1.1. Therefore,

$$\mathrm{Frob}(\chi) = \sum_{\lambda\vdash n} s_\lambda[t_1+\cdots+t_N]s_\lambda[X] = h_n\left[(t_1+\cdots+t_N)X\right].$$

When we take the m^{th} Kronecker power, we use as colors $[1^n,\ldots,N^n]^{\times m}$ giving

$$\mathrm{Frob}(\chi^m) = \sum_{\lambda\vdash n} s_\lambda[(t_1+\cdots+t_N)^m]s_\lambda[X] = h_n\left[(t_1+\cdots+t_N)^m X\right].$$

Setting $t_i = 1$ gives the multiplicities of irreducible representations in the m-fold tensor product of the representation arising from the action (on positions) of the symmetric group on words with N letters.

4.1.4. *Young tabloids.* We take the action of S_n on the set of row strict (increasing) tableaux $\mathrm{RST}(\mu)$ of shape μ and with entries $1,\ldots,n$. The action of σ on $T\in\mathrm{RST}(\mu)$ is given by sending i in T to σ_i , then rearranging the rows of the tableau in increasing order. We will denote the character of this action by ψ_μ. The character is given by sending $\sigma\in S_n$ to

$$\psi_\mu(\sigma) = |\{T\in\mathrm{RST}(\mu) : \sigma T = T\}|.$$

A row strict tableau T is fixed by σ if and only if the following holds:

If i and j are in the same cycle of σ, then i and j are in the same row of T.

For a partition μ, let $[\mu] = [1^{\mu_1},\ldots,n^{\mu_n}]$, with $p(f)=\rho(f)=1$ trivial. This color rule will then give the character ψ_μ. Our main result gives

$$\langle \chi_\mu, \chi^\lambda\rangle = |\,\mathrm{SYT}(\lambda,[\mu])| = K_{\lambda,\mu},$$

the Kostka numbers. Therefore,

$$\mathrm{Frob}(\psi_\mu) = \sum_{\lambda\vdash n} K_{\lambda,\mu}s_\lambda[X] = h_\mu.$$

For any two partitions $\mu,\nu\vdash n$, let

$$[\mu]\star[\nu] = \{[(a_1,b_1),\ldots,(a_n,b_n)] : [a_1,\ldots,a_n]=[\mu] \text{ and } [b_1,\ldots,b_n]=[\nu]\}$$

Then

$$\langle \psi_\mu\cdot\psi_\nu\ ,\chi^\lambda\rangle = |\mathrm{SYT}(\lambda,\mu\star\nu)| \ = \ \langle h_\mu * h_\nu, s_\lambda\rangle$$

where $\mathrm{SYT}(\lambda,[\mu]\star[\nu])$ is the set of semistandard tableaux whose entries, with multiplicities, are given by $[(a_1,b_1),\ldots,(a_n,b_n)]$ for some $[(a_1,b_1),\ldots,(a_n,b_n)]\in[\mu]\star[\nu]$. It is easier to see what we mean by example. So let $\mu=(3,2)$ and $\nu=(2,2,1)$. Then

$[1^{\mu_1},\ldots,n^{\mu_n}]=[1^3,2^2]=[1,1,1,2,2]$, and likewise $[1^{\nu_1},\ldots n^{\nu_n}]=[1,1,2,2,3]$. The $\star$ product produces the set $[\mu]\star[\nu]$

$$\begin{aligned}= \{&[(1,1),(1,1),(1,2),(2,2),(2,3)], \quad &&[(1,1),(1,2),(1,2),(2,1),(2,3)],\\ &[(1,1),(1,1),(1,3),(2,2),(2,2)], &&[(1,1),(1,2),(1,3),(2,1),(2,2)],\\ & &&[(1,2),(1,2),(1,3),(2,1),(2,1)]\}.\end{aligned}$$

So to compute $\langle h_{3,2}*h_{2,2,1},s_{2,1,1,1}\rangle$ we must compute the number of semistandard Young tableaux of shape $(2,1,1,1)$ in each of the above multisets. It is not difficult to check that the corresponding quantities are given respectively by $1,1,0,4$, and 0, the tableaux being listed in Figure 5.

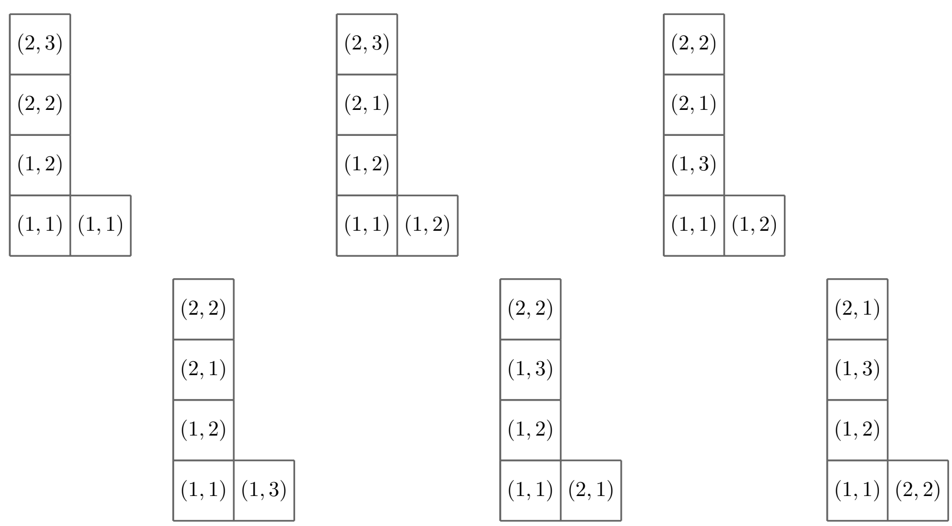


Figure 5. A list of all tableaux of shape $(2,1,1,1)$ with labels coming from $[(3,2)]\star[(2,2,1)]$.

Therefore,

$$\langle h_{3,2}*h_{2,2,1},s_{2,1,1,1}\rangle=6.$$

For $\mu^1,\ldots,\mu^s\vdash n$ we can similarly define the product $[\mu^1]\star[\mu^2]\star\cdots\star[\mu^s]$. We get

Corollary 4.2. *For $\mu^1,\ldots,\mu^s\vdash n$,*

$$\langle h_{\mu^1}*\cdots*h_{\mu^s},s_\lambda\rangle=|SYT(\lambda,[\mu^1]\star\cdots\star[\mu^s])|.$$

We should note that Corollary 4.2, when $s=2$, can also be seen by K Rule III in [10].

Another way to see why the action on $\mathrm{RST}(\mu)$ has character equal to that of ψ_μ, we note that every element $(f_{r_1},\ldots,f_{r_n})$ from $[\mu]$ which is fixed by the action of σ represents a tableaux in $\mathrm{RST}(\mu)$ which is fixed by σ: place the label i in row f_{r_i} of μ.

The action on $\mathrm{RST}(\mu)$ is transitive, and the stabilizer of any element is a row subgroup isomorphic to $S_{\mu_1}\times S_{\mu_2}\times\cdots\times S_{\mu_{\ell(\mu)}}$, which is sufficient to say the Frobenius image of this character is $h_{\mu_1}h_{\mu_2}\cdots h_{\mu_{\ell(\mu)}}$. On the other hand, Theorem 3.8 also implies the following well-known fact:

Proposition 4.3. *Let M be an S_n-module with character χ^M. Suppose S_n acts transitively on M, so that $M=\mathbb{C}[S_n]w$ is the orbit of a single element w. Suppose the* *stabilizer of w is isomorphic to $S_{\mu_1} \times \cdots S_{\mu_{\ell(\mu)}}$. Then*

$$\mathrm{Frob}(\chi^M) = h_\mu.$$

We next define a coloring rule which is similar to this example. We will color our points by $[\mu]$, but we instead allow each point to get a multiset of colors.

4.1.5. *Restrictions from GL_n.* Let $F = [[1^{d_1}, \dots, m^{d_m}]^n : d_i \geqslant 0]$ be the set of colors consisting of multisets $[1^{d_1}, \dots, m^{d_n}]$. For a coloring $f = (f_1, \dots, f_n)$ (so each f_i is a multiset), let

$$\rho(f) = t_1^{a_1} \cdots t_m^{a_m}$$

if, for each i, the number of occurrences of i in all of $(f_1, \dots, f_n)$ is a_i.

Take the polynomial ring $A = \mathbb{Q}[x_1, \dots, x_n]$ with the action of S_n which sends x_i under σ to x_{σ_i}. Then the degree d component of A, say A_d, is invariant under the action and produces a character ψ_d. Using the set of monomials $x_1^{r_1} \cdots x_n^{r_n}$ with $r_1 + \cdots + r_n = d$ as a basis for R_d gives a permutation action. To calculate

$$\psi_d(\sigma) = |\{x_1^{r_1} \cdots x_n^{r_n} : x_1^{r_1} \cdots x_n^{r_n} = x_{\sigma_1}^{r_1} \cdots x_{\sigma_n}^{r_n} \text{ and } r_1 + \cdots + r_n = d\}|,$$

we first note that $x_1^{r_1} \cdots x_n^{r_n}$ is fixed by σ provided the following holds:

$$\text{if } i \text{ and } j \text{ are in the same cycle of } \sigma, \text{ then } r_i = r_j.$$

We therefore must color the cycles of σ by distributing the multiset $[1^d]$ over the cycles. The number of 1's above i represents the power of x_i in a monomial. For example, the following colored permutation would produce the monomial $x_1^3 x_2 x_4^3 x_5 x_6^3 x_7^2 x_8^3 x_9 \in A_{17}$:

$$\begin{matrix} [1^3] & [1^3] & [1^3] & [1^3] & [1] & [1] & [1] & & [1^2] \\ (1 & 4 & 8 & 6) & (2 & 9 & 5) & (3) & (7) \end{matrix}$$

Since each element of a cycle has the same number of elements above, the produced monomial is fixed by σ.

The action of GL_n on A_1 gives an action on $\mathrm{Symm}^d(A_1) = A_d$ whose character we denote by ϕ_d. For $g \in GL_n$ with eigenvalues $\theta_1, \dots, \theta_n$, the character may be calculated by using the homogeneous symmetric function:

$$\phi_d(g) = h_d(\theta_1, \dots, \theta_n).$$

In particular, when g is a permutation matrix $\Pi(\sigma)$, where σ has cycle structure μ, then the eigenvalues of $\Pi(\sigma)$ with multiplicities are given by

$$\zeta(\sigma) = \left[1, \zeta_{\mu_1}, \dots, \zeta_{\mu_1}^{\mu_1 - 1}, 1, \zeta_{\mu_2}, \dots, \zeta_{\mu_2}^{\mu_2 - 1}, \dots, 1, \zeta_{\mu_{\ell(\mu)}}, \dots, \zeta_{\mu_{\ell(\mu)}}^{\mu_{\ell(\mu)} - 1}\right],$$

where ζ_r is an r^{th} primitive root of unity. We get that

$$\psi_d(\sigma) = h_d(\zeta(\sigma)).$$

It follows from our main result that

$$\begin{aligned} \langle \phi_{d_1} \cdots \phi_{d_m} \downarrow_{S_n}^{GL_n}, \chi^\lambda \rangle &= \langle h_{d_1} \cdots h_{d_m} \ [\zeta(\cdot)], \chi^\lambda \rangle \\ &= \langle \psi_{d_1} \cdots \psi_{d_m}, \chi^\lambda \rangle \\ &= \sum_{\cup M_i = [1^{d_1}, \dots, m^{d_m}]} \sum_{T \in \mathrm{SSYT}(\lambda, [M_1, \dots, M_n])} \rho(T) \\ &= s_\lambda \left[\frac{1}{(1 - t_1)(1 - t_2) \cdots (1 - t_m)} \right] \Big|_{t_1^{d_1} \cdots t_m^{d_m}}, \end{aligned}$$

where the sum runs over all semistandard with entries that are multisets M_i whose union is $[1^{d_1}, 2^{d_2}, \cdots, m^{d_m}]$. The last equality interprets each $M_i = [1^{a_1}, \dots, m^{a_m}]$ as coming from the series $1/((1-t_1)\cdots(1-t_m))$, and since the union equals of the M_i equals $[1^{d_1}, 2^{d_2}, \cdots, m^{d_m}]$, we must take the coefficient of $t_1^{d_1}\cdots t_m^{d_m}$.

This decomposes the action of S_n on the polynomial ring $\mathbb{Q}[x_{i,j} \colon i = 1, \ldots, m$ and $j = 1, \ldots, n]$, given by the Frobenius characteristic

$$\sum_{\lambda\vdash n} s_\lambda s_\lambda \left[\frac{1}{(1-t_1)\cdots(1-t_m)}\right] = h_n\left[\frac{X}{(1-t_1)\cdots(1-t_m)}\right].$$

This can be found in the work of Orellana and Zabrocki [18] and [21].

4.2. Examples on $Z_k \wr S_n$. One can generalize many examples coming from S_n to the wreath product case. However, we will focus on providing three main examples. The first is on tensor products of the defining representation; the second is on a combinatorial proof of the Murnaghan-Nakayama rule; and lastly, we will look at our main application to the affine semigroup algebras appearing in the subsequent section.

4.2.1. *The defining representation of $Z_k \wr S_n$.* Recall from Example 3.5 that the character of the defining representation is given by the color rule $F = [1, \varnothing^{n-1}]$ with value and weight function given by $p(1) = 1$, $p(\varnothing) = 0$, and $\rho = 1$. We immediately see from Theorem 3.8 that if $k > 1$, then $\operatorname{tr}\Pi_{\text{def}} = \chi^{((n-1),(1),\varnothing,\ldots,\varnothing)}$: there is only one semistandard tableau in the entries $[1, \varnothing^{n-1}]$ where 1 is placed in γ^1 and all $\varnothing$'s are placed in γ^0, and this tableau has shape $((n-1),(1),\varnothing,\ldots,\varnothing)$.

The product $[1,\varnothing^{n-1}]^{\times m}$ will then produce the colors for computing the trace of the m-fold tensor product of the defining representation, $\Pi_{\text{def}}^{\otimes m}$. For a given $\vec{\gamma} \vdash_k n$, the set $\mathrm{SSYT}_k(\vec{\gamma}, [1,\varnothing^{n-1}]^{\times m})$ can be interpreted as the set of all semistandard tableaux of shape $\vec{\gamma}$ whose entries form a set partition of $\{1,\ldots,m\}$, say $\{S_1,\ldots,S_n\}$, such that S_i appears in γ^r if and only if $|S_i| = r \mod k$. For instance, Figure 6 gives a tableau which contributes to the calculation of $\left\langle \chi^{((3,1),(2,2))}, \operatorname{tr}\Pi_{\text{def}}^{\otimes 10}\right\rangle$.

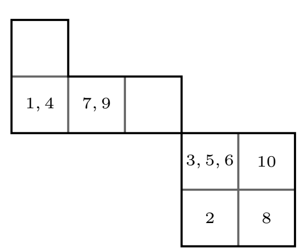


Figure 6. An element of $\mathrm{SSYT}_2\left(((3,1),(2,2)), [1,\varnothing^{n-1}]^{\times 10}\right)$ interpreted using set partitions. Sets are drawn without brackets, and they are ordered by minimal element.

We immediately note something important: every cell of γ^r for $r > 0$ must contain at least r elements; and at least $|\gamma^0| - \gamma^0_1$ cells in γ^0 must contain at least k elements. Therefore, $\langle \chi^{\vec{\gamma}}, \operatorname{tr}\Pi_{\text{def}}^{\otimes m}\rangle = 0$ if $m < N(\vec{\gamma})$, where

$$N(\vec{\gamma}) = k(|\gamma^0| - \gamma^0_1) + \sum_{i=1}^{k-1} i|\gamma^i|.$$

Usually, determining if an irreducible representation appears in a tensor product is very difficult. Remarkably in this case, we are able to determine precisely when an irreducible representation appears!

Corollary 4.4. *We have $\langle \operatorname{tr}\Pi_{\text{def}}^{\otimes m}, \chi^{\vec{\gamma}}\rangle \neq 0$ if and only if $m = N(\vec{\gamma}) + kr$ for some $r \geqslant 0$.*

Proof. From the previous observations, the sum of minimal cardinalities for each cell in $\vec{\gamma}$ gives the smallest m for which $\langle \chi^{\vec{\gamma}}, \operatorname{tr} \Pi_{\mathrm{def}}^{\otimes m} \rangle \neq 0$: $m = N(\vec{\gamma})$.

The cardinalities of the sets in each cell must always increase by at least k from these minimal cardinalities. Therefore, if $\langle \chi^{\vec{\gamma}}, \operatorname{tr} \Pi_{\mathrm{def}}^{\otimes m} \rangle \neq 0$, we must have $m = N(\vec{\gamma}) + kr$ for some $r \geqslant 0$. On the other hand, given such an m, we can always find a tableau in $\mathrm{SSYT}_k(\vec{\gamma}, [1, \varnothing^{n-1}]^{\times m})$. □

4.2.2. *A combinatorial proof of the Murnaghan-Nakayama rule, multisymmetric functions, and the Frobenius map for wreath products.* Suppose our color rule is given by $X^* = [x_{i,j}^n : i = 0, \ldots, k-1 \text{ and } j \geqslant 1]$. And suppose that the value of $x_{i,j}$ is given by $p(x_{i,j}) = i$, and that the weight is given by $\rho(x_{i,j}) = x_{i,j}$. Then if we let $X^i = x_{i,1} + x_{i,2} + \cdots$,

$$\begin{aligned}\sum_{(\sigma,f) \in X^*(Z_k \wr S_n)} \mathrm{weight}(\sigma, f) &= \prod_{r=0}^{k-1} \prod_{l=1}^{\ell(\lambda^r(\sigma))} \sum_{i=0}^{k-1} \sum_{j \geqslant 1} (u_r)^{p(x_{i,j})} \rho(x_{i,j})^{\lambda^r(\sigma)_l} \\ &= \prod_{r=0}^{k-1} \prod_{l=1}^{\ell(\lambda^r(\sigma))} p_{\lambda^r(\sigma)_l} \left[\sum_{i=0}^{k-1} \Psi^i(u_r) X^i \right].\end{aligned}$$

To simplify the notation, let us define for $\vec{\lambda}(\sigma) = \vec{\lambda}$ the multisymmetric function

$$P_{\vec{\lambda}(\sigma)}[X^0, \ldots, X^{k-1}] = \prod_{r=0}^{k-1} p_{\lambda^r} \left[\sum_{j=0}^{k-1} \Psi_r^j X^j \right].$$

Then Theorem 3.8 gives that

$$\frac{1}{n! k^n} \sum_{\sigma \in Z_k \wr S_n} \overline{\chi}^{\vec{\gamma}}(\sigma) P_{\vec{\lambda}(\sigma)}[X^0, \ldots, X^{k-1}] = \sum_{T \in \mathrm{SSYT}_k(\vec{\gamma}, X^*)} \rho(T).$$

But note that if we have the color $x_{i,j}$ in γ^r, then we are required to have $p(x_{i,j}) = i = r \mod k$. This means that each γ^r can only be filled with elements of $X^r = x_{r,1} + x_{r,2} + \cdots$. In other words, we have found that

Theorem 4.5. *For any $\vec{\gamma} \vdash_k n$, we have*

$$s_{\gamma^0}[X^0] s_{\gamma^1}[X^1] \cdots s_{\gamma^{k-1}}[X^{k-1}] = \sum_{\vec{\lambda} \vdash_k n} \overline{\chi}^{\vec{\gamma}}(\sigma) \frac{P_{\vec{\lambda}}[X^0, \ldots, X^{k-1}]}{Z_{\vec{\lambda}}},$$

where

$$Z_{\vec{\lambda}} = \frac{n! k^n}{|C_{\vec{\lambda}}|} = \prod_{i=0}^{k-1} \frac{z_{\lambda^i}}{k^{\ell(\lambda^i)}}.$$

This is the symmetric function version of the Murnaghan-Nakayama rule, a variation of the identity found in [15]. Here, we instead consider the homomorphism from class functions of $\mathbb{C}[Z_k \wr S_n]$ to the k-fold tensor product of symmetric functions (or multisymmetric functions) given by sending a conjugacy class

$$C_{\vec{\lambda}} \mapsto \frac{\overline{P}_{\vec{\lambda}}[X^0, \ldots, X^{k-1}]}{Z_{\vec{\lambda}}}.$$

Then the irreducible character is sent to

$$\chi^{\vec{\gamma}} \mapsto s_{\vec{\gamma}}[X] := \prod_{i=0}^{k-1} s_{\gamma^i}[X^i].$$

Note here, that to simplify the notation, we will henceforth write

$$b_{\vec{\lambda}}[X] := \prod_{i=0}^{k-1} b_{\lambda^i}[X^i]$$

when b_λ is some classical symmetric function basis, such as the Schur functions.

Proposition 4.6. *The Frobenius map,* Frob*, on characters of $Z_k \wr S_n$, given by sending the conjugacy class $C_{\vec{\lambda}}$ to $\overline{P}_{\vec{\lambda}}[X]/Z_{\vec{\lambda}}$, satisfies*

$$\mathrm{Frob}(\chi^{\vec{\gamma}}) = s_{\vec{\gamma}}[X].$$

For a given module M with (graded) character χ^M, we call Frob $\left(\chi^M\right)$ the *Frobenius characteristic* of M.

4.2.3. *A basis for all class functions.* Here, we show that every class function with values in $\mathbb{C}$ can be written as a sum of class functions given by color rules. For a given $\vec{\lambda} \vdash_k n$, let

$$F^{\vec{\lambda}} = \left[f_{i,j}^{\lambda_j^i} : 0 \leqslant i \leqslant k-1, 1 \leqslant j \leqslant \ell(\lambda^i) \right],$$

with

$$p(f_{i,j}) = i \qquad \text{and} \qquad \rho(f_{i,j}) = 1.$$

The multiplicity of $\chi^{\vec{\gamma}}$ is then the number of semistandard tableaux in the $f_{i,j}$, where $f_{i,j}$ appears only in γ^i, precisely λ_j^i times. Therefore

$$\chi^{F^{\vec{\lambda}}} = \chi^{\vec{\lambda}} + \sum_{\vec{\gamma} \geqslant \vec{\lambda}} |\,\mathrm{SSYT}_k(\vec{\gamma}, F^{\vec{\lambda}})|\; \chi^{\vec{\gamma}},$$

where we write $\vec{\gamma} \geqslant \vec{\lambda}$ to mean that for every i, $\lambda^i \leqslant \gamma^i$ in the dominance (or lexicographic) order. Therefore, these characters are triangularly related to irreducible characters, and every irreducible character can be written as a linear combination of the $\chi^{F^{\vec{\lambda}}}$. One can deduce that $\mathrm{Frob}(\chi^{F^{\vec{\lambda}}}) = h_{\vec{\lambda}}[X]$.

5. Projective toric varieties and their characters

This section will present the main application of our methods, and has two main objectives:

(1) To calculate the bigraded Frobenius characteristic of the projective coordinate ring of a general Segre product of projective toric varieties (see Proposition 5.6).
(2) To use this calculation, along with the theory of color rules developed in previous sections, to derive *equivariant* versions of Euler-Mahonian identities (see Subsection 5.4 and Theorem 5.25).

In order to achieve the first objective, we must define these rings and endow them with the structure of a bigraded $Z_k \wr S_n$-module. We do this in Subsections 5.1 and 5.2. Subsections 5.3 and 5.4 are interludes which relate our research to the existing literature.

Finally, we achieve the second objective in Subsections 5.5 and 5.6 by restricting our attention to the product of projective spaces and applying our methods. We give these results a geometric interpretation as quotients by regular sequences which directly generalize the ones described in [1] and [5], some of them conjecturally.

5.1. Preliminaries. This subsection includes a very minimal summary of relevant facts about semigroup algebras and toric varieties. For a more detailed exposition, see [17] or [8].

Note 5.1. *From now until the end of the section, we use the following conventions, unless specified otherwise:*

(1) *We work over the field of complex numbers $K = \mathbb{C}$. In particular, we work with algebraic varieties over $K = \mathbb{C}$.*
(2) *$P \subseteq \mathbb{R}^m$ is a full-dimensional convex lattice polytope. The lattice is $\mathbb{Z}^m$ except in Subsection* 5.3*, where it can be arbitrary.*

For a non-negative integer $d \in \mathbb{N}$, the set $dP = \{dx \in \mathbb{R}^m : x \in P\}$ is the d-th *dilation* of P. The *cone* over P is the set

$$C(P) \coloneqq \left\{ \sum_{i \in I} c_i(1, v_i) \in \mathbb{R}^{m+1} : \quad I \text{ finite, } c_i \in \mathbb{R}_{\geqslant 0}, v_i \in P \text{ for all } i \in I \right\}.$$

It is not hard to see that

$$C(P) \cap \mathbb{Z}^{m+1} = \coprod_{d \geqslant 0} \{d\} \times (dP \cap \mathbb{Z}^m). \tag{1}$$

For a point $v \in \mathbb{Z}^m$, write $v = \sum_{i=1}^m \pi_i(v) e_i$, where $\{e_i\}_{i=1}^m$ is the canonical basis of $\mathbb{R}^m$. The *affine semigroup algebra* of P is the set

$$K[P] \coloneqq K \left[x_{d,v} \coloneqq t^d \prod_{i=1}^m x_i^{\pi_i(v)} : \quad (d, v) \in C(P) \cap \mathbb{Z}^{m+1} \right] \subseteq K[t, x_1^{\pm 1}, \ldots, x_m^{\pm 1}].$$

Equivalently, by (1),

$$K[P] = K\left[x_{d,v} : \quad d \in \mathbb{N}, v \in dP \cap \mathbb{Z}^m\right]. \tag{2}$$

It follows from a well-known result, known as *Gordan's lemma*, that $K[P]$ is a finitely generated K-algebra. It admits a natural grading whose degree map is given by the (largest) exponent of t, which we call the *projective degree*. We let $\operatorname{pdeg}(f)$ denote the projective degree of $f \in K[P]$.

Geometrically, $X_P \coloneqq \operatorname{Proj} K[P]$ is a projective normal toric variety of (Krull) dimension m, and all such varieties arise in this way.

Let S be a finite set of generators of $K[P]$, and let $T_P \coloneqq K[S]$ be the K-algebra freely generated by the elements of S. The kernel $I_P \coloneqq \ker(\pi)$ of the natural surjection

$$\pi : T_P \twoheadrightarrow K[P] \tag{3}$$

is called the *toric ideal* of P and satisfies $K[P] \cong T_P / I_P$. We will presently endow $K[P]$ with multiple gradings, and this will be an isomorphism of multigraded K-algebras. Geometrically, $\operatorname{Proj} T_P$ is a weighted projective space, and the surjection π induces an embedding

$$X_P \hookrightarrow \operatorname{Proj} T_P.$$

Chapoton [7] and Adeyemo, Szendrői [1] study a refinement of the grading on $K[P]$. Fix $\boldsymbol{a} \in \mathbb{Z}^m$, which we call a *weight vector*. Let $d \in \mathbb{N}$ and $v \in P \cap \mathbb{Z}^m$. The map

$$\operatorname{cdeg}(x_{d,v}) \coloneqq \boldsymbol{a} \cdot v \in \mathbb{Z}$$

extends to $K[P]$ and endows it with an additional integral grading. Here, $\boldsymbol{a} \cdot v$ denotes the standard inner product of $\mathbb{R}^m$. We will call this map the *combinatorial degree*. It should be mentioned that Chapoton uses certain assumptions of positivity and genericity on the pair $(P, \boldsymbol{a})$ to develop his refined Ehrhart theory, but [1] does not use it, and neither do we. It is also worth noting that we can get a multigrading on $K[P]$ in a similar way by choosing a finite collection $\{\boldsymbol{a}_i\}_{i=1}^n \subseteq \mathbb{Z}^m$.

Another multigrading on $K[P]$ is studied by Reiner and Rhoades [20]. Their multigrading is based on Macaulay inverse systems for Minkowski sums of point configurations and is different from the one we study, already in small examples (see their Remark 3.20).

It will often be useful to assume that P is *normal*, i. e., that for all $d \in \mathbb{N}$ and $v \in dP \cap \mathbb{Z}^m$ there exist $v_1, v_2, \dots, v_d \in P \cap \mathbb{Z}^m$, such that $v = \sum_{i=1}^{d} v_i$. In particular, it can be shown that this is always the case if P is 2-dimensional. This notion of normality is also known as the *integer decomposition property (IDP)*. Note that, when P is normal, the affine semigroup algebra of P is simply

$$K[P] = K\left[x_v \coloneqq t\prod_{i=1}^{m} x_i^{\pi_i(v)} : \quad v \in P \cap \mathbb{Z}^m\right],$$

and (3) induces an embedding

$$X_P \hookrightarrow \operatorname{Proj}(T_P) \cong \mathbb{P}^{|P\cap\mathbb{Z}^m|-1}.$$

5.2. Actions on products. If $P' \subseteq \mathbb{R}^{m'}$ is another full-dimensional convex lattice polytope, the Cartesian product $P \times P'$ is again a polytope with these characteristics, and for all $d \in \mathbb{N}$,

$$d(P \times P') = dP \times dP'. \tag{4}$$

This operation translates nicely to the geometric point of view. If P and P' are *very ample* (a condition that is always satisfied by a dilation of P and P'), then, by [8, Theorem 2.4.7], the variety $X_{P\times P'} = \operatorname{Proj} K[P \times P']$ corresponds to the product of projective varieties $X_P \times X_{P'} = \operatorname{Proj} K[P] \times \operatorname{Proj} K[P']$ with the Segre embedding

$$X_P \times X_{P'} \hookrightarrow \mathbb{P}^{|P\cap\mathbb{Z}^m||P'\cap\mathbb{Z}^{m'}|-1}.$$

In particular, for $n \in \mathbb{N}_+$, we will be interested in $P^{\times n}$, the n-th Cartesian power of P. As in (2) (notice the additional subindices), and using (4), we may write

$$\begin{aligned} K[P^{\times n}] = K&\left[x_{d,v} \coloneqq t^d \prod_{i=1}^{n}\prod_{j=1}^{m} x_{i,j}^{\pi_j(v_i)} : \quad d \in \mathbb{N}, v_i \in dP \cap \mathbb{Z}^m \text{ for all } i = 1, \dots, n\right] \\ &\subseteq K[t, x_{1,1}^{\pm 1}, \dots, x_{n,m}^{\pm 1}]. \end{aligned}$$

The wreath product $Z_k \wr S_n$ acts linearly on $K[P^{\times n}]$ in a natural way, namely by permuting the lattice points of the n copies of P and multiplying by the corresponding roots of unity. More specifically, for $\sigma = u_{a_1}\sigma_1 \cdots u_{a_n}\sigma_n \in Z_k \wr S_n$, the action of σ fixes t and maps $x_{i,j}$ to $u_{a_i} x_{\sigma_i,j}$. Then, it is extended to $K[t, x_{1,1}^{\pm 1}, \dots, x_{n,m}^{\pm 1}]$ multiplicatively and linearly, and finally restricted to $K[P^{\times n}]$. This is a group action because if $\sigma' = u'_{a_1}\sigma'_1 \cdots u'_{a_n}\sigma'_n \in Z_k \wr S_n$, then

$$(\sigma'\sigma)(x_{i,j}) = u'_{a_{\sigma_i}} u_{a_i} x_{(\sigma'\sigma)_i,j} = \sigma'(u_{a_i} x_{\sigma_i,j}) = \sigma'(\sigma(x_{i,j})).$$

In order to better understand this action, we need to give a couple of definitions.

Definition 5.2 (Generalized permutation representations/matrices). *Let G be a group, S a set, and KS the K-vector space generated by the elements of S. We say KS carries a* generalized permutation representation *of G if there is a representation*

$$\Pi : G \to \mathrm{GL}(KS)$$

such that for all $g \in G$ and $s \in S$, there exist $u(g, s) \in K$ and $r \in S$ satisfying

$$\Pi(g)s = u(g, s)r.$$

In other words, the matrix $\Pi(g)$ *in the* standard basis *(the basis given by the elements of* S*) is a* generalized permutation matrix *(a matrix having only one non-zero entry in each row and column). Note that, if* $u(g,s)=1$ *for all* $g\in G$ *and* $s\in S$*, then* Π *is a permutation representation and* $\Pi(g)$ *in the standard basis is a permutation matrix.*

The following proposition follows.

Proposition 5.3. *Let* $d\in\mathbb{N}$*, and write* $K[\cdot]_d$ *for the* d*-th projective piece of* $K[\cdot]$*. The representation*

$$\Pi : Z_k \wr S_n \to \mathrm{GL}(K[t, x_{1,1}^{\pm 1}, \ldots, x_{n,m}^{\pm 1}]_d)$$

given by the linear action described above is a generalized permutation representation (S *is a monomial basis indexed by points in the lattice* $\mathbb{Z}^{mn}$*). Furthermore, in the standard basis, each matrix* $\Pi(\sigma)$ *is unitary.*

To endow $K[P^{\times n}]$ with the structure of a *multigraded* $Z_k \wr S_n$-module, we must impose conditions on the weight vectors $\boldsymbol{a}\in\mathbb{Z}^{mn}$ defining combinatorial degrees. The weight vectors that will work will turn out to be exactly the n-th Cartesian powers of vectors in $\mathbb{Z}^m$.

Proposition 5.4. *Let* I *be a finite set of indices. For each* $i\in I$*, let* $\boldsymbol{a}_i'\in\mathbb{Z}^m$ *be arbitrary. Then, the affine semigroup algebra* $K[P^{\times n}]$ *can be endowed with the structure of a multigraded* $Z_k\wr S_n$*-module, where the gradings are the projective degree and one combinatorial degree for each weight vector* $\boldsymbol{a}_i=(\boldsymbol{a}_i')^{\times n}\in\mathbb{Z}^{mn}$*.*

Conversely, if $\boldsymbol{a}\in\mathbb{Z}^{mn}$ *is a weight vector defining a combinatorial grading on* $K[P^{\times n}]$ *that is compatible with the representation* Π *in Proposition* 5.3*, then* $\boldsymbol{a}=(\boldsymbol{a}')^{\times n}$ *for some* $\boldsymbol{a}'\in\mathbb{Z}^m$*.*

Proof. Let Π be as in Proposition 5.3. The general case follows from the case in which there is only one weight vector $\boldsymbol{a}\in\mathbb{Z}^{mn}$. We will prove the forward direction first, but we will not use the fact that $\boldsymbol{a}=(\boldsymbol{a}')^{\times n}$ yet.

For $\sigma\in Z_k\wr S_n$, write $\tau=\sigma_1\cdots\sigma_n$ for the underlying permutation in S_n. Let $d\in\mathbb{N}$ and $x_{d,v}\in K[P^{\times n}]_d$. Since multiplying by a constant does not change the combinatorial degree:

$$\operatorname{cdeg}(\sigma(x_{d,v})) = \operatorname{cdeg}(\Pi(\sigma)(x_{d,v})) = \operatorname{cdeg}(\Pi\downarrow_{S_n}^{Z_k\wr S_n}(\tau)(x_{d,v})). \tag{5}$$

Let

$$\Pi' : S_n \to \mathrm{GL}_{mn}(\mathbb{R})$$

be the representation defined by the following rule: if $\tau\in S_n$, and $\{e_{i,j}\in\mathbb{R}^{mn} : i=1,\ldots,m,\ j=1,\ldots,n\}$ is the canonical basis of $\mathbb{R}^{mn}$, then $\tau e_{i,j}=e_{\tau(i),j}$. We have that $\Pi'\cong m\Pi_{\mathrm{def}}$, where Π_{def} is the defining representation of S_n. From the definition of $x_{d,v}$,

$$\Pi\downarrow_{S_n}^{Z_k\wr S_n}(\tau)(x_{d,v}) = x_{d,\Pi'(\tau)(v)}. \tag{6}$$

In particular, the restriction $\Pi\downarrow_{S_n}^{Z_k\wr S_n}$ is a permutation representation.

Now, since $\Pi(Z_k\wr S_n)$ fixes t, it leaves the projective degree invariant. We will see that, if $\boldsymbol{a}\in\mathbb{Z}^{mn}$ is $\Pi'(S_n)$-invariant, then $\Pi(Z_k\wr S_n)$ also leaves the combinatorial degree invariant. Indeed, let $\boldsymbol{a} \in \mathbb{Z}^{mn}$ be $\Pi'(S_n)$-invariant and $v \in \mathbb{Z}^{mn}$, then

$$\begin{aligned}
\operatorname{cdeg}(\sigma(x_{d,v})) &= \operatorname{cdeg}(\Pi(\sigma)(x_{d,v})) \\
&= \operatorname{cdeg}(\Pi \downarrow_{S_n}^{Z_k \wr S_n} (\tau)(x_{d,v})) && (5)\\
&= \operatorname{cdeg}(x_{d,\Pi'(\tau)(v)}) && (6)\\
&= \boldsymbol{a} \cdot (\Pi'(\tau)(v)) \\
&= (\Pi'(\tau^{-1})(\boldsymbol{a})) \cdot v && (\Pi' \text{ is orthogonal in the canonical basis})\\
&= \boldsymbol{a} \cdot v && (a \text{ is } \Pi'(S_n)\text{-invariant})\\
&= \operatorname{cdeg}(x_{d,v}).
\end{aligned}$$

In defining Π', we have ordered the canonical basis of $\mathbb{R}^{mn}$ in the following way:

$$e_{1,1} \prec e_{1,2} \prec \cdots \prec e_{1,m} \prec e_{2,1} \prec e_{2,2} \prec \cdots \prec e_{2,m} \prec \cdots \prec e_{n,m}.$$

This implies that $\boldsymbol{a}$ is $\Pi'(S_n)$-invariant if and only if $\boldsymbol{a} = (\boldsymbol{a}')^{\times n}$, where $\boldsymbol{a}' \in \mathbb{Z}^m$, so the forward direction follows by the previous discussion.

The backward direction is now much more direct: if $\boldsymbol{a}$ is not of the form $(\boldsymbol{a}')^{\times n}$ for some $\boldsymbol{a}' \in \mathbb{Z}^m$, then it is not $\Pi'(S_n)$-invariant, so there exist $\tau \in S_n$ and $v \in \mathbb{Z}^{mn}$ such that

$$\operatorname{cdeg}(\sigma(x_{d,v})) = (\Pi'(\tau^{-1})(\boldsymbol{a})) \cdot v \neq \boldsymbol{a} \cdot v = \operatorname{cdeg}(x_{d,v}).$$

□

Note 5.5. *We restrict ourselves to the bigraded case in Proposition* 5.4 *for the rest of the paper (i. e., the gradings are given by the projective degree and a single weight vector* $\boldsymbol{a} = (\boldsymbol{a}')^{\times n} \in \mathbb{Z}^{mn}$*).*

We will use the following result extensively for the rest of this section:

Proposition 5.6. *Consider the bigraded* $Z_k \wr S_n$*-module structure on* $K[P^{\times n}]$ *as in Proposition* 5.4*. For* $v \in \mathbb{Z}^m$*, let* $|v| = \sum_{j=1}^m \pi_j(v)$*. Then, for* $\sigma = u_{a_1}\sigma_1 \cdots u_{a_n}\sigma_n \in Z_k \wr S_n$ *of cycle type* $\vec{\lambda}(\sigma) = \vec{\lambda}$*, we have the following character:*

$$\chi^{K[P^{\times n}]}(\sigma) = \sum_{d \geqslant 0} t^d \prod_{i=0}^{k-1} \prod_{j=1}^{\ell(\lambda^i)} \sum_{v \in dP \cap \mathbb{Z}^m} u_i^{|v|} q^{(\lambda_j^i)(\boldsymbol{a}' \cdot v)}.$$

Proof. Fix $d \in \mathbb{N}$. Let $K[P^{\times n}]_d$ be the d-th projective piece of $K[P^{\times n}]$. For $i = 1, \ldots, n$ and $v_i \in dP \cap \mathbb{Z}^m$, write

$$x_{i,v_i} := \prod_{j=1}^m x_{i,j}^{\pi_j(v_i)}.$$

The set

$$\mathcal{B}_d := \left\{ t^d \prod_{i=1}^n x_{i,v_i} : \quad v_i \in dP \cap \mathbb{Z}^m, \text{ for all } i = 1, \ldots, n \right\}$$

is the standard (monomial) basis of $K[P^{\times n}]_d$ (in particular, $|\mathcal{B}_d| = |dP^{\times n} \cap \mathbb{Z}^{mn}|$). Let $x = t^d \prod_{i=1}^n x_{i,v_i} \in \mathcal{B}_d$. By definition,

$$\sigma x = t^d \prod_{i=1}^n u_{a_i}^{|v_i|} x_{\sigma_i, v_i}. \tag{7}$$

This gives a generalized permutation representation

$$\Pi_d : Z_k \wr S_n \to \mathrm{GL}(K\mathcal{B}_d).$$

To calculate the coefficient of t^d in the character $\chi^{K[P^{\times n}]}(\sigma)$ it suffices to count the fixed points x_{fixed} of the restriction $\Pi_d\downarrow_{S_n}^{Z_k\wr S_n}(\tau)$ with their corresponding roots of unity $u(x_{\text{fixed}})$, where $\tau\in S_n$ is the permutation underlying σ.

Let $D=\{(i,j)\in\mathbb{Z}^2:\quad 0\leqslant i\leqslant k-1, 1\leqslant j\leqslant \ell(\lambda^i)\}$. It is not hard to see that these fixed points are the elements of $\mathcal{B}_d$ of the form

$$x_{\text{fixed}}=x_{\text{fixed}}\left((v_{ij})_{(i,j)\in D}\right)=t^d\prod_{(i,j)\in D}\prod_{u_{a_k}\sigma_k\in c_j^i}x_{\sigma_k,v_{ij}},\qquad v_{ij}\in dP\cap\mathbb{Z}^m,\forall(i,j)\in D,$$

where c_j^i is the C_i-cycle of σ indexed by j. In other words, to construct a fixed point x_{fixed}, we must choose a point in $dP\cap\mathbb{Z}^m$ for each cycle c_j^i of σ.

By (7), their corresponding roots of unity are

$$u(x_{\text{fixed}})=\prod_{(i,j)\in D}\prod_{u_{a_k}\sigma_k\in c_j^i}u_{a_{\sigma_k}}^{|v_{ij}|}=\prod_{(i,j)\in D}u_i^{|v_{ij}|}.$$

Their combinatorial degrees are

$$\begin{aligned}\operatorname{cdeg}(x_{\text{fixed}})&=\operatorname{cdeg}\left(\prod_{(i,j)\in D}\prod_{u_{a_k}\sigma_k\in c_j^i}x_{\sigma_k,v_{ij}}\right)\\&=\sum_{(i,j)\in D}\sum_{u_{a_k}\sigma_k\in c_j^i}\operatorname{cdeg}(x_{\sigma_k,v_{ij}})\\&=\sum_{(i,j)\in D}(\lambda_j^i)(\boldsymbol{a}'\cdot v_{ij}).\end{aligned}$$

Therefore,

$$\begin{aligned}[t^d]\chi^{K[P^{\times n}]}(\sigma)&=\sum_{x_{\text{fixed}}}u(x_{\text{fixed}})q^{\operatorname{cdeg}(x_{\text{fixed}})}\\&=\sum_{(v_{ij})_{i,j}\in(dP\cap\mathbb{Z}^m)^D}\prod_{(i,j)\in D}u_i^{|v_{ij}|}q^{(\lambda_j^i)(\boldsymbol{a}'\cdot v_{ij})}\\&=\prod_{(i,j)\in D}\sum_{v_{ij}\in dP\cap\mathbb{Z}^m}u_i^{|v_{ij}|}q^{(\lambda_j^i)(\boldsymbol{a}'\cdot v_{ij})}.\end{aligned}$$

The result follows. ☐

It is possible to find a plethystic expression for the character in Proposition 5.6 for any k (see Theorem 5.11). However, the story is specially nice for $k=1$ (i. e., if the wreath product $Z_k\wr S_n$ is just S_n). Before giving the statement, however, we need to recall a few definitions from [7] or [1]:

Definition 5.7 (Refined Ehrhart polynomial, series). *Let $\boldsymbol{a}'\in\mathbb{Z}^m$ be a weight vector. The* refined Ehrhart polynomial *of P is the Laurent polynomial*

$$L_{P,d}^{\boldsymbol{a}'}(q)=\sum_{i\in\mathbb{Z}}(\#\{v\in dP\cap\mathbb{Z}^m:\quad \boldsymbol{a}'\cdot v=i\})q^i\in K[q,q^{-1}].$$

The refined Ehrhart series *of P is the formal power series*

$$E_P^{\boldsymbol{a}'}(t,q)=\sum_{d\geqslant 0}L_{P,d}^{\boldsymbol{a}'}(q)t^d\in K[q,q^{-1}][\![t]\!].$$

These definitions are justified as their specializations at $q=1$ are the classical Ehrhart polynomial and series, respectively. In [1], explicit expressions are given in the case of the simplex, cross polytope, and hypercube.

Corollary 5.8. *Consider the natural bigraded S_n-module structure on $K[P^{\times n}]$ as in Proposition 5.4 with $k=1$. Then, the bigraded Frobenius characteristic satisfies*

$$\mathrm{Frob}(K[P^{\times n}]) = \sum_{d\geqslant 0} t^d \sum_{\lambda\vdash n} s_\lambda\left[L^{\boldsymbol{a}'}_{P,d}(q)\right] s_\lambda[X],$$

where $L^{\boldsymbol{a}'}_{P,d}(t)$ is the refined Ehrhart polynomial of P.

Proof. Proposition 5.6 implies the following equalities for $k=1$:

$$\begin{aligned}\mathrm{Frob}(K[P^{\times n}]) &= \frac{1}{n!}\sum_{\sigma\in S_n} \chi^{K[P^{\times n}]}(\sigma) p_{\lambda(\sigma)}[X] \\ &= \frac{1}{n!}\sum_{\sigma\in S_n}\sum_{d\geqslant 0} t^d \prod_{j=1}^{\ell(\lambda(\sigma))} \sum_{v\in dP\cap\mathbb{Z}^m} q^{(\lambda_j)(\boldsymbol{a}'\cdot v)} p_{\lambda(\sigma)}[X] \\ &= \frac{1}{n!}\sum_{\sigma\in S_n}\sum_{d\geqslant 0} t^d p_{\lambda(\sigma)}\left[\sum_{v\in dP\cap\mathbb{Z}^m} q^{(\boldsymbol{a}'\cdot v)} X\right] \\ &= \sum_{d\geqslant 0} t^d \sum_{\lambda\vdash n} s_\lambda\left[\sum_{v\in dP\cap\mathbb{Z}^m} q^{(\boldsymbol{a}'\cdot v)}\right] s_\lambda[X] \\ &= \sum_{d\geqslant 0} t^d \sum_{\lambda\vdash n} s_\lambda\left[L^{\boldsymbol{a}'}_{P,d}(q)\right] s_\lambda[X]. \qquad\square\end{aligned}$$

Example 5.9. Let $P=\Xi_m\subseteq\mathbb{R}^m$ be the m-dimensional cross-polytope, defined as the convex hull of the set

$$\{e_1,-e_1,e_2,-e_2,\dots,e_m,-e_m\}.$$

By definition,

$$\begin{aligned}K[(\Xi_m)^{\times n}] &= K\left[t\prod_{i\in I} x_{i,j_i}^{\pm 1}:\quad I\subseteq\{1,\dots,n\}, j_i\in\{1,\dots,m\}\text{ for all } i\in I\right]\\ &\subseteq K\left[t, x_{1,1}^{\pm1},\dots,x_{n,m}^{\pm1}\right].\end{aligned}$$

For example,

$$\begin{aligned}K[(\Xi_2)^{\times 2}] &= K\begin{bmatrix} t, & & & \\ tx_{1,1}, & tx_{1,1}^{-1}, & tx_{1,2}, & tx_{1,2}^{-1},\\ tx_{2,1}, & tx_{2,1}^{-1}, & tx_{2,2}, & tx_{2,2}^{-1},\\ tx_{1,1}x_{2,1}, & tx_{1,1}x_{2,1}^{-1}, & tx_{1,1}x_{2,2}, & tx_{1,1}x_{2,2}^{-1},\\ tx_{1,1}^{-1}x_{2,1}, & tx_{1,1}^{-1}x_{2,1}^{-1}, & tx_{1,1}^{-1}x_{2,2}, & tx_{1,1}^{-1}x_{2,2}^{-1},\\ tx_{1,2}x_{2,1}, & tx_{1,2}x_{2,1}^{-1}, & tx_{1,2}x_{2,2}, & tx_{1,2}x_{2,2}^{-1},\\ tx_{1,2}^{-1}x_{2,1}, & tx_{1,2}^{-1}x_{2,1}^{-1}, & tx_{1,2}^{-1}x_{2,2}, & tx_{1,2}^{-1}x_{2,2}^{-1}\end{bmatrix}\\ &\subseteq K\left[t,x_{1,1},x_{1,1}^{-1},x_{1,2},x_{1,2}^{-1},x_{2,1},x_{2,1}^{-1},x_{2,2},x_{2,2}^{-1}\right].\end{aligned}$$

If the weight vector is $\boldsymbol{a}'=(1,1)$, then

$$L^{\boldsymbol{a}'}_{\Xi_2,d}(q) = L^{\boldsymbol{a}'}_{\Xi_2,d-1}(q) + (d+1)(q^{-d}+q^d) + 2\left(q^{-d+2}+q^{-d+4}+\cdots+q^{d-2}\right).$$

It follows that

$$\begin{aligned}E^{\boldsymbol{a}'}_{\Xi_2}(t,q) = \sum_{d\geqslant 0} t^d L^{\boldsymbol{a}'}_{\Xi_2,d}(q) &= \frac{(1-t^2)(1+t)}{(1-qt)^2(1-t/q)^2}\\ &= \sum_{d\geqslant 0} t^d h_d\left[2\left(q+\frac{1}{q}-\epsilon\right)-1\right],\end{aligned}$$

where ϵ is treated as a variable within the plethystic brackets then evaluated at -1 outside of the brackets. Therefore,

$$\mathrm{Frob}(K[X_{\Xi_2}^{\times 2}])(X) = \sum_{d\geqslant 0} t^d \sum_{\lambda\vdash 2} s_\lambda\left[h_d\left[2\left(q+\frac{1}{q}-\epsilon\right)-1\right]\right] s_\lambda[X].$$

The Schur coefficients can then be calculated. For instance, since $s_{1,1} = p_1^2/2 - p_2/2$, the coefficient of $s_{1,1}$ is given by

$$\begin{aligned}\frac{1}{2}\sum_{d\geqslant 0} t^d &\left(h_d\left[2\left(q+\frac{1}{q}-\epsilon\right)-1\right]^2 - h_d\left[2\left(q^2+\frac{1}{q^2}-\epsilon\right)-1\right]\right)\\ &= t\left(q^2+2q+6+\frac{2}{q}+\frac{1}{q^2}\right)\\ &\qquad + t^2\left(q^4+6q^3+14q^2+12q+14+\frac{12}{q}+\frac{14}{q^2}+\frac{6}{q^3}+\frac{3}{q^4}\right)+\cdots.\end{aligned}$$

Example 5.10. Let $P = \Delta_m \subseteq \mathbb{R}^m$ be the m-dimensional unit simplex, defined as the convex hull of the set

$$\{0, e_1, e_2, \ldots, e_m\}.$$

By definition,

$$\begin{aligned}K[(\Delta_m)^{\times n}] &= K\left[t\prod_{i\in I} x_{i,j_i} : \quad I\subseteq\{1,\ldots,n\}, j_i\in\{1,\ldots,m\} \text{ for all } i\in I\right]\\ &\subseteq K[t, x_{1,1},\ldots,x_{n,m}].\end{aligned}$$

If the weight vector is $\boldsymbol{a}' = (1,2,\ldots,m)$, [1] shows that

$$L_{\Delta_m,i}^{(1,2,\ldots,m)} = \begin{bmatrix} i+m \\ m\end{bmatrix}_q,$$

where $\begin{bmatrix} i+m \\ m\end{bmatrix}_q$ is a q-binomial coefficient. Therefore, Corollary 5.8 implies that

$$\mathrm{Frob}(K[\Delta_m^{\times n}]) = \sum_{i\geqslant 0} t^i \sum_{\lambda\vdash n} s_\lambda\left[\begin{bmatrix} i+m \\ m\end{bmatrix}_q\right] s_\lambda[X].$$

We will refer to this example repeatedly in Subsection 5.5.

The following application of Theorem 3.8 gives a combinatorial description of the multiplicities of irreducible representations of $Z_k \wr S_n$ appearing in $K[P^{\times n}]$ for any P. We will use this for Theorem 5.25, where we give explicit combinatorially-defined statistics counting multiplicities in the quotient of $K[P^{\times n}]$ by a regular sequence when $P = \Delta_1 = [0,1] \subseteq \mathbb{R}$.

Theorem 5.11. *For $d\in\mathbb{N}$, let $F_d = [v^n : v \in dP\cap\mathbb{Z}^m]$ be a color rule with value $p(v) = |v|$ and weight $\rho(v) = q^{\boldsymbol{a}'\cdot v}$. Let χ^{F_d} be the character given by this color rule. Then,*

$$\chi^{K[P^{\times n}]} = \sum_{d\geqslant 0} t^d \chi^{K[P^{\times n}]_d} = \sum_{d\geqslant 0} t^d \chi^{F_d}.$$

In particular, for any $\vec{\gamma} \vdash_k n$,

$$\langle \chi^{K[P^{\times n}]}, \chi^{\vec{\gamma}}\rangle = \sum_{d\geqslant 0} t^d \sum_{T\in \mathrm{SSYT}_k(\vec{\gamma}, F_d)} \rho(T).$$

Proof. The first statement follows from follows from Proposition 5.6. The second statement follows from the first and Theorem 3.8. ☐

5.3. Interlude 1: Equivariant Ehrhart theory. *Equivariant Ehrhart theory* was introduced by Stapledon [23] with motivations from algebraic geometry and mirror symmetry. Its main object of study is the *equivariant Ehrhart series* which arises when a finite group G acts linearly on the lattice points of a G-invariant convex lattice polytope P. We recommend [9] as a reference, where the theory is further developed.

Corollary 5.8 may be seen as a statement in a *refined* version of this equivariant Ehrhart theory. The goal of this subsection is to make this relation to the existing literature explicit. The first step is to describe a setup that is equivalent to the one in [9].

Definition 5.12 (Equivariant Ehrhart series). *Assume the following setup:*

(1) $L \subseteq \mathbb{R}^N$ *is a lattice (i. e., a free abelian subgroup) of rank* N.
(2) G *is a finite group.*
(3) $\Pi' : G \to \mathrm{GL}(L)$ *is a lattice representation (i. e., a representation* $\Pi' : G \to \mathrm{GL}_N(\mathbb{R})$ *such that* L *is* $\Pi'(G)$*-invariant).*
(4) $P \subseteq \mathbb{R}^N$ *is a full-dimensional convex lattice polytope that is* $\Pi'(G)$*-invariant.*
(5) *For* $d \in \mathbb{N}$, $\Pi_{P,d} : G \to \mathrm{GL}(K(dP \cap L))$ *is the permutation representation carried by the vector space* $K(dP \cap L)$ *such that, for all* $g \in G$ *and* $x \in dP \cap L$,
$$\Pi_{P,d}(g)(x) = \Pi'(g)(x).$$
(6) $\Pi_P = \bigoplus_{d \in \mathbb{N}} \Pi_{P,d} : G \to \mathrm{GL}(K(C(P) \cap (L \times \mathbb{Z})))$. *In particular,* Π_P *is a graded representation.*

Then, the equivariant Ehrhart series *of* P *is the formal power series*
$$EE_P(t) = \sum_{d \geqslant 0} \chi(\Pi_P)_d t^d = \sum_{d \geqslant 0} \chi(\Pi_{P,d}) t^d \in R(G)[\![t]\!],$$
where $R(G)$ *is the ring of* K*-valued class functions of* G.

To refine this series we have to endow Π_P with a bigrading.

Proposition 5.13. *Assume the setup in Definition* 5.12*. Additionally:*

(7) Π' *is orthogonal with respect to the standard inner product of* $\mathbb{R}^N$ *(this choice of inner product can be made without loss of generality).*
(8) Π' *contains the trivial representation* Π_{triv} *of* G *with multiplicity* $m_{\mathrm{triv}} \geqslant 1$. *Equivalently,* $\Pi'(G)$ *acts trivially on a subspace* $V_{\mathrm{triv}} \subseteq \mathbb{R}^N$ *of dimension* $\geqslant 1$.
(9) $\boldsymbol{a} \in V_{\mathrm{triv}} \cap L$,
(10) $\mathbb{Z}_{\boldsymbol{a},L} \coloneqq \boldsymbol{a} \cdot L$, *where* $\cdot$ *denotes the standard inner product of* $\mathbb{R}^N$.

Then, we have a decomposition
$$\Pi_P = \bigoplus_{d \in \mathbb{N}} \bigoplus_{i \in \mathbb{Z}_{\boldsymbol{a},L}} \Pi_{P,d,i},$$
where, for each $i \in \mathbb{Z}_{\boldsymbol{a},L}$ *and* $g \in G$, $\Pi_{P,d,i}(g)$ *is the restriction of* $\Pi_{P,d}(g)$ *to the subspace* $K\{v \in dP \cap L : \ \boldsymbol{a} \cdot v = i\}$. *In particular,* Π_P *is a* bigraded representation.

Proof. It suffices to prove that $K\{v \in dP \cap L : \ \boldsymbol{a} \cdot v = i\}$ is a submodule of $K(dP \cap L)$. Let $g \in G, d \in \mathbb{N}, i \in \mathbb{Z}_{\boldsymbol{a},L}$, and $v \in dP \cap L$ such that $\boldsymbol{a} \cdot v = i$. Then,
$$\begin{aligned} & \boldsymbol{a} \cdot (\Pi_{P,d}(g)(v)) = \boldsymbol{a} \cdot (\Pi'(g)(v)) \\ (\Pi' \text{ is orthogonal}) \quad & = (\Pi'(g^{-1})(\boldsymbol{a})) \cdot v \\ (\boldsymbol{a} \in V_{\mathrm{triv}}) \quad & = \boldsymbol{a} \cdot v \\ & = i. \end{aligned}$$
This proves the claim. ∎

From now on, we write *REET* for the setup described in Definition 5.12 and Proposition 5.13. We may now define the refinement.

Definition 5.14 (Refined equivariant Ehrhart series). *Assume REET. Then, the* refined equivariant Ehrhart series *of P is the formal power series*

$$EE_P(t,q) = \sum_{d\geqslant 0}\sum_{i\in\mathbb{Z}_{\boldsymbol{a},L}} \chi(\Pi_P)_{d,i} t^d q^i = \sum_{d\geqslant 0}\sum_{i\in\mathbb{Z}_{\boldsymbol{a},L}} \chi(\Pi_{P,d,i}) t^d q^i \in R(G)[q,q^{-1}][\![t]\!].$$

Remark 5.15. Assume REET. We have

$$K[P] \cong K(C(P)\cap(L\times\mathbb{Z}))$$

as bigraded G-modules.

If $G = S_n$, we can relate the bigraded Ehrhart series to the bigraded Frobenius character:

Corollary 5.16. *Assume REET. Let $n \in \mathbb{N}_+$ and $G = S_n$. Then, the bigraded Frobenius characteristic satisfies*

$$\operatorname{Frob}(K[P]) = \operatorname{Frob}(K(C(P)\cap(L\times\mathbb{Z}))) = \frac{1}{n!}\sum_{\sigma\in S_n} EE_P(t,q)(\sigma) p_{\lambda(\sigma)}[X],$$

where $EE_P(t,q)(\sigma)$ is the refined equivariant Ehrhart series of P evaluated at σ.

We have achieved our goal, since Corollary 5.8 gives a more explicit expression for the character in Corollary 5.16 in the following particular case:

- $L = \mathbb{Z}^{mn}$ $(N = mn)$,
- $P = (P')^{\times n} \subseteq \mathbb{R}^{mn}$, where $P' \subseteq \mathbb{R}^m$ is a full-dimensional convex lattice polytope, and
- Π' decomposes (up to reordering coordinates so that P is $\Pi'(S_n)$-invariant) as $m\Pi_{\mathrm{def}}$, where Π_{def} is the defining representation of S_n. This implies that $\boldsymbol{a} = (\boldsymbol{a}')^{\times n}$, where $\boldsymbol{a}' \in \mathbb{Z}^m$.

Note that, even in this particular case, not all $\Pi'(S_n)$-invariant polytopes $P \subseteq \mathbb{R}^{mn}$ are Cartesian products $(P')^{\times n} \subseteq \mathbb{R}^{mn}$, where $P' \subseteq \mathbb{R}^m$. It would be very interesting to find refined analogues of other results and techniques in [9] for polytopes which are not Cartesian products. We are currently working in this direction.

Also note that if $k \neq 1$, the linear action we consider in Subsection 5.2 is *not* a permutation representation. However, most of the statements in this subsection still hold after slight modifications.

5.4. Interlude 2: Euler-Mahonian identities and the projective coinvariant algebra.

Before continuing, we would like to put the results of this section into context with the current literature. We will use Tables 1 and 2 as an aid. The graded rings $R_{m,n,k}$ in the first column will be defined in Subsection 5.5.

Braun and Olsen [5] call the formulas in Table 1 *Euler-Mahonian identities.* These formulas have appeared in many different contexts within algebraic combinatorics (see the discussion after Theorem 1.1 in [5]). In their paper, they study quotients of the form

$$R_{n,k} := K[(\Delta_1)^{\times n}]/(J_{k,n}),$$

where $K[(\Delta_1)^{\times n}]$ is the n-dimensional unit hypercube, and $J_{k,n}$ is an ideal generated by a regular sequence which is invariant under a linear action of the wreath product $Z_k \wr S_n$ on $K[(\Delta_1)^{\times n}]$. They define a grading on $K[(\Delta_1)^{\times n}]$ such that $J_{k,n}$ is homogeneous, and obtain a Gröbner basis for $R_{n,k}$. Then, they interpret the identities by Carlitz, Bagno, and Bagno, Biagioli in Table 1 as the bigraded Hilbert series of $R_{n,1}$

Graded ring	**Identity**	**Reference**	**Notes**
R_n	$\frac{1}{(1-q)^n} = \frac{\sum_{\sigma\in S_n} q^{\mathrm{maj}(\sigma)}}{\prod_{j=1}^{n}(1-q^j)}$	MacMahon [13]	maj is the major index.
$R_{1,n,1}$	$\sum_{i\geqslant 0}([i+1]_q)^n t^i = \frac{\sum_{\sigma\in S_n} t^{\mathrm{des}(\sigma)} q^{\mathrm{maj}(\sigma)}}{\prod_{j=0}^{n}(1-tq^j)}$	Carlitz [6]	des is the number of descents.
$R_{1,n,k}$	$\sum_{i\geqslant 0}([i+1]_q)^n t^i = \frac{\sum_{\sigma\in Z_k\wr S_n} t^{\mathrm{ndes}(\sigma)} q^{\mathrm{nmaj}(\sigma)}}{(1-t)\prod_{j=1}^{n}(1-t^k q^{kj})}$	Bagno [3]	ndes is the number of negative descents, nmaj is the negative major index.
	$= \frac{\sum_{\sigma\in Z_k\wr S_n} t^{\mathrm{fdes}(\sigma)} q^{\mathrm{fmaj}(\sigma)}}{(1-t)\prod_{j=1}^{n}(1-t^k q^{kj})}$	Bagno, Biagioli [4]	fdes is the number of flag descents, fmaj is the flag major index.

Table 1. Euler-Mahonian identities.

Taking the Hilbert series of the graded rings in the first column gives the numerators in the identities from the second column.

and $R_{n,k}$, respectively. Finally, in Section 6, they also hint at a relation between $R_{n,1}$ and the classical *coinvariant algebra*

$$R_n \coloneqq K[x_1, x_2, \dots, x_n]/K[x_1, x_2, \dots, x_n]_+^{S_n}, \tag{8}$$

where $K[x_1, x_2, \dots, x_n]_+^{S_n}$ is the ideal generated by all homogeneous invariants of positive degree. Nevertheless, the idea of using coinvariant algebras such as (8) to study multivariate statistics is much older, and has been used successfully in [2], for example, where it is credited to Ira Gessel.

Remark 5.17. The linear action studied in [5] is exactly the action we study in Subsection 5.2 in the special case when $P = (\Delta_1)^{\times n}$, and their bigrading is given, in our language, by the projective degree and the weight vector $\boldsymbol{a} = (1)^{\times n}$. Moreover, their ideals $J_{k,n}$ are exactly the ideals $I_{1,n,k}$ we will define after our Conjecture 5.18, and their quotients $R_{k,n}$ are the quotients $R_{1,k,n}$.

Since all graded rings in Table 1 admit the structure of graded S_n or $Z_k\wr S_n$-modules, it is natural to take their Frobenius characteristic. We call the resulting formulas *equivariant* Euler-Mahonian identities (see Table 2). For example, the identity for R_n is a classical result by Lusztig, Stanley. In a privately shared preprint [25], Szendrői notes that the relation between $R_{n,1}$ and R_n can be exploited to refine this identity, as in Table 2, row 2. For this reason, he refers to $R_{n,1}$ as the *projective coinvariant algebra*. This was, in fact, the starting point of our work in this section.

Raicu, Sam, Weyman [19] also prove the identity in Table 2, row 2 while studying modules supported on Chow varieties. In the proof of Theorem 5.25, we sketch a proof using classical results on q-binomial coefficients. It also follows from the identity in Table 2, row 3.

The last three rows of Table 2 are new results we will see later in this section.

Graded ring	**Identity**	**Reference**	**Notes**
R_n	$\sum_{\lambda\vdash n} s_\lambda[X] s_\lambda\left[\frac{1}{1-q}\right] = \frac{\sum_{\lambda\vdash n}\sum_{T\in\mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} s_\lambda[X]}{\prod_{j=1}^{n}(1-q^j)}$	Lusztig, Stanley	Equivariant version of MacMahon's identity.
$R_{1,n,1}$	$\sum_{i\geqslant 0} t^i \sum_{\lambda\vdash n} s_\lambda\left[[i+1]_q\right] s_\lambda[X] = \frac{\sum_{\lambda\vdash n}\sum_{T\in\mathrm{SYT}(\lambda)} t^{\mathrm{des}(T)} q^{\mathrm{comaj}(T)} s_\lambda[X]}{\prod_{j=0}^{n}(1-tq^j)}$	Szendrői [25], Raicu, Sam, Weyman [19], Theorem 5.25	Equivariant version of Carlitz's identity.
$R_{1,n,k}$	$\sum_{r\geqslant 0} t^r \sum_{\vec{\lambda}\vdash_k n} \prod_{i=0}^{k-1}\prod_{j=1}^{\ell(\lambda^i)} [r+1]_{q_{i,j}} \overline{P}_{\vec{\lambda}}[X]/Z_{\vec{\lambda}} = \frac{\sum_{\vec{\lambda}\vdash_k n}\sum_{T\in\mathrm{SYT}(\vec{\lambda})} t^{\mathrm{wdes}(T)} q^{\mathrm{wcomaj}(T)} s_{\vec{\lambda}}[X]}{(1-t)\prod_{j=1}^{n}(1-t^k q^{kj})}$	Theorem 5.25	Equivariant version of Bagno's and Bagno, Biagioli's identities.
$R_{m,n,1}$	$\sum_{i\geqslant 0} t^i \sum_{\lambda\vdash n} s_\lambda\left[\begin{bmatrix} i+m \\ m\end{bmatrix}_q\right] s_\lambda[X] = \frac{\sum_{\lambda\vdash n} h_{S_n,\lambda}(t,q) s_\lambda[X]}{\prod_{j=0}^{mn}(1-tq^j)}$	Prop. 5.21, 1.	$h_{S_n,\lambda}\in\mathbb{N}[t,q]$. The statistics are not necessarily Euler-Mahonian.
$R_{m,n,k}$	$\mathrm{Frob}\left(K[(\Delta_m)^{\times n}]\right) = \frac{\sum_{\vec{\lambda}\vdash_k n} h_{Z_k\wr S_n,\vec{\lambda}}(t,q) s_{\vec{\lambda}}[X]}{(1-t)\prod_{j=1}^{mn}(1-t^k q^{kj})}.$	Prop. 5.21, 2.	$h_{Z_k\wr S_n,\vec{\lambda}}\in\mathbb{N}[t,q]$ (conjecturally for large k). The statistics are not necessarily Euler-Mahonian.

Table 2. Equivariant Euler-Mahonian identities.

Taking the Frobenius characteristic of the graded S_n or $Z_k \wr S_n$-modules in the first column gives the numerators in the identities from the second column.

5.5. Products of projective spaces. We now resume the discussion we started in Subsection 5.2. The wreath product action and bigraded algebras we have defined so far exhibit very interesting behaviour. For example, it is evident, from the formulas in Proposition 5.6 or Corollary 5.8, that the characters of these algebras feature negative exponents of q. This is also the case in Example 5.9.

If $P = \Delta_m \subseteq \mathbb{R}^m$, the m-dimensional unit simplex (see Example 5.10), then the corresponding product variety is $(\mathbb{P}^m)^{\times n}$ with the Segre embedding. Recall that, by a theorem of Hochster [11], the affine semigroup algebra of a (normal) polytope is Cohen-Macaulay. Results in [1] and [5], along with computational experiments, suggest the following:

Conjecture 5.18. *Let $m,n,k \in \mathbb{N}_+, P = \Delta_m \subseteq \mathbb{R}^m$, and $\boldsymbol{a}' = (1,2,\dots,m)$. Then, the elements*

$$e^0 = t, \quad e^i_{m,n,k} = \sum_{\substack{x \in K[(\Delta_m)^{\times n}] \\ x \text{ monic monomial} \\ \operatorname{pdeg}(x)=1, \operatorname{cdeg}(x)=i}} x^k, i = 1,\dots,mn$$

form a regular sequence in $K[(\Delta_m)^{\times n}]$ of maximal length.

Define $I_{m,n,k}$ to be the ideal generated by the regular sequence in Conjecture 5.18. Note that $I_{m,n,k}$ is invariant under the linear action of $Z_k \wr S_n$ and homogeneous with respect to the bigrading. Define

$$R_{m,n,k} \coloneqq K[P^{\times n}]/I_{m,n,k}$$

for the corresponding quotient. If Conjecture 5.18 holds, $R_{m,n,k}$ is a finite-dimensional K-vector space.

The statement and idea of the proof of point 2 in the following proposition was hinted to us by Balázs Szendrői.

Proposition 5.19. *Conjecture 5.18 is true in the following cases:*

(1) $k = m = 1$ *and* $n \in \mathbb{N}_+$;
(2) $k = 1$ *and* $m,n \in \mathbb{N}_+$;
(3) $m = 1$ *and* $k,n \in \mathbb{N}_+$;
(4) i. $m = 2$ *and* $n,k \leqslant 10$;
  ii. $m = 3$, $n = 2$ *and* $k \leqslant 7$;
  iii. $m = 4$, $n = 2$ *and* $k \leqslant 3$.

Proof.

(1) For a geometric proof, see [1], Proposition 2.7. Alternatively, this point follows from point 3, and we will see it is equivalent to point 2.
(2) We will give an isomorphism

$$\varphi : K[(\Delta_1)^{\times mn}]^{S_{(m^n)}} \xrightarrow{\sim} K[(\Delta_m)^{\times n}].$$

Here

$$S_{(m^n)} = S_{I_1} \times S_{I_2} \times \cdots \times S_{I_n}$$

is the standard *Young subgroup* of S_{mn}, where, for $i \in \{1,\dots,n\}$,

$$I_i \coloneqq \{(i-1)m+1, (i-1)m+2, \dots, im\}.$$

We now describe the isomorphism φ. The relevant algebras are

$$\begin{aligned} K[(\Delta_1)^{\times mn}]^{S_{(m^n)}} &= K\left[t\prod_{i\in I} x_{i,1} : \quad I \subseteq \{1,\dots,mn\}\right]^{S_{(m^n)}} \\ &\subseteq K\left[t\prod_{i\in I} x_{i,1} : \quad I \subseteq \{1,\dots,mn\}\right] \\ &\subseteq K[t, x_{1,1}, x_{2,1}, \dots, x_{mn,1}] \end{aligned}$$

and

$$\begin{aligned} K[(\Delta_m)^{\times n}] &= K\left[t\prod_{i\in I} x_{i,j_i} : \quad I \subseteq \{1,\dots,n\}, j_i \in \{1,\dots,m\} \text{ for all } i \in I\right] \\ &\subseteq K[t, x_{1,1}, x_{1,2}, \dots, x_{1,m}, x_{2,1}, x_{2,2}, \dots, x_{2,m}, \dots, x_{n,m}] \end{aligned}$$

(see Example 5.10). We then see that $K[(\Delta_1)^{\times mn}]^{S_{(m^n)}}$ is generated as a K-algebra by elements of the form

$$x(j_1, j_2, \dots, j_n) = t\left(\prod_{i=1}^{n} \sum_{\substack{J_i \subseteq I_i \\ |J_i|=j_i}} \prod_{k\in J_i} x_{k,1}\right),$$

where $0 \leqslant j_i \leqslant |I_i| = m$ for all $i = 1, \dots, n$. The isomorphism φ is now clear:

$$x(j_1, j_2 \dots, j_n) \mapsto t\prod_{i=1}^{n} x_{i,j_i}.$$

It is not hard, although a bit tedious, to show that the relations are the same on both sides.

By point 1, there is a regular sequence of invariants in $K[(\Delta_1)^{\times mn}]$:

$$(t, t(x_{1,1}+x_{2,1}+\cdots+x_{mn,1}), t(x_{1,1}x_{2,1}+x_{1,1}x_{3,1}+\cdots x_{mn-1,1}x_{mn,1}), \dots, tx_{1,1}x_{2,1}\cdots x_{mn,1}).$$

Since $S_{(m^n)} \subseteq S_{mn}$, this regular sequence is already in $K[(\Delta_1)^{\times mn}]^{S_{(m^n)}}$. Explicitly, for $i = 1, \dots, mn$, the elements of this regular sequence can be written as

$$e^i_{1,mn,1} = \sum_{\lambda\in C_{m,n}(i)} x(\lambda),$$

where $C_{m,n}(i)$ is the set of weak compositions of i whose Young diagram is contained in the rectangle $[0, m] \times [0, n]$. Then, for $i = 1, \dots, mn$,

$$\varphi(e^i_{1,mn,1}) = e^i_{m,n,1}.$$

The claim follows.

(3) This follows from [5], proof of Theorem 4.10.

(4) These cases were verified using Macaulay2. □

We see that the inclusion $I_{m,n,k} \subseteq K[(\Delta_m)^{\times n}]_+^{Z_k \wr S_n}$ is proper when $m \geqslant 2$, as the following example shows.

Example 5.20. Let $m = n = 2$. By Example 5.10,

$$K[(\Delta_2)^{\times 2}] = K\begin{bmatrix} tx_{2,2} & tx_{1,1}x_{2,2} & tx_{1,2}x_{2,2} \\ tx_{2,1} & tx_{1,1}x_{2,1} & tx_{1,2}x_{2,1} \\ t & tx_{1,1} & tx_{1,2} \end{bmatrix} \subseteq K[t, x_{1,1}, \dots, x_{2,2}].$$

Note that S_2 acts on the generators of $K[(\Delta_2)^{\times 2}]$ by reflecting along the diagonal going from southwest to northeast.

The regular sequence for $k = 1$ is

$$(t,\quad tx_{1,1} + tx_{2,1},\quad tx_{1,2} + tx_{1,1}x_{2,1} + tx_{2,2},\quad tx_{1,2}x_{2,1} + tx_{1,1}x_{2,2},\quad tx_{1,2}x_{2,2}).$$

We see that the monomial $tx_{1,1}x_{2,1} \in K[(\Delta_2)^{\times 2}]_+^{S_2}$ but $tx_{1,1}x_{2,1} \notin I_{2,2,1}$.

Even though $K[(\Delta_m)^{\times n}]$ may have too many invariants, studying $R_{m,n,k}$ is still very fruitful. Indeed, if R is a Cohen-Macaulay (bi)graded K-algebra, and $e_0, e_1, \dots, e_N$ is a homogeneous regular sequence of maximal length, it is a well-known fact that

$$R \cong R/(e_0, e_1, \dots, e_N) \otimes K[e_0, e_1, \dots, e_N]$$

as (bi)graded K-vector spaces. In particular, if Conjecture 5.18 holds,

$$K[(\Delta_m)^{\times n}] \cong R_{m,n,k} \otimes K[e^0, e^1_{m,n,k}, e^2_{m,n,k}, \dots, e^{mn}_{m,n,k}]$$

as (bi)graded K-vector spaces. This can be seen as an analogue of Chevalley's theorem for reflection groups. Since $I_{m,n,k}$ is invariant and homogeneous, this is also an isomorphism of bigraded $Z_k \wr S_n$-modules. The Frobenius characteristic then satisfies

$$\operatorname{Frob}(K[(\Delta_m)^{\times n}]) = \frac{\operatorname{Frob}(R_{m,n,k})}{(1-t)\prod_{i=1}^{mn}(1-t^k q^{ki})}. \tag{9}$$

We will use this to give a geometric interpretation to equivariant Euler-Mahonian identities. This discussion, together with Corollary 5.8, also imply the following:

Proposition 5.21. *Let $P = \Delta_m \subseteq \mathbb{R}^m$ and $\boldsymbol{a}' = (1, 2, \ldots, m)$. Then:*

(1) *If $k = 1$ and $m, n \in \mathbb{N}_+$, there exists a polynomial $h_{S_n,\lambda}(t, q) \in \mathbb{N}[t, q]$ such that the Frobenius characteristic of $K[(\Delta_m)^{\times n}]$ is given by*

$$\sum_{i \geqslant 0} t^i \sum_{\lambda \vdash n} s_\lambda \left[\begin{bmatrix} i+m \\ m \end{bmatrix}_q \right] s_\lambda[X] = \frac{\sum_{\lambda \vdash n} h_{S_n,\lambda}(t,q) s_\lambda[X]}{\prod_{j=0}^{mn}(1-tq^j)}.$$

(2) *In general, if Conjecture 5.18 is true, there exists a polynomial $h_{Z_k \wr S_n,\vec{\lambda}}(t, q) \in \mathbb{N}[t, q]$ such that*

$$\operatorname{Frob}\left(K[(\Delta_m)^{\times n}]\right) = \frac{\sum_{\vec{\lambda} \vdash_k n} h_{Z_k \wr S_n,\vec{\lambda}}(t,q) s_{\vec{\lambda}}(X)}{(1-t)\prod_{j=1}^{mn}(1-t^k q^{kj})}.$$

If $m = 1$, we can use our theory of color rules to give an explicit combinatorial formula for $h_{Z_k \wr S_n,\vec{\lambda}}(t, q)$. The case $m = 1, k = 1$ will already be very interesting. To proceed, we must define the statistics involved in this formula.

Definition 5.22 (Wreath statistics for tableaux). *Fix a sequence of partitions $\vec{\gamma} \vdash_k n$ (drawn corner to corner as in Figure 1). For $T \in \operatorname{SYT}(\vec{\gamma})$, let $W(T)$ be the tableau one gets from T by the following procedure: replace the label i in T by a_i, where*

(1) *if the label 1 is in γ^r, we set $a_1 = r$. Then,*
(2) *suppose we have just replaced the label i in T, located in γ^r. Suppose $i + 1$ is in γ^s.*
   (a) *If $i + 1$ is strictly to the right of i, then we set $a_{i+1} = a_i + s - r$.*
   (b) *If $i + 1$ is weakly to the left of i, then set $a_{i+1} = a_i + k + s - r$.*

Now define $\operatorname{wcomaj}(T) = |W(T)|$ *and* $\operatorname{wdes}(T) = \max(W(T))$ *is the maximal label in $W(T)$. See Figure 7 for an example.*

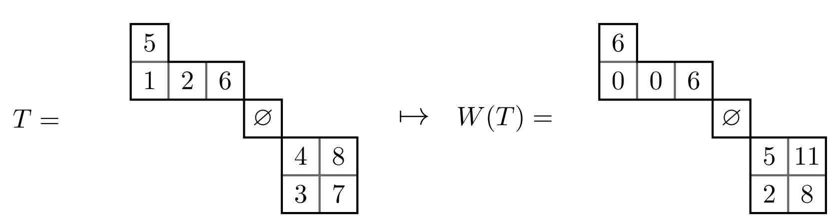


Figure 7. The standard tableau T and image $W(T)$ give $\operatorname{wcomaj}(T) = |W(T)| = 0 + 0 + 2 + 5 + 6 + 6 + 8 + 11$ and $\operatorname{wdes}(T) = \max(W(T)) = 11$.

This notion of wreath comajor index and number of wreath descents can also be defined for elements in $Z_k \wr S_n$.

Definition 5.23 (Wreath statistics for colored permutations). *The wreath descent set of an element $\sigma = u_{a_1}\sigma_1 \cdots u_{a_n}\sigma_n$ is an ordered list of sets $\operatorname{Des}(\sigma) = (D^1, \ldots, D^k)$ constructed in the following way:*

(1) *If* $a_{i+1} - a_i = r \mod k$*, with* $r \in \{1, \dots, k-1\}$*, then we say* i *is an* r*-descent and* $i \in D^r$*.*
(2) *If* $a_i = a_{i+1}$ *and* $\sigma_i > \sigma_{i+1}$*, then we say that* i *is a* k*-descent, and* $i \in D^k$*.*

The wreath comajor index and the number of wreath descents is given by

$$\mathrm{wcomaj}(\sigma) = \sum_{r=0}^{k} \sum_{i \in D^r} r \cdot (n-i) \qquad \text{and} \qquad \mathrm{wdes}(\sigma) = \sum_{r=0}^{k} |D^r| \cdot r.$$

Remark 5.24. One can define an RSK algorithm

$$\sigma \in Z_k \wr S_n \mapsto (P(\sigma), Q(\sigma)) \in \cup_{\vec{\gamma} \vdash_k n} \mathrm{SYT}(\vec{\gamma})^{\times 2}$$

by taking $\sigma = u_{a_1}\sigma_1 \cdots u_{a_n}\sigma_n$ and applying the usual bumping algorithm, only that the index σ_i is inserted into γ^{a_i}. The recording tableau $Q(\sigma)$ then has the property that $\mathrm{wcomaj}(\sigma) = \mathrm{wcomaj}(Q(\sigma))$ and $\mathrm{wdes}(\sigma) = \mathrm{wdes}(Q(\sigma))$.

Theorem 5.25. *Let* $P = \Delta_1 = [0,1] \subseteq \mathbb{R}^1$ *and* $\boldsymbol{a}' = (1)$*. Then, for all* $k, n \in \mathbb{N}_+$*, the bigraded Frobenius characteristic of* $K[(\Delta_1)^{\times n}]$ *is given by*
(10)

$$\sum_{r \geqslant 0} t^r \sum_{\vec{\lambda} \vdash_k n} \prod_{i=0}^{k-1} \prod_{j=1}^{\ell(\lambda^i)} [r+1]_{q_{i,j}} \overline{P}_{\vec{\lambda}}[X]/Z_{\vec{\lambda}} = \frac{\sum_{\vec{\lambda} \vdash_k n} \sum_{T \in \mathrm{SYT}(\vec{\lambda})} t^{\mathrm{wdes}(T)} q^{\mathrm{wcomaj}(T)} s_{\vec{\lambda}}[X]}{(1-t)\prod_{j=1}^{n}(1-t^k q^{kj})},$$

where $q_{i,j} = u_i q^{\lambda(\sigma)^i_j}$ *and* $[n]_q = 1 + q + \cdots + q^{n-1}$ *is the* q*-integer.*

In particular, for $k = 1$*,*

$$\sum_{i \geqslant 0} t^i \sum_{\lambda \vdash n} s_\lambda \left[[i+1]_q\right] s_\lambda[X] = \frac{\sum_{\lambda \vdash n} \sum_{T \in \mathrm{SYT}(\lambda)} t^{\mathrm{des}(T)} q^{\mathrm{comaj}(T)} s_\lambda[X]}{\prod_{j=0}^{n}(1-tq^j)}. \tag{11}$$

These identities can be interpreted geometrically in view of Equation 9*.*

Proof. We give the proof for the general case in Subsection 5.6.

However, the consequence for $k=1$ can be proven on its own with more classical methods (see Subsection 5.4 for a historical discussion). Indeed, note that

$$L^{(1)}_{\Delta_1,i}(q) = 1 + q + \cdots + q^i = \frac{1-q^{i+1}}{1-q} = [i+1]_q.$$

Then, the left-hand side of Equation (11) is $\mathrm{Frob}((\Delta_1)^{\times n})$ in Corollary 5.8. To obtain the right hand side, use [22], Proposition 7.9.12, and standard properties of the q-binomial coefficient. □

Remark 5.26. The wreath statistics in Theorem 5.25 are related to the negative and flag statistics in [3], [4], [5], since after projecting the equivariant formulas to the non-equivariant ones we see that their joint distribution must be the same.

5.6. Proof of Theorem 5.25. By Proposition 5.6, the left-hand side of Equation 10 gives

$$\mathrm{Frob}(\chi^{K[(\Delta_1)^{\times n}]}) = \frac{1}{n!k^n} \sum_{\sigma \in Z_k \wr S_n} \chi^{K[(\Delta_1)^{\times n}]}(\sigma) \overline{P}_{\vec{\lambda}(\sigma)}[X].$$

Since $d\Delta_1 \cap \mathbb{Z} = \{0, \dots, d\}$, by Theorem 5.11, we know that

$$\frac{1}{n!k^n} \sum_{\sigma \in Z_k \wr S_n} \chi^{K[(\Delta_1)^{\times n}]}(\sigma) \overline{P}_{\vec{\lambda}(\sigma)}[X] = \sum_{\vec{\gamma} \vdash_k n} s_{\vec{\gamma}}[X] \sum_{d \geqslant 0} t^d \sum_{T \in \mathrm{SSYT}_k(\vec{\gamma}, \mathsf{N}_{n,d})} \rho(T),$$

where $\mathrm{SSYT}_k(\vec{\gamma}, \mathsf{N}_{n,d})$ is the set of semistandard tableaux with entries no greater than d, and i appears in γ^j only if $i = j \mod k$. These tableaux have a q-weight given by the sum of elements in T, $\rho(T) = |T|$. Let

$$\max T = \max\{\beta : \ \beta \text{ appears in } T\}.$$

We first note that

$$\sum_{d\geqslant 0} t^d \sum_{T\in \mathrm{SSYT}_k(\vec{\gamma},\mathsf{N}_{n,d})} q^{|T|} = \frac{1}{1-t} \sum_{T\in\mathrm{SSYT}_k(\vec{\gamma},\mathbb{N})} t^{\max T} q^{|T|}.$$

Let $R_n = \sum_{T\in\mathrm{SSYT}_k(\vec{\gamma},\mathbb{N})} t^{\max T} q^{|T|}$. For $i = n, n-1, \ldots, 1$, we set

$$R_{i-1} = (1 - t^k q^{ki}) R_i. \tag{12}$$

We will construct a sequence of sets of tableaux U_i for which

$$R_i = \sum_{T\in U_i} t^{\max T} q^{|T|}.$$

To this end, for each i, we define an injection

$$\varphi_i : U_i \to U_i$$

with the property that

- $|\varphi(T)| = ki + |T|$, and
- $\max \varphi(T) = k + \max T$,

meaning that Equation 12 is satisfied if we set

$$U_{i-1} = U_i \smallsetminus \varphi_i(U_i).$$

In the end, we will have found that

$$\prod_{i=1}^{n} (1 - t^k q^{ki}) R_n = \sum_{T\in U_0} t^{\max T} q^{|T|}.$$

For a given semistandard tableau T, we let $\mathrm{stand}(T)$ be the standardization of T as described in Section 2. Let $c_i = c_i(T)$ be the column in which the label i is located in $\mathrm{stand}(T)$; and let $a_i = a_i(T)$ be the corresponding label in T. In this notation, columns 1 to γ_1^1 are columns of γ^1; columns $\gamma_1^1 + 1, \ldots, \gamma_1^1 + \gamma_1^2$ are columns of γ^2; and so on. We will then say that the column c_i is in γ^j if the label i of $\mathrm{stand}(T)$ is in γ^j.

Define $\varphi_i(T)$ to be the resulting tableau one gets by adding k to the labels in T corresponding to the labels $n, n-1, \ldots, n-i+1$ of $\mathrm{stand}(T)$. Note that $\varphi(U_n)$ is the set of all tableaux in U_n whose minimal label is at least k. This means U_{n-1} consists of all tableaux in U_n with the following property:

$$\text{if } c_1 \text{ is in } \gamma^i, \text{ then } a_i = i.$$

In general, the set U_{n-j-1} is attained from U_{n-j} by adding an additional condition: suppose c_j is in γ^r and c_{j+1} is in γ^s.

- If $c_j < c_{j+1}$ and $r = s$, then $a_{j+1} = a_j + s - r$.
- If $c_j \geqslant c_{j+1}$, then $a_{j+1} = a_j + k + s - r$.

The set U_0 satisfies these two conditions for each label j in $\mathrm{stand}(T)$. Thus, each $S \in \mathrm{SYT}(\vec{\gamma})$ corresponds to a unique element $W(T) \in U_0$, where the map W is defined in Definition 5.22.

We have found that, for given n, k and $\vec{\gamma} \vdash_k n$,

$$(1-t)\left(\prod_{j=1}^{n} 1 - t^k q^{kj}\right) \sum_{d\geqslant 0} t^d \sum_{T\in\mathrm{SSYT}_k(\vec{\gamma},\mathsf{N}_{n,d})} q^{|T|} t^{\max(T)} = \sum_{T\in\mathrm{SYT}(\vec{\gamma})} q^{\mathrm{wcomaj}(T)} t^{\mathrm{wdes}(T)}.$$

This proves the result.

6. Proof of the main result

Here, we prove the main result regarding class functions computed by color rules, Theorem 3.8.

6.1. The combinatorial objects. We start with a class function χ of $Z_k \wr S_n$ computed by a color rule $F = [f_1^{m_1}, f_2^{m_2}, \dots]$ with value and weight functions p and ρ, as in Definition 3.3. Let $\gamma \vdash_k n$. The multiplicity of $\chi^{\vec{\gamma}}$ in χ is given by the inner product

$$\langle \chi, \chi^{\vec{\gamma}} \rangle = \frac{1}{|Z_k \wr S_n|} \sum_{\sigma \in Z_k \wr S_n} \chi(\sigma) \overline{\chi}^{\vec{\gamma}}(\sigma).$$

We will show that

$$\sum_{\sigma \in Z_k \wr S_n} \chi(\sigma) \overline{\chi}^{\vec{\gamma}}(\sigma) = n! k^n \sum_{T \in \mathrm{SSYT}_k(\vec{\gamma}, F)} \rho(T),$$

as in Definition 3.7. We start by constructing a set of combinatorial objects $\mathcal{P}_{\vec{\gamma},F}$ and a weight function w for which

$$\sum_{P \in \mathcal{P}_{\vec{\gamma},F}} \mathrm{weight}(P) = \sum_{\sigma \in Z_k \wr S_n} \chi(\sigma) \overline{\chi}^{\vec{\gamma}}(\sigma).$$

The set $\mathcal{P}_{\vec{\gamma},F}$ is created by the following procedure. One can follow Example 6.2 to see how the steps create these objects:

(1) Write σ in cycle notation.
(2) Over each cycle, choose a color f_i and place it over each index in the cycle so that the resulting coloring satisfies the color rule.
(3) Rearrange the cycles first in increasing order by color, so that cycles colored by f_j appear to the right of cycles colored by f_i when $i < j$. Then, for cycles with the same color, rearrange in decreasing cycle order (as described in Section 2).
Let $(c_1, \dots, c_l)$ be the cycles from *right* to *left* with lengths given by $\alpha = (\alpha_1, \dots, \alpha_l)$ and C-types given by $a = (a_1, \dots, a_l)$; and let $(g_1, \dots, g_l)$ be the colors over these respective cycles.
(4) Pick a rim hook tableau T of shape $\vec{\gamma}$ and type (α, a).
(5) If $\zeta_1, \dots, \zeta_l$ are the rim hooks in T in the order in which they were placed, write over ζ_j by placing the first index of c_j and its root of unity in the first cell of ζ_j; place the second index of c_j and its root of unity in the second cell of ζ_j; and so on. Similarly, include the colors so that each cell in ζ_j also contains the color g_j. Let P be the resulting tableau.
(6) Suppose $u_{r_i} i$ appears in γ^{b_i} with color f_{s_i} over it. Let $d(T)$ be the total number of South steps in all of the rim hooks appearing in T. Define

$$\mathrm{weight}(P) := (-1)^{d(T)} \prod_{i=1}^{n} (u_{r_i})^{p(f_{s_i}) - b_i} \rho(f_{s_i}).$$

We call $\mathcal{P}_{\vec{\gamma},F}$ the collection of rim hook tableau fillings with colors in F. By construction, we have the following.

Proposition 6.1. *For any $\vec{\gamma} \vdash_k n$ and color rule F,*

$$\sum_{P \in \mathcal{P}_{\vec{\gamma},F}} \mathrm{weight}(P) = \sum_{\sigma \in Z_k \wr S_n} \chi(\sigma) \overline{\chi}^{\vec{\gamma}}(\sigma).$$

Example 6.2. As an example of this construction, for the case when $n = 8$, $k = 3$, and $\vec{\gamma} = ((2,1),(2,2),(1))$, consider the element

$$(1)(2\ u_1 6\ 4)(u_2 3)(u_1 5)(u_2 7\ u_1 8),$$

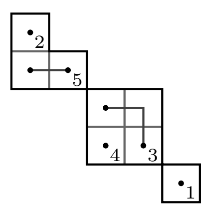
We choose where to place the colors from the color rule $[f_1^6, f_2, f_3]$. One such possibility is

$$\begin{array}{ccccc} f_3 & f_1\ \ f_1\ \ f_1 & f_1 & f_2 & f_1\ \ \ f_1 \\ (1) & (2\ u_1 6\ 4) & (u_2 3) & (u_1 5) & (u_2 7\ u_1 8) \end{array}.$$

We then rearrange the cycles so colors are increasing left to right, and the minimal indices in cycles with the same color are decreasing from left to right:

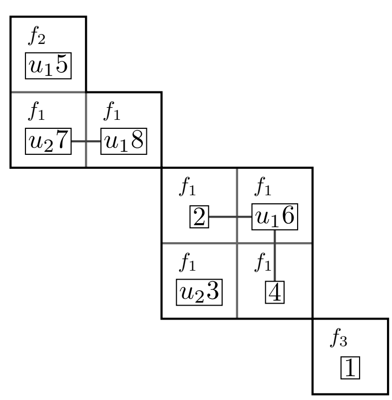
$$\begin{array}{ccccc} f_1\ \ \ f_1 & f_1 & f_1\ \ f_1\ \ f_1 & f_2 & f_3 \\ (u_2 7\ u_1 8) & (u_2 3) & (2\ u_1 6\ 4) & (u_1 5) & (1) \end{array}.$$

We now pick a rim hook tableau of shape $((2,1),(2,2),(1))$ whose rim hooks have lengths given by these cycles read right to left: $(1,1,3,1,2)$. Say we pick



Finish by filling in the shape by the entries in the cycles to obtain

$P =$



We calculate the weights as follows:

$$\begin{aligned}\text{weight}(P) = (-1)^1 &(u_0^{p(f_3)-2}\rho(f_3))(u_0^{p(f_1)-1}\rho(f_1))(u_2^{p(f_1)-1}\rho(f_1)) \\ &\times (u_0^{p(f_1)-1}\rho(f_1))(u_1^{p(f_2)-0}\rho(f_2))(u_1^{p(f_1)-1}\rho(f_1))(u_2^{p(f_1)-0}\rho(f_1))(u_1^{p(f_1)-0}\rho(f_1)).\end{aligned}$$

To make it easier to follow, we first start with the sign given by the number of South steps in the rim hooks. There is only one, giving $(-1)^1$. Then, for each $i = 1,\dots,8$ (in this order), we write $u_{r_i}^{p(f_{s_i}-b_i)}\rho(f_{s_i})$, where $u_{r_i} i$ appears in γ^{b_i} with color f_{s_i}. For instance, since $u_1 6$ appears in γ^1 with color f_1, we get $u_1^{p(f_1)-1}\rho(f_1)$.

6.2. A weight-preserving, sign-reversing involution and a map of order k. To evaluate the first sum in Proposition 6.1, we must eliminate all non-integral and non-positive weights. To do this, we must first use a weight-preserving, sign-reversing involution to eliminate a certain possibility for our objects' fillings. We will then employ another fact of the cyclic group to eliminate a broader range of possibilities. To facilitate our discussion, we will say that a cell is disconnected from another cell if a rim hook between the two cells is severed, and we will say that a cell is connected to another cell if a rim hook from one cell is extended to the other.

We first define a sign-reversing involution ψ on $\mathcal{P}_{\vec{\gamma},F}$. Examples of the operation ψ can be found in Figures 8 and 9. For a given $P \in \mathcal{P}_{\vec{\gamma},F}$, define $\psi(P)$ by the following process:

(1) Scan along the southernmost row, from left to right, until one finds the first cell c such that one of two things hold:
  (a) The cell above c is in the same rim hook as c or
  (b) The cell above c contains the end of a rim hook with the same color in c.
(2) If no such cell is found in this row, move inductively to the next row up.
(3) If no such cell is found in any row, leave the object fixed.

Suppose we now have such a cell c, then for the given cases, we do the following:

Case (a): If c is in the same rim hook as the cell above, then
1. Disconnect c from the cell above it.
2. Suppose there is a rim hook ζ which ends one cell west of c and has the same color as c. If the index in c is larger than the smallest index in ζ, then connect c with ζ.
3. Let ξ be the rim hook which now contains c. Read the indices in ξ from right to left, looking for the first index i in ξ which is smaller than every index in ξ on its left. If such an index is found, disconnect the index from ξ from the left. Iterate this procedure with the remaining portion of ξ. This step ensures that each rim hook begins with its smallest index.

Case (b): If c is one cell south the end of a rim hook containing the same color, then
1. If c is in the same rim hook as the cell to its west, disconnect these two cells.
2. Connect c with the cell above c.
3. Let ζ be the rim hook which now contains c. Suppose there is a rim hook ξ which begins one cell east of the end of ζ. If the smallest index in ζ is smaller than the smallest index in ξ, connect ζ and ξ. Iterate this process until either there is no rim hook ξ that begins one cell east of the rim hook containing c or the smallest index in ζ is larger than the smallest index in ξ.

As an example of this operation, consider the tableau filling in Example 6.2. We see in Figure 8 that the first cell c that ψ would locate is the cell containing 4. We then disconnect from above, and connect to the cell on the left, since $3 < 4$. Figure 9 gives a more drastic example.

Proposition 6.3. *The map* $\psi : \mathcal{P}_{\vec{\gamma},F} \to \mathcal{P}_{\vec{\gamma},F}$ *is well-defined. Furthermore,* ψ *is an involution, and if* P *is not fixed, then* $\operatorname{weight}(P) = -\operatorname{weight}(\psi(P))$.

Proof. The act of connecting and disconnecting rim hooks ensures that the cycles are inserted in the proper order, meaning that we indeed get a map from $\mathcal{P}_{\vec{\gamma},F}$ to itself. Furthermore, if c is the cell located by the map ψ, then Case (a) is sent to Case (b) at c, and Case (b) is sent to Case (a) at c. This ensures that when we apply ψ to $\psi(P)$, we locate the same cell c to perform the required operation. So ψ is an involution.

Secondly, no color or root of unity is changed, so the only difference between $\psi(P)$ and P will be a sign given by the new underlying rim hook tableau.

Lastly, at most one South step in the rim hooks will be deleted or added by the operation. This is because there cannot be any South steps below c, since otherwise ψ

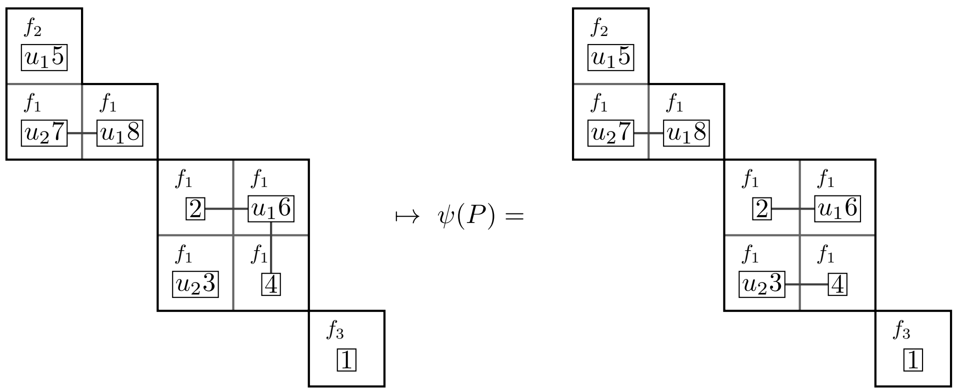


FIGURE 8. The involution ψ on the object in Example 6.2.

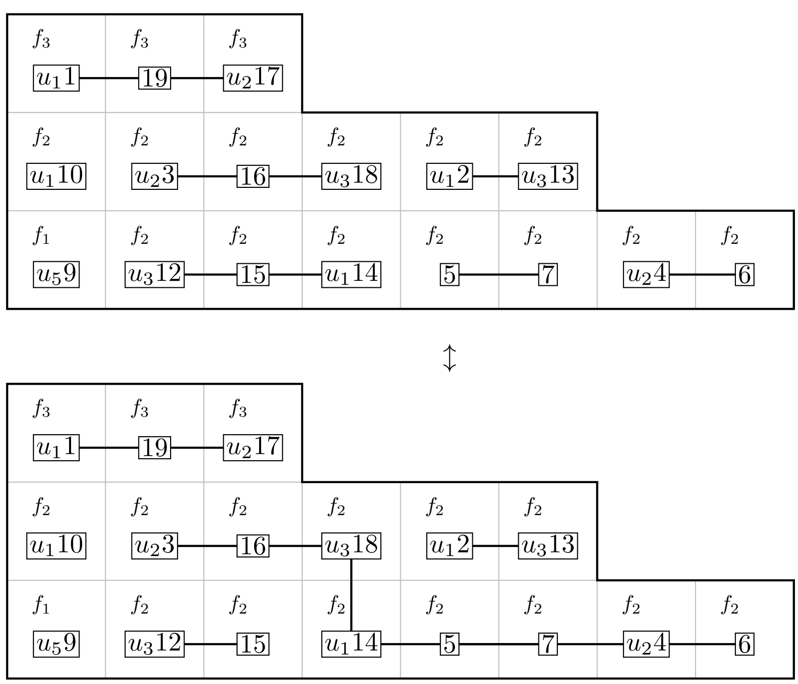


FIGURE 9. A larger example of the involution ψ.

would have first located a cell below c. So Case (a), point 3 will never delete a South step. For the same reason, Case (b), point 3 never encounters the end of a rim hook to the right of c that ends above a cell with the same color (otherwise ψ would locate this cell first instead of c). If P is not fixed by ψ, precisely one South step in a rim hook is either added or removed, which reverses the sign. □

REMARK 6.4. Let $\mathcal{P}'_{\vec{\gamma},F}$ be the fixed points of ψ. We first note that the underlying rim hook of any $P \in \mathcal{P}'_{\vec{\gamma},F}$ cannot have a South step, otherwise we would find a cell satisfying Case (a) in ψ. Secondly, we see that we cannot have two cells, one on top of the other, with the same color, since either we would have a South step in a rim hook, or there would be the end of a rim hook falling into Case (b) of ψ. This means that the underlying colors form a semistandard tableau.

We will now use the following fact:

Lemma 6.5. *For any s and u_r, we have*

$$u_r^s + (u_1 u_r)^s + \cdots + (u_1^{k-1} u_r)^s = k\ \delta_{(s=0 \mod k)}$$

Proof. If $s = 0 \mod k$, then we get $u_r^s = 1$ for any r, and the sum reduces to k. Otherwise, we can rewrite the sum as

$$u_r^s (u_1^0 + u_1^s + \cdots + u_1^{(k-1)s}) = u_r^s \frac{1-(u_1^s)^k}{1-u_1^s} = 0.$$ □

Proposition 6.6. *There is a map $\psi' : \mathcal{P}'_{\vec{\gamma},F} \to \mathcal{P}'_{\vec{\gamma},F}$ of order k such that if P is not fixed, then P, $\psi'(P)$,..., ${\psi'}^{k-1}(P)$ are all distinct, and* $\mathrm{weight}(P) + \mathrm{weight}(\psi'(P)) + \cdots + \mathrm{weight}({\psi'}^{k-1}(P)) = 0$.

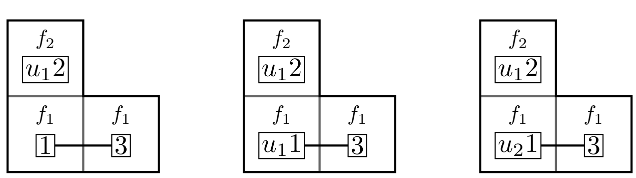
Proof. Given $P \in \mathcal{P}'_{\vec{\gamma},F}$, scan from bottom to top, left to right, for the first cell c in γ^r containing a color f such that $p(f) \neq r \mod k$. Suppose this cell contains $u_i j$. Let $\psi'(P)$ be the tableau filling obtained by replacing u_i in c by $u_1 u_i$. This clearly has order k since $u_1^k = 1$.

The weight of the image of this map is given by $\mathrm{weight}(\psi'(P)) = u_1^{p(f)-r}\,\mathrm{weight}(P)$. So if $p(f) - r \neq 0 \mod k$,

$$\begin{aligned}\mathrm{weight}(P) + \mathrm{weight}(\psi'(P)) + \cdots + \mathrm{weight}({\psi'}^{k-1}(P))& \\ = \mathrm{weight}(P)(1 + u_1^{p(f)-r} + u_2^{p(f)-r} + \cdots + u_{k-1}^{p(f)-r}) &= 0\end{aligned}$$

by Lemma 6.5. □

For a simple example of an orbit of ψ', we have Figure 10. The sum of weights is displayed below each tableau filling.



$u_2 +$ $u_2 u_1 +$ $u_2 u_1^2 = u_2 + 1 + u_1 = 0.$

Figure 10. An orbit of ψ' when $\vec{\lambda} = ((2,1),(0),(0))$, $F = [f_1^2, f_2]$, and $p(f_i) = i$.

6.3. The final fixed points. Let $\mathcal{T}_{\vec{\gamma},F}$ be the fixed points of ψ'. For $P \in \mathcal{T}_{\vec{\gamma},F}$, let $S(P)$ be the tableau obtained by removing everything except for the colors. From Remark 6.4, since $\mathcal{T}_{\vec{\gamma},F} \subseteq \mathcal{P}'_{\vec{\gamma},F}$, we must have that $S(P)$ is a semistandard tableau in the colors of F. Since P is fixed by ψ', we must have that if the color f_i appears in γ^r, then $p(f_i) = r \mod k$. All together, this means that $S(P) \in \mathrm{SSYT}_k(\vec{\gamma}, F)$. We also have that $\mathrm{weight}(P)$ only depends on the colors, so $\mathrm{weight}(P) = \rho(P)$ as described in Theorem 3.8.

Given $T \in \mathrm{SSYT}_k(\vec{\gamma}, F)$, we now want to construct all elements $P \in \mathcal{T}_{\vec{\gamma},F}$ such that $S(P) = T$. But this can be done in a very simple way:

Choose any $\sigma = u_{a_1}\sigma_1 \cdots u_{a_n}\sigma_n$ written in one-line notation. Read the cells of T from top to bottom, left to right, and place $u_{a_1}\sigma_1$ in the first cell, $u_{a_2}\sigma_2$ in the second cell, and so on. Now for each row in which the color f_i appears consecutively, there is a unique way to connect this sequence of cells colored by f_i with rim hooks so that we have a valid element of $\mathcal{P}_{\vec{\gamma},F}$. This is given precisely by the decreasing cycle order, as in Remark 2.1, so that we only connect c to its left if its index is not a left-to-right minimum in this sequence of cells. In the end, we find that

Proposition 6.7. *For any $\vec{\gamma} \vdash_k n$ and color rule F,*

$$\sum_{P \in \mathcal{T}_{\vec{\gamma},F}} \operatorname{weight}(P) = n! k^n \sum_{T \in \mathrm{SSYT}_k(\vec{\gamma},F)} \rho(F).$$

Theorem 3.8 then follows.

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