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Resurgence numbers and convex regions associated to pairs of graded families of ideals

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Abstract We discuss how to understand the asymptotic resurgence number of a pair of graded families of ideals from combinatorial data of their associated convex regions. When the families consist of monomial ideals, the convex regions being considered are the Newton–Okounkov regions of the families. When ideals in the second family are classical invariant ideals, for instance, determinantal ideals or ideals of Pfaffians, these convex regions are constructed from the associated Rees packages.

1. Introduction

Let R be a Noetherian commutative ring, and let $\mathfrak{a}_\bullet = \{\mathfrak{a}_i\}_{i \geqslant 1}$ and $\mathfrak{b}_\bullet = \{\mathfrak{b}_i\}_{i \geqslant 1}$ be graded families of ideals in R. The *resurgence* and *asymptotic resurgence* numbers, $\rho(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet)$ and $\widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet)$, are measures for the non-containment between elements in the families $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$; see [22]. These invariant are generalizations of the resurgence and asymptotic resurgence numbers of an ideal, that have been much investigated in the *Ideal Containment Problem* and have shown many exciting applications (cf. [3, 4, 5, 6, 7, 8, 13, 12, 14, 16, 18, 19, 20, 23, 25, 31, 32, 34] and references therein). Particularly,

$$\rho(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = \sup \left\{ \frac{s}{r} \;\middle|\; s, r \geqslant 1, \mathfrak{a}_s \not\subseteq \mathfrak{b}_r \right\}, \text{ and}$$
$$\widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = \sup \left\{ \frac{s}{r} \;\middle|\; s, r \geqslant 1, \mathfrak{a}_{st} \not\subseteq \mathfrak{b}_{rt} \text{ for } t \gg 0 \right\},$$

with the convention that $\sup \varnothing = -\infty$. Resurgence and asymptotic resurgence a priori are very difficult to compute, even for an ideal.

Our goal is to provide a combinatorial understanding of the asymptotic resurgence of a pair of graded families of ideals through associated convex regions. Our first result addresses the case when the families $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ consist of monomial ideals. In this case, the convex regions of interest are the *Newton–Okounkov regions* $\Delta(\mathfrak{a}_\bullet)$ and $\Delta(\mathfrak{b}_\bullet)$ associated to $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$; see Section 2 for precise definitions. Newton–Okounkov regions are defined in a similar manner as Newton–Okounkov bodies, but are often not compact and designed to handle families of non-$\mathfrak{m}$-primary ideals. In general, the relationship between algebraic invariant and properties of a graded family of ideals and combinatorial data from its associated Newton–Okounkov body has been much studied (cf. [10, 11, 21, 27, 28, 30] and references thereafter). Our work adds another layer to this extensive research program.

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Keywords. family of ideals, resurgence number, asymptotic resurgence, convex body, convex region, Newton polyhedron, monomial ideals.

Leaving out the detailed definitions and terminology until later, our first main result is stated as follows. (Note that $\Delta(\mathfrak{b}_\bullet)^o$ is the *polar set* of $\Delta(\mathfrak{b}_\bullet)$, as defined in Definition 2.2.)

Theorem 1.1 (Theorem 2.7 below). *Let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be graded families of monomial ideals in $R = \Bbbk[x_1, \dots, x_n]$.*

(1) *If $\mathcal{R}(\mathfrak{b}_\bullet)$ is Noetherian, then*

$$\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) = \sup\{\lambda > 0 \mid \lambda \cdot \Delta(\mathfrak{a}_\bullet) \not\subseteq \Delta(\mathfrak{b}_\bullet)\}.$$

(2) *If $\{\lambda > 0 \mid \lambda \cdot \Delta(\mathfrak{a}_\bullet) \not\subseteq \Delta(\mathfrak{b}_\bullet)\} \neq \varnothing$, then*

$$\sup\{\lambda > 0 \mid \lambda \cdot \Delta(\mathfrak{a}_\bullet) \not\subseteq \Delta(\mathfrak{b}_\bullet)\} = \frac{1}{\inf\{\langle \mathbf{a}, \mathbf{b}\rangle \mid \mathbf{a} \in \Delta(\mathfrak{a}_\bullet), \mathbf{b} \in \Delta(\mathfrak{b}_\bullet)^o\}}.$$

The statement of Theorem 2.7 is inspired by an attempt to understand the main theorem of [34]. More precisely, for an ideal $I \subseteq R$, write $I^{(\bullet)} = \{I^{(i)}\}_{i\geqslant 1}$ and $\overline{I^\bullet} = \{\overline{I^i}\}_{i\geqslant 1}$ for the graded families of symbolic and integral closure of ordinary powers of I. [34, Theorem 3.7] shows that if I is a *squarefree monomial* ideal in a polynomial ring $R = \Bbbk[x_1, \dots, x_n]$, then

$$\frac{1}{\rho(I^{(\bullet)}, \overline{I^\bullet})} = \min\{\langle u, v\rangle \mid u \in \mathrm{SP}(I), v \in \mathrm{SP}(I^\vee)\},$$

where $I^\vee$ is the Alexander dual of I, and $\mathrm{SP}(I)$ represents the *symbolic polyhedron* of I. We will see in Corollary 2.12 that this formula is equivalent to

$$\widehat{\rho}(I^{(\bullet)}, \overline{I^\bullet}) = \sup\{\lambda > 0 \mid \lambda \cdot \mathrm{SP}(I) \not\subseteq \mathrm{NP}(I)\}; \tag{1}$$

here, $\mathrm{NP}(I)$ denotes the Newton polyhedron of I. In many important situations, by [22, Corollary 3.9], we have $\widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = \widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) = \rho(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet})$; particularly, in a polynomial ring, $\widehat{\rho}(I^{(\bullet)}, I^\bullet) = \widehat{\rho}(I^{(\bullet)}, \overline{I^\bullet}) = \rho(I^{(\bullet)}, \overline{I^\bullet})$ — see also [13].

The problem is much harder for graded families of arbitrary ideals. To get similar statements for more general ideals, we shall focus on the class of *Nagata rings*; those are Noetherian *domains* R satisfying the property that every finitely generated domain over R has a module-finite integral closure in any finite extension of its quotient field, i.e., R is *universally Japanese*. The restriction to Nagata rings is to avoid strange behaviors, for instance, where $\mathcal{R}(\overline{\mathfrak{b}})$ is not a finitely generated $\mathcal{R}(\mathfrak{b})$-module, as exhibited in Nagata's example (cf. [22, Example 3.5]).

Let $\mathfrak{b}$ be an ideal in a Nagata ring R that has a *Rees package* $\mathcal{R} = (\mathcal{B}, \mathbf{v}, \Gamma)$. The notion and existence of Rees package was introduced and studied recently in [2]. To a graded family $\mathfrak{a}_\bullet = \{\mathfrak{a}_i\}_{i\geqslant 1}$ of ideals in R, we associate a closed set $\Gamma_{\mathcal{R}}(\mathfrak{a}_\bullet)$, which contains $\mathbf{v}$-values of the $\mathcal{B}$-monomials appearing in elements of $\mathfrak{a}_i$, $i \geqslant 1$; see Section 3 for the precise definitions. A graded family $\mathfrak{b}_\bullet = \{\mathfrak{b}_i\}_{i\geqslant 1}$ is called $\mathfrak{b}$-*equivalent* if there exists an integer $k \in \mathbb{N}$ such that for all $i \geqslant 1$,

$$\mathfrak{b}_{i+k} \subseteq \overline{\mathfrak{b}^i} \subseteq \mathfrak{b}_i.$$

Our next result is stated as follows.

Theorem 1.2 (Theorems 3.6 and 3.8 below). *Let $\mathfrak{b} \subseteq R$ be an ideal in a Nagata ring R with Rees package $\mathcal{R} = (\mathcal{B}, \mathbf{v}, \Gamma)$, in which $\mathbf{v}$ is a vector of valuations. Let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be graded families of nonzero ideals in R such that either*

(1) *$\mathfrak{b}_\bullet = \mathfrak{b}^\bullet$ is the family of ordinary powers of $\mathfrak{b}$; or*

(2) *$\mathfrak{a}_\bullet$ is a filtration and $\mathfrak{b}_\bullet$ is $\mathfrak{b}$-equivalent.*

Let $\Gamma_{\mathcal{R}}(\mathfrak{a}_\bullet)$ be the closed set obtained from $\mathbf{v}$-values of $\mathcal{B}$-monomials in elements of $\mathfrak{a}_\bullet$ as in Definition 3.3. Then,

$$\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) = \sup\{\lambda > 0 \mid \lambda \cdot \Gamma_{\mathcal{R}}(\mathfrak{a}_\bullet) \not\subseteq \Gamma\}.$$

Results in [2] show that the following classes of ideals possess Rees packages:

(i) monomial ideals in affine semigroup rings;
(ii) sums of products of determinantal ideals of generic matrices;
(iii) sums of products of determinantal ideals of symmetric matrices;
(iv) sums of products of ideals of Pfaffians of skew-symmetric matrices;
(v) products of determinantal ideals of Hankel matrices.

Thus, Theorems 3.6 and 3.8 are applicable when $\mathfrak{b} \subseteq R$ is an ideal belonging to the classes (i) – (v). In these cases, $\Gamma_{\mathcal{R}}(\mathfrak{a}_\bullet)$ is a closed convex set.

Note that in Theorems 3.6 and 3.8, we impose the condition that $\mathfrak{b}_\bullet$ is either the family of ordinary powers $\{\mathfrak{b}^i\}_{i\geqslant 1}$ or $\mathfrak{b}$-equivalent, for an ideal $\mathfrak{b}$, so that the second convex region Γ in the expression of $\widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet)$ is already given by the hypothesis. The actual generalization of Theorem 2.7 is Theorem 3.7, in which Γ is replaced by Γ_k. This Γ_k plays the same role as $\Delta(\mathfrak{b}_\bullet)$, when $\mathfrak{b}_\bullet$ is Noetherian; see [21, Theorem 3.4].

We remark here that it was pointed out to us by an anonymous referee that, by considering families $\mathfrak{b}_\bullet$ that is "power-like", namely, there exists an ideal I, constants $c, C > 0$, such that

$$I^{Ci} \subseteq \mathfrak{b}_i \subseteq I^{ci} \text{ for all } i \geqslant 1,$$

one may be able to obtain estimates for $\widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet)$ in a similar fashion as our results. However, in this case, the formula is not as precise as those in Theorems 2.7, 3.6 and 3.8.

Our study in Sections 2 and 3 assumes the graded family $\mathfrak{b}_\bullet$ to be Noetherian; that is, the Rees algebra $\mathcal{R}(\mathfrak{b}_\bullet)$ is a Noetherian ring. In general, $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ are not necessarily Noetherian. For a non-Noetherian graded family $\mathfrak{a}_\bullet$, one can approximate it by a collection of Noetherian families, the *truncations* of $\mathfrak{a}_\bullet$; see Section 4 for precise definitions. Our last main result shows that resurgence and asymptotic resurgence numbers, $\rho(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet)$ and $\widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet)$, can be computed via those of truncations of $\mathfrak{a}_\bullet$. On the other hand, examples exist to exhibit that truncations of $\mathfrak{b}_\bullet$ will not give the same conclusion; see Example 4.4.

Theorem 1.3 (Theorem 4.3 below). *Let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be graded families of ideals in a Noetherian commutative ring R. For $n \in \mathbb{N}$, let $\mathfrak{a}_{n,\bullet}$ denote the n-th truncation graded family of $\mathfrak{a}_\bullet$. We have*

$$\rho(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = \sup_{n\geqslant 1} \rho(\mathfrak{a}_{n,\bullet}, \mathfrak{b}_\bullet) = \lim_{n\to\infty} \rho(\mathfrak{a}_{n,\bullet}, \mathfrak{b}_\bullet), \text{ and}$$
$$\widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = \sup_{n\geqslant 1} \widehat{\rho}(\mathfrak{a}_{n,\bullet}, \mathfrak{b}_\bullet) = \lim_{n\to\infty} \widehat{\rho}(\mathfrak{a}_{n,\bullet}, \mathfrak{b}_\bullet).$$

2. Asymptotic resurgence and Newton–Okounkov regions

This section is devoted to demonstrating a connection between the asymptotic resurgence number $\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet})$ of a pair of graded families of ideals and Newton–Okounkov regions associated with these families.

We collect some important notations and terminology. If $R = \Bbbk[x_1, \dots, x_n]$ is a polynomial ring over a field $\Bbbk$ and $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geqslant 0}$, then we shall denote by $x^{\mathbf{a}}$ the monomial $x_1^{a_1} \dots x_n^{a_n}$ in R. The Newton polyhedron of a monomial ideal is a well-known notion. The construction of a symbolic polyhedron for a monomial ideal is from [9].

Definition 2.1. *Let $R = \Bbbk[x_1, \dots, x_n]$ and let $I \subseteq R$ be a monomial ideal.*

(1) *The* Newton polyhedron *of I is defined to be*

$$\mathrm{NP}(I) = \mathrm{convexhull}\langle \{\mathbf{a} \in \mathbb{Z}^n_{\geqslant 0} \mid x^{\mathbf{a}} \in I\}\rangle \subseteq \mathbb{R}^n_{\geqslant 0}.$$

(2) *Suppose, in addition, that I is squarefree. The* symbolic polyhedron *of I is given by*

$$\mathrm{SP}(I) = \bigcap_{\mathfrak{p}\in \mathrm{Ass}(I)} \mathrm{NP}(\mathfrak{p}) = \bigcap_{\mathfrak{p}\in \mathrm{Ass}(I)} \left\{ (a_1,\dots,a_n) \in \mathbb{R}^n_{\geqslant 0} \;\middle|\; \sum_{x_i\in\mathfrak{p}} a_i \geqslant 1 \right\} \subseteq \mathbb{R}^n_{\geqslant 0}.$$

We introduce a notion of polar sets that contrasts with the commonly known concept in analysis. For clarity and simplicity, we will still refer to them as polar sets throughout this paper, avoiding the need for new terminology.

Definition 2.2. *For a convex set $\mathcal{C} \subseteq \mathbb{R}^n$ not containing the origin, the* polar set *of $\mathcal{C}$ is defined by*

$$\mathcal{C}^o = \{\mathbf{a} \in \mathbb{R}^n \mid \langle \mathbf{a}, \mathbf{b}\rangle \geqslant 1 \text{ for all } \mathbf{b} \in \mathcal{C}\},$$

where $\langle \bullet, \bullet \rangle$ denotes the usual dot product in $\mathbb{R}^n$.

Remark 2.3. From the discussion in [15, 17, 19], it can be observed that, for a squarefree monomial ideal $I \subseteq \Bbbk[x_1,\dots,x_n]$,

$$\mathrm{SP}(I)^o = \mathrm{NP}(I^\vee) \text{ and } \mathrm{NP}(I) = \mathrm{SP}(I^\vee)^o,$$

where $I^\vee = \langle \prod_{x_i\in\mathfrak{p}} x_i \mid \mathfrak{p} \in \mathrm{Ass}(I)\rangle$ denotes the *Alexander dual* of I.

The "Bipolar Theorem" is a well-known result in analysis (cf. [33]). Since our notion of polar sets differs from the analytical definition, we will provide the proof of a similar bipolar statement in a simple case relevant to our study.

Lemma 2.4. *Let $\mathcal{C} \subseteq \mathbb{R}^n_{\geqslant 0}$ be a closed convex set that does not contain the origin and absorbs $\mathbb{R}^n_{\geqslant 0}$, i.e., $\mathcal{C} + \mathbb{R}^n_{\geqslant 0} \subseteq \mathcal{C}$. Then,*

$$\mathcal{C}^{oo} = C.$$

Proof. It is easy to see from the definition that $\mathcal{C} \subseteq \mathcal{C}^{oo}$. We shall establish the reverse containment. Suppose, by contradiction, that there exists $\mathbf{b} \in \mathcal{C}^{oo} \smallsetminus \mathcal{C}$.

It is a standard result from convex geometry that there exists a hyperplane H which *strongly* separates $\mathbf{b}$ and $\mathcal{C}$, whose equation is of the form $\langle \mathbf{c}, \mathbf{x}\rangle = 1$, where $\mathbf{c} = (c_1,\dots,c_n)$. Since $\mathcal{C} \subseteq \mathbb{R}^n_{\geqslant 0}$, we may assume that H contains points in $\mathbb{R}^n_{\geqslant 0}$ which are not the origin. This implies that there exists i such that $c_i > 0$.

Since $\mathcal{C}$ absorbs $\mathbb{R}^n_{\geqslant 0}$, by increasing the i-th coordinate of the points in $\mathcal{C}$, it follows that $\mathcal{C}$ lies in the positive half-space determined by H. That is, $\langle \mathbf{c}, \mathbf{y}\rangle \geqslant 1$ for all $\mathbf{y} \in \mathcal{C}$. This implies that $\mathbf{c} \in \mathcal{C}^o$. On the other hand, since $\mathbf{b}$ lies on the other side of H, we must have $\langle \mathbf{c}, \mathbf{b}\rangle < 1$. As a consequence, $\mathbf{b} \notin (\mathcal{C}^o)^o = \mathcal{C}^{oo}$, which is a contradiction. ☐

A *graded family* of ideals in a commutative ring R is a collection of ideals $\{\mathfrak{a}_i\}_{i\geqslant 1}$ satisfying the condition that $\mathfrak{a}_p \cdot \mathfrak{a}_q \subseteq \mathfrak{a}_{p+q}$ for all $p, q \geqslant 1$. A graded family $\{\mathfrak{a}_i\}_{i\geqslant 1}$ is called a *filtration* if $\mathfrak{a}_i \supseteq \mathfrak{a}_{i+1}$ for all $i \geqslant 1$. The *Rees algebra* associated to a graded family $\mathfrak{a}_\bullet = \{\mathfrak{a}_i\}_{i\geqslant 1}$ of ideals in R is defined to be

$$\mathcal{R}(\mathfrak{a}_\bullet) = \bigoplus_{i\geqslant 0} \mathfrak{a}_i t^i \subseteq R[t],$$

where, by convention, $\mathfrak{a}_0 = R$. The graded family $\mathfrak{a}_\bullet$ is called *Noetherian* if the Rees algebra $\mathcal{R}(\mathfrak{a}_\bullet)$ is a Noetherian ring.

The *Newton–Okounkov region* associated with a graded family of ideals, in general, is defined based on the existence of *good valuations*; see [28]. It is similar to the *Newton–Okounkov body* for graded families of $\mathfrak{m}$-primary ideals considered in [10, 11, 27, 28], but is not compact in general. This construction becomes more transparent for graded families of monomial ideals; see, for instance, [21].

Definition 2.5. *Let $R = \Bbbk[x_1, \ldots, x_n]$ and let $\mathfrak{a}_\bullet = \{\mathfrak{a}_i\}_{i \geqslant 1}$ be a graded family of* monomial *ideals in R. The* Newton–Okounkov region *associated to $\mathfrak{a}_\bullet$ is*

$$\Delta(\mathfrak{a}_\bullet) = \overline{\bigcup_{k \geqslant 1} \frac{1}{k} \operatorname{NP}(\mathfrak{a}_k)} \subseteq \mathbb{R}^n_{\geqslant 0}.$$

Remark 2.6. Observe that if $\mathfrak{a}_\bullet = I^\bullet$, for a monomial ideal I, then $\Delta(\mathfrak{a}_\bullet) = \operatorname{NP}(I)$, and if $\mathfrak{a}_\bullet = I^{(\bullet)}$, for a squarefree monomial ideal I, then $\Delta(\mathfrak{a}_\bullet) = \operatorname{SP}(I)$.

By construction, $\Delta(\mathfrak{a}_\bullet)$ is a closed convex set that absorbs $\mathbb{R}^n_{\geqslant 0}$. It was also pointed out in [21, Example 3.2] that for any nonempty closed convex set $P \subseteq \mathbb{R}^n_{\geqslant 0}$ absorbing $\mathbb{R}^n_{\geqslant 0}$, there is a graded family of monomial ideal $\mathfrak{a}_\bullet$ such that $\Delta(\mathfrak{a}_\bullet) = P$.

We are now ready to present our first main result.

Theorem 2.7. *Let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be graded families of monomial ideals in $R = \Bbbk[x_1, \ldots, x_n]$.*

(1) *If $\mathcal{R}(\mathfrak{b}_\bullet)$ is Noetherian, then*

$$\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) = \sup\{\lambda > 0 \mid \lambda \cdot \Delta(\mathfrak{a}_\bullet) \not\subseteq \Delta(\mathfrak{b}_\bullet)\}.$$

(2) *If $\{\lambda > 0 \mid \lambda \cdot \Delta(\mathfrak{a}_\bullet) \not\subseteq \Delta(\mathfrak{b}_\bullet)\} \neq \varnothing$, then*

$$\sup\{\lambda > 0 \mid \lambda \cdot \Delta(\mathfrak{a}_\bullet) \not\subseteq \Delta(\mathfrak{b}_\bullet)\} = \frac{1}{\inf\{\langle \mathbf{a}, \mathbf{b} \rangle \mid \mathbf{a} \in \Delta(\mathfrak{a}_\bullet), \mathbf{b} \in \Delta(\mathfrak{b}_\bullet)^o\}}.$$

Proof. (1) Since $\mathcal{R}(\mathfrak{b}_\bullet)$ is Noetherian, it follows from [21, Theorem 3.4] that there exists $k \geqslant 1$ such that $\overline{\mathfrak{b}_k^t} = \overline{\mathfrak{b}_{kt}}$ and $\Delta(\mathfrak{b}_\bullet) = \frac{1}{k} \operatorname{NP}(\mathfrak{b}_k) = \frac{1}{kt} \operatorname{NP}(\mathfrak{b}_{kt})$ for all $t \in \mathbb{N}$.

We will first consider the trivial case, when $\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) = -\infty$. By definition, this happens if only if for any integers $s, r \geqslant 1$ and any $M \in \mathbb{N}$, there exists $\theta \geqslant M$ such that $\mathfrak{a}_{s\theta} \subseteq \overline{\mathfrak{b}_{r\theta}}$. Particularly, $\overline{\mathfrak{a}_{s\theta}} \subseteq \overline{\mathfrak{b}_{r\theta}}$, and so

$$\frac{s}{r} \cdot \frac{1}{s\theta} \operatorname{NP}(\mathfrak{a}_{s\theta}) \subseteq \frac{1}{r\theta} \operatorname{NP}(\mathfrak{b}_{r\theta}) \subseteq \Delta(\mathfrak{b}_\bullet).$$

Thus, for any $\lambda = \frac{p}{q} \in \mathbb{Q}_{>0}$ and any $\ell \in \mathbb{N}$, by taking $s = p\ell$ and $r = q\ell$, we obtain

$$\lambda \cdot \frac{1}{\ell} \operatorname{NP}(\mathfrak{a}_\ell) = \frac{p\ell}{q\ell} \cdot \frac{1}{\ell} \operatorname{NP}(\mathfrak{a}_\ell) \subseteq \frac{p\ell}{q\ell} \cdot \frac{1}{p\ell\theta} \operatorname{NP}(\mathfrak{a}_{p\ell\theta}) = \frac{s}{r} \cdot \frac{1}{s\theta} \operatorname{NP}(\mathfrak{a}_{s\theta}) \subseteq \Delta(\mathfrak{b}_\bullet).$$

Therefore, $\lambda \cdot \Delta(\mathfrak{a}_\bullet) \subseteq \Delta(\mathfrak{b}_\bullet)$ for any $\lambda \in \mathbb{Q}_{>0}$. Hence, we get $\lambda \cdot \Delta(\mathfrak{a}_\bullet) \subseteq \Delta(\mathfrak{b}_\bullet)$ for any $\lambda > 0$, i.e.,

$$\sup\{\lambda > 0 \mid \lambda \cdot \Delta(\mathfrak{a}_\bullet) \not\subseteq \Delta(\mathfrak{b}_\bullet)\} = -\infty.$$

Conversely, suppose that for any $\lambda > 0$, we have $\lambda \cdot \Delta(\mathfrak{a}_\bullet) \subseteq \Delta(\mathfrak{b}_\bullet)$. As a consequence, for any integers $s, r \geqslant 1$, we have

$$\frac{s}{r} \cdot \frac{1}{skt} \operatorname{NP}(\mathfrak{a}_{skt}) \subseteq \Delta(\mathfrak{b}_\bullet) = \frac{1}{rkt} \operatorname{NP}(\mathfrak{b}_{rkt}) \text{ for all } t \in \mathbb{N}.$$

Particularly, $\operatorname{NP}(\mathfrak{a}_{skt}) \subseteq \operatorname{NP}(\mathfrak{b}_{rkt})$. It then follows that $\mathfrak{a}_{skt} \subseteq \overline{\mathfrak{a}_{skt}} \subseteq \overline{\mathfrak{b}_{rkt}}$. Therefore, for any $M \in \mathbb{N}$, by choosing $t > \frac{M}{k}$ and setting $\theta = kt > M$, we have $\mathfrak{a}_{s\theta} \subseteq \overline{\mathfrak{b}_{r\theta}}$. Hence,

$$\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) = -\infty.$$

Consider the case that $\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) \neq -\infty$. Let $s, r \geqslant 1$ be such that $\mathfrak{a}_{st} \not\subseteq \overline{\mathfrak{b}_{rt}}$ for $t \gg 0$; the existence of s and r are due to the assumption on $\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet})$. Particularly, for $t \gg 0$, there exists a monomial $x_1^{u_{t,1}} \cdots x_n^{u_{t,n}} \in \mathfrak{a}_{st} \smallsetminus \overline{\mathfrak{b}_{rt}}$. This implies that $u_t = (u_{t,1}, \ldots, u_{t,n}) \in \operatorname{NP}(\mathfrak{a}_{st}) \smallsetminus \operatorname{NP}(\mathfrak{b}_{rt})$.

By replacing t with kt, it follows that for $t \gg 0$,

$$u_{kt} = (u_{kt,1}, \ldots, u_{kt,n}) \in \operatorname{NP}(\mathfrak{a}_{skt}) \smallsetminus \operatorname{NP}(\mathfrak{b}_{rkt}).$$

We now have

$$\frac{u_{kt}}{skt} \in \frac{1}{skt}\operatorname{NP}(\mathfrak{a}_{skt}) \subseteq \Delta(\mathfrak{a}_\bullet) \text{ and } \frac{s}{r}\cdot\frac{u_{kt}}{skt} = \frac{u_{kt}}{rkt} \notin \frac{1}{rkt}\operatorname{NP}(\mathfrak{b}_{rkt}) = \Delta(\mathfrak{b}_\bullet).$$

Therefore, $\frac{s}{r}\cdot\Delta(\mathfrak{a}_\bullet) \not\subseteq \Delta(\mathfrak{b}_\bullet)$. This is true for any "admissible" quotient s/r defining $\widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet)$. Hence,

$$\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) \leqslant \sup\{\lambda > 0 \mid \lambda\cdot\Delta(\mathfrak{a}_\bullet) \not\subseteq \Delta(\mathfrak{b}_\bullet)\}.$$

Conversely, let $s, r \geqslant 1$ be such that $\frac{s}{r}\cdot\Delta(\mathfrak{a}_\bullet) \not\subseteq \Delta(\mathfrak{b}_\bullet)$. Then, there exists $q \in \mathbb{N}$ such that

$$\frac{s}{r}\cdot\frac{1}{q}\operatorname{NP}(\mathfrak{a}_q) \not\subseteq \Delta(\mathfrak{b}_\bullet) = \frac{1}{kt}\operatorname{NP}(\mathfrak{b}_{kt}) \text{ for all } t \in \mathbb{N}.$$

This implies that there is a rational vector $v = (v_1, \ldots, v_n) \in \operatorname{NP}(\mathfrak{a}_q)$ such that $\frac{s}{r}\cdot\frac{1}{q}v \notin \frac{1}{kt}\operatorname{NP}(\mathfrak{b}_{kt})$ for all t. Particularly,

$$\frac{s}{r}\cdot\frac{1}{q}v \notin \frac{1}{kq}\operatorname{NP}(\mathfrak{b}_{kq}).$$

By replacing v and q with vL and qL, respectively, where L is the least common multiple of the denominators of $v_1, \ldots, v_n$, we may assume that v is an integral vector.

Since $v \in \operatorname{NP}(\mathfrak{a}_q)$, $x_1^{v_1}\ldots x_n^{v_n} \in \overline{\mathfrak{a}_q}$, which further implies that $x_1^{pv_1}\ldots x_n^{pv_n} \in \mathfrak{a}_q^p \subseteq \mathfrak{a}_{pq}$ for some $p \in \mathbb{N}$. We then have

$$(x_1^{pv_1}\ldots x_n^{pv_n})^{skt} \in \mathfrak{a}_{pq}^{skt} \subseteq \mathfrak{a}_{sktpq} \text{ and } sktp\cdot v \notin rtp\cdot\operatorname{NP}(\mathfrak{b}_{kq}).$$

It follows that

$$(x^v)^{sktp} = (x_1^{pv_1}\ldots x_n^{pv_n})^{stk} \in \mathfrak{a}_{sktpq} \smallsetminus \overline{\mathfrak{b}_{kq}^{rtp}} = \mathfrak{a}_{sktpq} \smallsetminus \overline{\mathfrak{b}_{rtpkq}}.$$

Thus, $\mathfrak{a}_{sktpq} \not\subseteq \overline{\mathfrak{b}_{rktpq}}$ for all $t \in \mathbb{N}$. Therefore, $\frac{s}{r} = \frac{skpq}{rkpq} \leqslant \widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet})$. Hence,

$$\sup\{\lambda > 0 \mid \lambda\cdot\Delta(\mathfrak{a}_\bullet) \not\subseteq \Delta(\mathfrak{b}_\bullet)\} \leqslant \widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}),$$

and the first assertion of the theorem is proved.

(2) To establish the second equality, we use the fact $\lambda\cdot\Delta(\mathfrak{a}_\bullet) \subseteq \gamma\cdot\Delta(\mathfrak{a}_\bullet)$ for any $0 < \gamma \leqslant \lambda$ from [15]. It follows that

$$\sup\{\lambda > 0 \mid \lambda\cdot\Delta(\mathfrak{a}_\bullet) \not\subseteq \Delta(\mathfrak{b}_\bullet)\} = \inf\{\lambda > 0 \mid \lambda\cdot\Delta(\mathfrak{a}_\bullet) \subseteq \Delta(\mathfrak{b}_\bullet)\}.$$

We continue to show that

$$\inf\{\lambda > 0 \mid \lambda\cdot\Delta(\mathfrak{a}_\bullet) \subseteq \Delta(\mathfrak{b}_\bullet)\} = \inf\{\lambda > 0 \mid \langle\lambda\mathbf{a}, \mathbf{b}\rangle \geqslant 1, \forall\mathbf{a} \in \Delta(\mathfrak{a}_\bullet), \mathbf{b} \in \Delta(\mathfrak{b}_\bullet)^o\}.$$

Indeed, if $\lambda\cdot\Delta(\mathfrak{a}_\bullet) \subseteq \Delta(\mathfrak{b}_\bullet)$, then for any $\mathbf{a} \in \Delta(\mathfrak{a}_\bullet)$ (whence $\lambda\mathbf{a} \in \Delta(\mathfrak{b}_\bullet)$) and $\mathbf{b} \in \Delta(\mathfrak{b}_\bullet)^o$, by definition, $\langle\lambda\mathbf{a}, \mathbf{b}\rangle \geqslant 1$. On the other hand, if $\langle\lambda\mathbf{a}, \mathbf{b}\rangle \geqslant 1$ for all $\mathbf{a} \in \Delta(\mathfrak{a}_\bullet)$ and $\mathbf{b} \in \Delta(\mathfrak{b}_\bullet)^o$, then $\lambda\cdot\Delta(\mathfrak{a}_\bullet) \subseteq \Delta(\mathfrak{b}_\bullet)^{oo} = \Delta(\mathfrak{b}_\bullet)$, where the last equality follows from Lemma 2.4.

Finally, it follows that

$$\begin{aligned}
\sup\{\lambda > 0 \mid \lambda\cdot\Delta(\mathfrak{a}_\bullet) \not\subseteq \Delta(\mathfrak{b}_\bullet)\} &= \inf\{\lambda > 0 \mid \langle\lambda\mathbf{a}, \mathbf{b}\rangle \geqslant 1, \forall\mathbf{a} \in \Delta(\mathfrak{a}_\bullet), \mathbf{b} \in \Delta(\mathfrak{b}_\bullet)^o\} \\
&= \inf\left\{\lambda \;\middle|\; \frac{1}{\langle\mathbf{a}, \mathbf{b}\rangle} \leqslant \lambda, \forall\mathbf{a} \in \Delta(\mathfrak{a}_\bullet), \mathbf{b} \in \Delta(\mathfrak{b}_\bullet)^o\right\} \\
&= \frac{1}{\inf\{\langle\mathbf{a}, \mathbf{b}\rangle \mid \mathbf{a} \in \Delta(\mathfrak{a}_\bullet), \mathbf{b} \in \Delta(\mathfrak{b}_\bullet)^o\}}.
\end{aligned}$$

The assertion is established. ☐

Remark 2.8. In a different context, for graded families of polyhedra, a similar looking statement to that of Theorem 2.7 is obtained in [15].

We illustrate Theorem 2.7 with the following example. This example appeared in [21] for a different purpose.

Example 2.9. Let $I = (x,y)^2 \cap (y,z)^3 \cap (x,z)^4 \subseteq \Bbbk[x,y,z]$. Consider the graded families $\mathfrak{a}_\bullet = I^{(\bullet)}$ and $\mathfrak{b}_\bullet = I^\bullet$. Then, it follows from [21, Remark 2.14] that $\Delta(\mathfrak{a}_\bullet) = \mathrm{SP}(I)$ and $\Delta(\mathfrak{b}_\bullet) = \mathrm{NP}(I)$. It is known from [21, Example 5.5] that

$$\mathrm{SP}(I) = \text{convexhull}\left\{(4,3,0),(2,0,3),(0,2,4),\left(\frac{3}{2},\frac{1}{2},\frac{5}{2}\right)\right\} + \mathbb{R}^3_{\geqslant 0}.$$

Note that $I = (xyz^3, x^2z^3, x^2yz^2, y^2z^4, x^3y^2z, x^4y^3)$. Therefore,

$$\mathrm{NP}(I) = \text{convexhull}\{(1,1,3),(2,0,3),(2,1,2),(0,2,4),(3,2,1),(4,3,0)\} + \mathbb{R}^3_{\geqslant 0}.$$

Observe that the first three vertices of $\mathrm{SP}(I)$ belong to $\mathrm{NP}(I)$. Thus,

$$\sup\{\lambda > 0 \mid \lambda \cdot \mathrm{SP}(I) \not\subseteq \mathrm{NP}(I)\} = \sup\left\{\lambda > 0 \;\middle|\; \lambda \cdot \left(\frac{3}{2},\frac{1}{2},\frac{5}{2}\right) \notin \mathrm{NP}(I)\right\}.$$

This value is then determined by where the line connecting the origin and $\left(\frac{3}{2},\frac{1}{2},\frac{5}{2}\right)$ *first* intersects $\mathrm{NP}(I)$. This point can be computed to be $\left(\frac{5}{3},\frac{5}{9},\frac{25}{9}\right)$. Thus,

$$\sup\left\{\lambda > 0 \;\middle|\; \lambda \cdot \left(\frac{3}{2},\frac{1}{2},\frac{5}{2}\right) \notin \mathrm{NP}(I)\right\} = \frac{10}{9}.$$

Hence, due to Theorem 2.7, we get $\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) = \sup\{\lambda > 0 \mid \lambda \cdot \Delta(\mathfrak{a}_\bullet) \not\subseteq \Delta(\mathfrak{b}_\bullet)\} = \frac{10}{9}$.

It is worth noting that, in [1, Theorem 3], the resurgence number was also given as $\rho(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = \frac{10}{9}$. Thus, there are equalities

$$\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) = \widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = \rho(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) = \rho(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = \frac{10}{9}.$$

Question 2.10. *Does the statement of Theorem 2.7(1) still hold without the condition that $\mathcal{R}(\mathfrak{b}_\bullet)$ is Noetherian? We do not know of any example suggesting otherwise.*

As a consequence of Theorem 2.7, we obtain the following generalization of [34, Theorem 3.7].

Corollary 2.11. *Let $\mathfrak{a}_\bullet$ be graded family of monomial ideals in $R = \Bbbk[x_1,\ldots,x_n]$ and $\mathfrak{b}$ be a squarefree monomial ideal in R. Then,*

$$\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}^\bullet}) = \sup\{\lambda > 0 \mid \lambda \cdot \Delta(\mathfrak{a}_\bullet) \not\subseteq \mathrm{NP}(\mathfrak{b})\} = \frac{1}{\inf\{\langle \mathbf{a}, \mathbf{b}\rangle \mid \mathbf{a} \in \Delta(\mathfrak{a}_\bullet), \mathbf{b} \in \mathrm{SP}(\mathfrak{b}^\vee)\}}.$$

Proof. By Remark 2.3 (see also [15, 17]), we have

$$\mathrm{NP}(\mathfrak{b}) = \mathrm{SP}(\mathfrak{b}^\vee)^o = \left\{\mathbf{a} \in \mathbb{R}^n_{\geqslant 0} \;\middle|\; \langle \mathbf{a}, \mathbf{b}\rangle \geqslant 1 \text{ for all } \mathbf{b} \in \mathrm{SP}(\mathfrak{b}^\vee)\right\}.$$

Thus, applying Theorem 2.7, we get the desired result. ☐

As a consequence of Corollary 2.11, [34, Theorem 3.7] is now recovered by letting $\mathfrak{a}_\bullet = \mathfrak{a}^{(\bullet)}$ for an ideal $\mathfrak{a}$, and noting that $\widehat{\rho}(\mathfrak{a}^{(\bullet)}, \mathfrak{a}^\bullet) = \widehat{\rho}(\mathfrak{a}^{(\bullet)}, \overline{\mathfrak{a}^\bullet}) = \rho(\mathfrak{a}^{(\bullet)}, \overline{\mathfrak{a}^\bullet})$ by [13].

Corollary 2.12. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be squarefree monomial ideals in $R = \Bbbk[x_1,\ldots,x_n]$. Then,*

$$\widehat{\rho}(\mathfrak{a}^{(\bullet)}, \overline{\mathfrak{b}^\bullet}) = \frac{1}{\min\{\langle \mathbf{a}, \mathbf{b}\rangle \mid \mathbf{a} \in \mathrm{SP}(\mathfrak{a}), \mathbf{b} \in \mathrm{SP}(\mathfrak{b}^\vee)\}} = \widehat{\rho}((\mathfrak{b}^\vee)^{(\bullet)}, \overline{(\mathfrak{a}^\vee)^\bullet}).$$

In particular, for any squarefree monomial ideal $\mathfrak{a} \subseteq R$,

$$\widehat{\rho}(\mathfrak{a}^{(\bullet)}, \overline{\mathfrak{a}^\bullet}) = \widehat{\rho}((\mathfrak{a}^\vee)^{(\bullet)}, \overline{(\mathfrak{a}^\vee)^\bullet}).$$

Proof. The proof follows from Corollary 2.11, the facts that $\Delta(\mathfrak{a}^{(\bullet)}) = \mathrm{SP}(\mathfrak{a})$, $(\mathfrak{a}^\vee)^\vee = \mathfrak{a}$, and that both $\mathrm{SP}(\mathfrak{a}), \mathrm{SP}(\mathfrak{b}^\vee)$ are polyhedra. ☐

3. Invariant ideals, Rees packages and convex regions

In this section, we attempt to understand combinatorially the asymptotic resurgence from associated convex regions, and look for statements that are similar to Theorem 2.7 for more general classes of ideals. Throughout the section, we will assume that R is a Nagata algebra over a field $\Bbbk$. Our approach is based on the notion of *Rees package* introduced and studied recently in [2].

Definition 3.1 ([2, Definition 3.2]). *Fix a $\Bbbk$-vector space basis $\mathcal{B} = \{\mathbf{b}_i \mid i \in \Lambda\}$ for R. We say that $v : R \to \mathbb{Z}_{\geqslant 0}$ is a* $\mathcal{B}$-monomial function *if for every $f = \sum_{i\in\Lambda} c_i\mathbf{b}_i \in R$, we have $v(f) = \min\{v(\mathbf{b}_i) \mid c_i \neq 0\}$. A valuation of R, that is also a $\mathcal{B}$-monomial function, is called a* B-monomial valuation*.*

Recall that a hyperplane in $\mathbb{R}^d$ defined by $\langle \mathbf{h}, \mathbf{x}\rangle = c$, where $\mathbf{h} = (h_1, \dots, h_d) \in \mathbb{R}^d$, $\mathbf{x} = (x_1, \dots, x_d)$ and $c \in \mathbb{R}$ is called *non-coordinate* if $c \neq 0$.

Definition 3.2 ([2, Proposition 3.4 and Definition 3.5]). *Let $v_1, \dots, v_d$ be $\mathcal{B}$-monomial functions of R and set $\mathbf{v} = (v_1, \dots, v_d)$. Let $\Gamma \subseteq \mathbb{R}^d_{\geqslant 0}$ be an integral polyhedron such that for every $s \in \mathbb{N}$, the integral closure $\overline{I^s}$ is spanned as a $\Bbbk$-vector space by the set*

$$\{\mathbf{b} \in \mathcal{B} \mid \mathbf{v}(\mathbf{b}) \in s\Gamma\}.$$

Let $H_1, \dots, H_r$ be the non-coordinate supporting hyperplanes of Γ, where H_k is given by $\langle \mathbf{h}_k, \mathbf{x}\rangle = c_k$ for some $\mathbf{h}_k \in \mathbb{Z}^d_{\geqslant 0}$ and $c_k \in \mathbb{Z}_{>0}$. We call $(\mathcal{B}, \mathbf{v}, \Gamma)$ a Rees package *for I if the $\mathcal{B}$-monomial function $V_k := \langle \mathbf{h}_k, \mathbf{v}\rangle$ is a valuation of R for any $k = 1, \dots, r$. In this case, $\{V_1, \dots, V_r\}$ are precisely the* Rees valuations *of I and for any $u \in \mathbb{Q}_{\geqslant 0}$, the* rational power *$\overline{I^u}$ is spanned as a $\Bbbk$-vector space by*

$$\{\mathbf{b} \in \mathcal{B} \mid \mathbf{v}(\mathbf{b}) \in u\Gamma\}.$$

See, for example, [24] *for the definitions of Rees valuations and rational powers. Rees packages were shown to exist for several classes of monomial and invariant ideals in* [2].

Our next definition constructs a closed set associated to a graded family of ideals with respect to an ideal with a given Rees package.

Definition 3.3. *Let $\mathfrak{b} \subseteq R$ be an ideal and assume that $\mathcal{R} = (\mathcal{B}, \mathbf{v}, \Gamma)$ is a Rees package for $\mathfrak{b}$. For any graded family $\mathfrak{a}_\bullet = \{\mathfrak{a}_i\}_{i\geqslant 1}$, define*

$$V_{\mathcal{R}}(\mathfrak{a}_i) = \text{convexhull}\big(\{\mathbf{v}(\mathbf{b}) \mid f \in \mathfrak{a}_i \textit{ and } \mathbf{b} \textit{ is a } \mathcal{B}\textit{-monomial in } f\}\big) \subseteq \mathbb{R}^d_{\geqslant 0},$$

and set

$$\Gamma_{\mathcal{R}}(\mathfrak{a}_\bullet) = \overline{\bigcup_{k=1}^{\infty} \frac{1}{k} V_{\mathcal{R}}(\mathfrak{a}_k)}.$$

Remark 3.4. We say that $\mathbf{v}$ is a *vector of valuations* if the $\mathcal{B}$-monomial functions $v_1, \dots, v_d$ are valuations of R. In this case, the condition that V_k is a valuation, for $k = 1, \dots, r$, for $(\mathcal{B}, \mathbf{v}, \Gamma)$ to be a Rees package in Definition 3.2 is automatically satisfied. It is not hard to see that if $\mathbf{v}$ is a vector of valuations then $\Gamma_{\mathcal{R}}(\mathfrak{a}_\bullet)$ is a closed convex set. This is the case in all known examples where Rees packages exist; see [2].

We shall use an example given in [2] to illustrate the construction introduced in Definition 3.3.

Example 3.5. Let $Y = [y_{i,j}]$ be a $m \times m$ symmetric matrix consisting of indeterminates $y_{i,j}$ with $1 \leqslant i, j \leqslant m$ and $m \geqslant 3$. Let $R = \Bbbk[y_{i,j} \mid i, j \in [m]]$ and let $\mathfrak{b} = I_{m-1}(Y)$ be the ideal generated by all $(m-1) \times (m-1)$-minors of Y in R. Let $\mathcal{B}$ be the set of standard monomials in R, see [2, Setup 3.16].

It follows from [2, Theorem 3.17] that $\mathfrak{b}$ admits a Rees packages $\mathcal{R} = (\mathcal{B}, \mathbf{v}, \Gamma)$, where $\mathbf{v} = (\gamma_1, \dots, \gamma_m)$ is a vector of $\mathcal{B}$-monomial valuations, and

$$\Gamma = \text{convexhull}\{(m-1, m-2, \dots, 1, 0)\} + \mathbb{R}^m_{\geqslant 0}.$$

Consider the graded family $\mathfrak{a}_\bullet = \mathfrak{b}^{(\bullet)}$. By [26, Proposition 4.3], for all $s \in \mathbb{N}$, $\mathfrak{b}^{(s)}$ is generated by standard monomials, $\mathfrak{b}^{(2k)} = \left(\mathfrak{b}^{(2)}\right)^k$ and $\mathfrak{b}^{(2k+1)} = \mathfrak{b}\left(\mathfrak{b}^{(2)}\right)^k$ for all k. Also, $\mathfrak{b}^{(2)} = \mathfrak{b}^2 + I_m(Y)$. Thus, for all $k \geqslant 1$,

$$\mathbf{v}\left(\mathfrak{b}^{(2k)}\right) = k\mathbf{v}\left(\mathfrak{b}^{(2)}\right) \text{ and } \mathbf{v}\left(\mathfrak{b}^{(2k+1)}\right) = k\mathbf{v}\left(\mathfrak{b}^{(2)}\right) + \mathbf{v}(\mathfrak{b}).$$

By the construction of $\mathbf{v}$ given in [2, Theorem 3.17], we get

$$\mathbf{v}(\mathfrak{b}) = (m-1, m-2, \dots, 1, 0) \text{ and } \mathbf{v}(I_m(Y)) = (m, m-1, \dots, 2, 1).$$

Therefore,

$$V_{\mathcal{R}}(\mathfrak{a}_{2k}) = \text{convexhull}\left(\{k(m, m-1, \dots, 2, 1), 2k(m-1, m-2, \dots, 1, 0)\}\right) + \mathbb{R}^m_{\geqslant 0},$$

and

$$\begin{aligned} V_{\mathcal{R}}(\mathfrak{a}_{2k+1}) = \text{convexhull}\,\{&(k+1)(m, m-1, \dots, 2, 1) - (1, 1, \dots, 1), \\ &(2k+1)(m-1, m-2, \dots, 1, 0)\} + \mathbb{R}^m_{\geqslant 0}. \end{aligned}$$

Thus, taking the limit gives us

$$\begin{aligned} \Gamma_{\mathcal{R}}(\mathfrak{a}_\bullet) &= \overline{\bigcup_{k=1}^{\infty} \frac{1}{k} V_{\mathcal{R}}(\mathfrak{a}_k)} \\ &= \text{convexhull}\left(\left\{\frac{1}{2}(m, m-1, \dots, 2, 1), (m-1, m-2, \dots, 1, 0)\right\}\right) + \mathbb{R}^m_{\geqslant 0}. \end{aligned}$$

Our next result establishes $\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet})$ when $\mathfrak{b}_\bullet = \mathfrak{b}^\bullet$ is the family of ordinary powers of an ideal $\mathfrak{b} \subseteq R$ with Rees packages.

Theorem 3.6. *Let $\mathfrak{b} \subseteq R$ be a nonzero ideal with Rees package $\mathcal{R} = (\mathcal{B}, \mathbf{v}, \Gamma)$, in which $\mathbf{v}$ is a vector of valuations. Let $\mathfrak{a}_\bullet = \{\mathfrak{a}_i\}_{i \geqslant 1}$ be a graded family of nonzero ideals in R. Then,*

$$\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}^\bullet}) = \sup\{\lambda > 0 \mid \lambda \cdot \Gamma_{\mathcal{R}}(\mathfrak{a}_\bullet) \not\subseteq \Gamma\}.$$

Proof. Consider $s, r \in \mathbb{N}$ such that $\mathfrak{a}_{st} \not\subseteq \overline{\mathfrak{b}^{rt}}$ for $t \gg 0$. It follows that, for $t \gg 0$, $\mathfrak{a}_{st}$ is not spanned by the $\mathcal{B}$-monomials whose $\mathbf{v}$-values are in $rt\Gamma$. Thus, for $t \gg 0$, there exists $f \in \mathfrak{a}_{st}$ and a $\mathcal{B}$-monomial $\mathbf{b}$ in f such that $\mathbf{v}(\mathbf{b}) \notin rt\Gamma$. Since Γ is closed and convex, this implies that

$$\frac{1}{st} V_{\mathcal{R}}(\mathfrak{a}_{st}) \not\subseteq \frac{rt}{st}\Gamma \text{ for } t \gg 0.$$

That is, $\frac{s}{r} \cdot \frac{1}{st} V_{\mathcal{R}}(\mathfrak{a}_{st}) \not\subseteq \Gamma$ for $t \gg 0$. Therefore,

$$\frac{s}{r} \cdot \Gamma_{\mathcal{R}}(\mathfrak{a}_\bullet) \not\subseteq \Gamma.$$

Conversely, let $s, r \in \mathbb{N}$ be such that $\frac{s}{r}\Gamma_{\mathcal{R}}(\mathfrak{a}_\bullet) \not\subseteq \Gamma$. Since Γ is closed and convex, it follows that there exists $k \in \mathbb{N}$ with

$$\frac{s}{r} \cdot \frac{1}{k} V_{\mathcal{R}}(\mathfrak{a}_k) \not\subseteq \Gamma,$$

i.e., $s \cdot V_{\mathcal{R}}(\mathfrak{a}_k) \not\subseteq rk\Gamma$. Since $\mathbf{v}$ is a vector of valuations and $\mathfrak{a}_k^{st} \subseteq \mathfrak{a}_{kst}$, we have $stV_{\mathcal{R}}(\mathfrak{a}_k) \subseteq V_{\mathcal{R}}(\mathfrak{a}_{kst})$ for all $t \in \mathbb{N}$. This implies that, for all $t \in \mathbb{N}$, $V_{\mathcal{R}}(\mathfrak{a}_{kst}) \not\subseteq rkt\Gamma$, and so there exists $f_t \in \mathfrak{a}_{kst}$ and $\mathbf{b}_t$ a $\mathcal{B}$-monomial in f_t such that $\mathbf{v}(\mathbf{b}_t) \notin rkt\Gamma$. Particularly, $\mathfrak{a}_{kst} \not\subseteq \overline{\mathfrak{b}^{rkt}}$ for all $t \in \mathbb{N}$. Hence,

$$\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}^\bullet}) \leqslant \frac{s}{r}.$$

The result is established. ☐

Theorem 3.6 has a generalization when $\mathfrak{b}_\bullet$ is a Noetherian family. For a graded family $\mathfrak{b}_\bullet = \{\mathfrak{b}_i\}_{i \geqslant 1}$ and a positive integer k, set

$$\mathcal{R}^{[k]}(\mathfrak{b}_\bullet) = \bigoplus_{i \geqslant 0} \mathfrak{b}_{ik} t^i \subseteq R[t].$$

This is the k-th Veronese subalgebra of the Rees algebra $\mathcal{R}(\mathfrak{b}_\bullet)$.

Theorem 3.7. *Let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be graded families of nonzero ideals in R. Assume that for some $k \in \mathbb{N}$, $\mathcal{R}^{[k]}(\mathfrak{b}_\bullet)$ is a standard graded R-algebra, and let $\mathfrak{b}_k^\bullet$ denote the graded family $\{\mathfrak{b}_k^i\}_{i \geqslant 1}$. Suppose that $\mathfrak{b}_k$ has a Rees package $\mathcal{R}_k = (\mathcal{B}_k, \mathbf{v}_k, \Gamma_k)$, in which $\mathbf{v}_k$ is a vector of valuations. Then,*

$$\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) = \frac{1}{k} \sup\{\lambda > 0 \mid \lambda \cdot \Gamma_{\mathcal{R}_k}(\mathfrak{a}_\bullet) \not\subseteq \Gamma_k\}.$$

Proof. By [22, Lemma 4.14], we have

$$\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) = \frac{1}{k} \widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_k^\bullet}).$$

The result then follows from Theorem 3.6. ☐

Recall that a graded family $\mathfrak{b}_\bullet = \{\mathfrak{b}_i\}_{i \geqslant 1}$ is said to be $\mathfrak{b}$-equivalent for an ideal $\mathfrak{b} \subseteq R$ if there exists a positive integer k such that for any $i \geqslant 1$,

$$\mathfrak{b}_{i+k} \subseteq \mathfrak{b}^i \subseteq \mathfrak{b}_i.$$

Our next result computes $\widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = \widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet})$ when $\mathfrak{b}_\bullet$ is $\mathfrak{b}$-equivalent.

Theorem 3.8. *Let $\mathfrak{b} \subseteq R$ be a nonzero ideal with Rees package $\mathcal{R} = (\mathcal{B}, \mathbf{v}, \Gamma)$, in which $\mathbf{v}$ is a vector of valuations. Let $\mathfrak{a}_\bullet = \{\mathfrak{a}_i\}_{i \geqslant 1}$ be a filtration and let $\mathfrak{b}_\bullet = \{\mathfrak{b}_i\}_{i \geqslant 1}$ be a graded family of nonzero ideals in R. Suppose that $\mathfrak{b}_\bullet$ is $\mathfrak{b}$-equivalent. Then,*

$$\widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = \rho(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) = \widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) = \sup\{\lambda > 0 : \lambda \cdot \Gamma_{\mathcal{R}}(\mathfrak{a}_\bullet) \not\subseteq \Gamma\}.$$

Proof. The first two equalities follow from [22, Theorem 4.8]. Also, as observed in [22, Theorem 4.8], since $\mathfrak{b}_\bullet$ is $\mathfrak{b}$-equivalent, by [22, Corollary 3.3], we have

$$\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) = \widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}^\bullet}).$$

The result now follows from Theorem 3.6. ☐

Here, we illustrate Theorems 3.6 and 3.8 using Example 3.5.

Example 3.9. As we saw in Example 3.5, $\Gamma = \text{convexhull}\{(m-1, m-2, \ldots, 1, 0)\} + \mathbb{R}^m_{\geqslant 0}$ and

$$\begin{aligned}\Gamma_{\mathcal{R}}(\mathfrak{a}_\bullet) &= \overline{\bigcup_{k=1}^{\infty} \frac{1}{k} V_{\mathcal{R}}(\mathfrak{a}_k)} \\ &= \text{convexhull}\left(\left\{\frac{1}{2}(m, m-1, \ldots, 2, 1), (m-1, m-2, \ldots, 1, 0)\right\}\right) + \mathbb{R}^m_{\geqslant 0}.\end{aligned}$$

Observe that $\Gamma \subseteq \Gamma_{\mathcal{R}}(\mathfrak{a}_\bullet)$ and the supremum value of λ such that $\lambda \cdot \Gamma_{\mathcal{R}}(\mathfrak{a}_\bullet) \not\subseteq \Gamma$ is determined by where the line connecting the origin and $\frac{1}{2}(m, m-1, \ldots, 2, 1)$ first intersects Γ. This point is computed to be $\left(m-1, \frac{(m-1)^2}{m}, \frac{(m-1)(m-2)}{m}, \dots, \frac{m-1}{m}\right)$. Therefore,

$$\sup\left\{\lambda > 0 \;\middle|\; \lambda \cdot \Gamma_{\mathcal{R}}(\mathfrak{a}_\bullet) \not\subseteq \Gamma\right\} = \frac{2(m-1)}{m}.$$

Hence, due to Theorem 3.6, we get

$$\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) = \frac{2(m-1)}{m}.$$

Note that in this case, it is actually the usual asymptotic resurgence of the ideal $\mathfrak{b}$, which has been recently computed in [29, Section 4].

Coupling Theorems 3.6, 3.8, and the work in [2] on Rees packages, we obtain the following corollary.

Corollary 3.10. *Let $\mathfrak{b} \subseteq R$ be an ideal that belongs to one of the following classes:*

(i) *monomial ideals in affine semigroup rings;*
(ii) *sums of products of determinantal ideals of generic matrices;*
(iii) *sums of products of determinantal ideals of symmetric matrices;*
(iv) *sums of products of ideals of Pfaffians of skew-symmetric matrices;*
(v) *products of determinantal ideals of Hankel matrices.*

Let $\mathcal{R} = (\mathcal{B}, \mathbf{v}, \Gamma)$ be a Rees package of $\mathfrak{b}$. Let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be graded families of ideals in R. Assume either

(1) *$\mathfrak{b}_\bullet = \mathfrak{b}^\bullet$ is the family of ordinary powers of $\mathfrak{b}$; or*
(2) *$\mathfrak{a}_\bullet$ is a filtration and $\mathfrak{b}_\bullet$ is $\mathfrak{b}$-equivalent.*

Then,

$$\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) = \sup\{\lambda > 0 \mid \lambda \cdot \Gamma_{\mathcal{R}}(\mathfrak{a}_\bullet) \not\subseteq \Gamma\}.$$

4. Truncation of graded families

In this section, we investigate whether resurgences of not necessarily Noetherian graded families can be approximated by those of Noetherian families. Particularly, we will look at resurgences of *truncated* families. We shall begin with a necessary definition.

Definition 4.1. *Let $\mathfrak{a}_\bullet$ be a graded family of ideals in R. The n-th* truncation *of $\mathfrak{a}_\bullet$, denoted by $\mathfrak{a}_{n,\bullet} = \{\mathfrak{a}_{n,k}\}_{k\in\mathbb{N}}$, is a graded family of ideals in R, where*

$$\mathfrak{a}_{n,k} = \begin{cases} \mathfrak{a}_k & \text{if } k \leqslant n \\ \sum\limits_{i,j>0,\ i+j=k} \mathfrak{a}_{n,i}\mathfrak{a}_{n,j} & \text{if } k > n. \end{cases}$$

Observe that for $k > n$, $\mathfrak{a}_{n,k}$ is given by sum of products of the form $\prod_{\lambda_1+\cdots+\lambda_s=k} \mathfrak{a}_{\lambda_i}$, where $(\lambda_1, \dots, \lambda_s)$ ranges over the partitions of k with all summands at most n. Our next result shows that truncation works well for understanding the so-called skew Waldschmidt constant of a graded family of ideals. Observe that if $\mathfrak{a}_\bullet$ is a graded family of ideals in a domain R and v is a valuation of the fraction field of R, then $\{v(\mathfrak{a}_n)\}_{n\in\mathbb{N}}$ is subadditive, and so the following limit exists and is called the *skew Waldschmidt* constant of $\mathfrak{a}_\bullet$ with respect to v:

$$\widehat{v}(\mathfrak{a}_\bullet) = \lim_{n\to\infty} \frac{v(\mathfrak{a}_n)}{n} = \inf_{n\in\mathbb{N}} \frac{v(\mathfrak{a}_n)}{n}.$$

The valuation v is said to be supported on R if $v(x) \geqslant 0$ for all $x \in R$.

Theorem 4.2. *Let $\mathfrak{a}_\bullet$ be a graded family of ideals in a domain R.*

(1) *If v is a valuation of fraction field of R that is supported on R, then*

$$\widehat{v}(\mathfrak{a}_\bullet) = \inf_{n \geqslant 1} \widehat{v}(\mathfrak{a}_{n,\bullet}) = \lim_{n\to\infty} \widehat{v}(\mathfrak{a}_{n,\bullet}).$$

(2) *If R is a polynomial ring and $\mathfrak{a}_\bullet$ is a graded family of monomial ideals, then*

$$\Delta(\mathfrak{a}_\bullet) = \overline{\bigcup_{n=1}^{\infty} \Delta(\mathfrak{a}_{n,\bullet})}.$$

Proof. Consider any integers $n \geqslant m \geqslant 1$. Note that $\mathfrak{a}_{m,k} \subseteq \mathfrak{a}_{n,k} \subseteq \mathfrak{a}_k$, for every $k \geqslant 1$. Therefore, $v(\mathfrak{a}_k) \leqslant v(\mathfrak{a}_{n,k}) \leqslant v(\mathfrak{a}_{m,k})$ for every $k \geqslant 1$. This implies that

$$\widehat{v}(\mathfrak{a}_\bullet) \leqslant \widehat{v}(\mathfrak{a}_{n,\bullet}) \leqslant \widehat{v}(\mathfrak{a}_{m,\bullet}) \text{ for all } n \geqslant m \geqslant 1.$$

Furthermore, $\{\widehat{v}(\mathfrak{a}_{n,\bullet})\}_{n\geqslant 1}$ is a non-increasing sequence, hence,

$$\widehat{v}(\mathfrak{a}_\bullet) \leqslant \inf_{n \geqslant 1} \widehat{v}(\mathfrak{a}_{n,\bullet}) = \lim_{n\to\infty} \widehat{v}(\mathfrak{a}_{n,\bullet}).$$

On the other hand, let $\epsilon > 0$ be any positive real number. Then, there exists k_0 such that, for all $k \geqslant k_0$,

$$\frac{v(\mathfrak{a}_k)}{k} < \widehat{v}(\mathfrak{a}_\bullet) + \epsilon.$$

For $n \geqslant k_0$, we then have

$$\widehat{v}(\mathfrak{a}_{n,\bullet}) \leqslant \frac{v(\mathfrak{a}_{n,n})}{n} = \frac{v(\mathfrak{a}_n)}{n} < \widehat{v}(\mathfrak{a}_\bullet) + \epsilon.$$

This is true for any $\epsilon > 0$, so $\lim\limits_{n\to\infty} \widehat{v}(\mathfrak{a}_{n,\bullet}) \leqslant \widehat{v}(\mathfrak{a}_\bullet)$. We have established the equality

$$\lim_{n\to\infty} \widehat{v}(\mathfrak{a}_{n,\bullet}) = \widehat{v}(\mathfrak{a}_\bullet).$$

To prove the second statement, again, consider any integers $n \geqslant m \geqslant 1$. Since $\mathfrak{a}_{m,k} \subseteq \mathfrak{a}_{n,k} \subseteq \mathfrak{a}_k$, for every $k \geqslant 1$, we have

$$\mathrm{NP}(\mathfrak{a}_{m,k}) \subseteq \mathrm{NP}(\mathfrak{a}_{n,k}) \subseteq \mathrm{NP}(\mathfrak{a}_k).$$

Consequently,

$$\Delta(\mathfrak{a}_{m,\bullet}) \subseteq \Delta(\mathfrak{a}_{n,\bullet}) \subseteq \Delta(\mathfrak{a}_\bullet).$$

Thus, $\Delta(\mathfrak{a}_{1,\bullet}) \subseteq \Delta(\mathfrak{a}_{2,\bullet}) \subseteq \cdots$ is an ascending sequence of closed sets.

It is easy to note that

$$\overline{\bigcup_{n=1}^{\infty} \Delta(\mathfrak{a}_{n,\bullet})} \subseteq \Delta(\mathfrak{a}_\bullet).$$

Now consider any $\lambda \in \Delta(\mathfrak{a}_\bullet)$. Then, there exists a sequence $\left\{\frac{\alpha_{n_k}}{n_k}\right\}_{k\geqslant 1}$ with $\alpha_{n_k} \in \mathrm{NP}(\mathfrak{a}_{n_k})$ such that $\lim\limits_{k\to\infty} \frac{\alpha_{n_k}}{n_k} = \lambda$. Note that for every $k \geqslant 1$,

$$\frac{\alpha_{n_k}}{n_k} \in \frac{1}{n_k} \mathrm{NP}(\mathfrak{a}_{n_k,n_k}) \subseteq \Delta(\mathfrak{a}_{n_k,\bullet}).$$

Thus,

$$\frac{\alpha_{n_k}}{n_k} \in \overline{\bigcup_{n=1}^{\infty} \Delta(\mathfrak{a}_{n,\bullet})} \text{ for all } k \geqslant 1.$$

Since $\overline{\bigcup_{n=1}^{\infty} \Delta(\mathfrak{a}_{n,\bullet})}$ is a closed set, we get that $\lambda \in \overline{\bigcup_{n=1}^{\infty} \Delta(\mathfrak{a}_{n,\bullet})}$. Hence, the assertion follows. ☐

Our last main result of the paper is stated as follows.

Theorem 4.3. *Let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be graded families of ideals in a Noetherian commutative ring R. For $n \in \mathbb{N}$, let $\mathfrak{a}_{n,\bullet}$ denote the n-th truncation graded family of $\mathfrak{a}_\bullet$. We have*

$$\rho(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = \sup_{n\geqslant 1} \rho(\mathfrak{a}_{n,\bullet}, \mathfrak{b}_\bullet) = \lim_{n\to\infty} \rho(\mathfrak{a}_{n,\bullet}, \mathfrak{b}_\bullet), \text{ and}$$
$$\widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = \sup_{n\geqslant 1} \widehat{\rho}(\mathfrak{a}_{n,\bullet}, \mathfrak{b}_\bullet) = \lim_{n\to\infty} \widehat{\rho}(\mathfrak{a}_{n,\bullet}, \mathfrak{b}_\bullet).$$

Proof. Since $\mathfrak{a}_{n,\bullet} \subseteq \mathfrak{a}_\bullet$, we have $\rho(\mathfrak{a}_{n,\bullet}, \mathfrak{b}_\bullet) \leqslant \rho(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet)$ for all $n \in \mathbb{N}$. Moreover, since $\mathfrak{a}_{n,\bullet} \subseteq \mathfrak{a}_{n+1,\bullet}$, the sequence $\{\rho(\mathfrak{a}_{n,\bullet}, \mathfrak{b}_\bullet)\}_{n\in\mathbb{N}}$ is a non-decreasing sequence. Thus, the limit $\lim_{n\to\infty} \rho(\mathfrak{a}_{n,\bullet}, \mathfrak{b}_\bullet)$ exists and

$$\rho(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) \geqslant \sup_{n\geqslant 1} \rho(\mathfrak{a}_{n,\bullet}, \mathfrak{b}_\bullet) = \lim_{n\to\infty} \rho(\mathfrak{a}_{n,\bullet}, \mathfrak{b}_\bullet).$$

Suppose that $\rho(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) > \sup\limits_{n\geqslant 1} \rho(\mathfrak{a}_{n,\bullet}, \mathfrak{b}_\bullet)$, then there exists $s, r \in \mathbb{N}$ such that

$$\rho(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) \geqslant \frac{s}{r} > \sup_{n\geqslant 1} \rho(\mathfrak{a}_{n,\bullet}, \mathfrak{b}_\bullet) \text{ and } \mathfrak{a}_s \not\subseteq \mathfrak{b}_r.$$

From $\frac{s}{r} > \sup\limits_{n\geqslant 1} \rho(\mathfrak{a}_{n,\bullet}, \mathfrak{b}_\bullet)$, we have $\mathfrak{a}_{n,s} \subseteq \mathfrak{b}_r$, for every $n \in \mathbb{N}$. In particular, $\mathfrak{a}_s = \mathfrak{a}_{s,s} \subseteq \mathfrak{b}_r$, which is a contradiction. Therefore, $\rho(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = \sup\limits_{n\geqslant 1} \rho(\mathfrak{a}_{n,\bullet}, \mathfrak{b}_\bullet)$. The statement for $\widehat{\rho}$ is proved identically. ☐

We end the paper with an example exhibiting that a similar statement to that of Theorem 4.3 may not hold if truncations of $\mathfrak{b}_\bullet$ are taken instead of those of $\mathfrak{a}_\bullet$.

Example 4.4. In general, $\lim\limits_{n\to\infty} \rho(\mathfrak{a}_\bullet, \mathfrak{b}_{n,\bullet})$ and $\rho(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet)$ may be different (and similarly, $\lim\limits_{n\to\infty} \widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_{n,\bullet})$ and $\widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet)$ may be different, etc.). Consider the families $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ with $\mathfrak{a}_i = I^i$ and

$$\mathfrak{b}_i = \begin{cases} I & \text{if } i \neq 2 \\ I^2 & \text{if } i = 2. \end{cases}$$

By [22, Example 2.10 (2)], we have

$$\rho(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = \frac{1}{2} \text{ and } \widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = \widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) = -\infty < \frac{\widehat{v}(\mathfrak{b}_\bullet)}{\widehat{v}(\mathfrak{a}_\bullet)}.$$

For $n \geqslant 5$, by induction on k, we have the following formula for the elements of the n-th truncation $\mathfrak{b}_{n,\bullet}$:

$$\mathfrak{b}_{n,k} = \begin{cases} I^{\lceil \frac{k}{n} \rceil} & \text{if } k \neq 2 \\ I^2 & \text{if } k = 2. \end{cases}$$

Hence, for each $n \geqslant 5$,

$$\rho(\mathfrak{a}_\bullet, \mathfrak{b}_{n,\bullet}) = \sup\left\{ \frac{s}{r} \;\middle|\; I^s \not\subseteq I^{\lceil \frac{r}{n} \rceil} \right\} = \sup_s \left\{ \frac{s}{sn+1} \right\} = \frac{1}{n}.$$

Therefore, $\lim\limits_{n\to\infty} \rho(\mathfrak{a}_\bullet, \mathfrak{b}_{n,\bullet}) = 0 \neq \rho(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = \dfrac{1}{2}$.

On the other hand, for each $n \geqslant 5$,

$$\widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_{n,\bullet}) = \sup\left\{ \frac{s}{r} \;\middle|\; I^{st} \not\subseteq I^{\lceil \frac{rt}{n} \rceil}, t \gg 1 \right\} = \sup_s \left\{ \frac{s}{sn+1} \right\} = \frac{1}{n}.$$

The second equality holds as $I^{st} \not\subseteq I^{\lceil \frac{rt}{n} \rceil}$ implies that $\frac{rt}{n} > st$, i.e., $rt > stn$, and so $r > sn$; hence, $r \geqslant sn+1$, i.e., $rt \geqslant stn + t$. Therefore, $\lim\limits_{n\to\infty} \widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_{n,\bullet}) = 0 \neq \widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = -\infty$.

Furthermore, if we pick I to be a normal ideal, then $\overline{\mathfrak{b}_{n,i}} = \mathfrak{b}_{n,i}$ and $\overline{\mathfrak{b}_i} = \mathfrak{b}_i$, so

$$\lim_{n\to\infty} \widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_{n,\bullet}}) = \lim_{n\to\infty} \widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_{n,\bullet}) \neq \widehat{\rho}(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = \widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}).$$

Note that in this case if I is a monomial ideal, then we still have

$$\widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}) = \sup\{\lambda \mid \lambda \cdot \Delta(\mathfrak{a}_\bullet) \not\subseteq \Delta(\mathfrak{b}_\bullet), \lambda > 0\}.$$

In fact, we see that $\Delta(\mathfrak{a}_\bullet) = \mathrm{NP}(I)$ and that

$$\Delta(\mathfrak{b}_\bullet) = \overline{\bigcup_{k=1}^{\infty} \frac{1}{k} \mathrm{NP}(\mathfrak{b}_k)} = \overline{\bigcup_{k=1}^{\infty} \frac{1}{k} \mathrm{NP}(I)} = \mathbb{R}^n_{\geqslant 0}.$$

Thus,

$$\sup\{\lambda > 0 \mid \lambda \cdot \Delta(\mathfrak{a}_\bullet) \not\subseteq \Delta(\mathfrak{b}_\bullet), \lambda > 0\} = -\infty = \widehat{\rho}(\mathfrak{a}_\bullet, \overline{\mathfrak{b}_\bullet}).$$

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