ALGEBRAIC COMBINATORICS

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Volume 9, issue 4 (2026), p. 953-972.
https://doi.org/10.5802/alco.499

*Algebraic Combinatorics is published by The Combinatorics Consortium
and is a member of the Centre Mersenne for Open Scientific Publishing*
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

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Double orthodontia formulas and Lascoux positivity

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Abstract We give a new formula for double Grothendieck polynomials based on Magyar's orthodontia algorithm for diagrams. Our formula implies a similar formula for double Schubert polynomials $\mathfrak{S}_w(\mathbf{x};\mathbf{y})$. We also prove a curious positivity result: for vexillary permutations $w \in S_n$, the polynomial $x_1^n \dots x_n^n \mathfrak{S}_w(x_n^{-1}, \dots, x_1^{-1}; 1, \dots, 1)$ is a graded nonnegative sum of Lascoux polynomials. We conjecture that this positivity result holds for all $w \in S_n$. This conjecture would follow from a problem of independent interest regarding Lascoux positivity of certain products of Lascoux polynomials.

1. Introduction

1.1. Orthodontia and flagged Weyl modules. The Schubert polynomials, denoted $\mathfrak{S}_w(x_1, \dots, x_n)$ and introduced by Lascoux and Schützenberger [17], are distinguished representatives of Schubert varieties in the cohomology of the flag variety of $\mathbb{C}^n$. These polynomials have very rich combinatorial structure [1, 2, 6, 11, 14, 9] and play a central role in algebraic combinatorics.

Schubert polynomials $\mathfrak{S}_w(\mathbf{x})$ are indexed by permutations $w \in S_n$. The Rothe diagram $D(w)$ of $w \in S_n$ encodes a plethora of information about $\mathfrak{S}_w(\mathbf{x})$. More generally, for any %-avoiding diagram D, there is a representation of the group B of invertible upper triangular matrices called the *flagged Weyl module* $\mathcal{M}_D$. When $D = D(w)$ is a Rothe diagram, the dual character χ_D of the representation $\mathcal{M}_D$ is equal to the Schubert polynomial $\mathfrak{S}_w(\mathbf{x})$ [12, 13]. The study of dual characters χ_D as a whole has shed light on Schubert and related polynomials [5, 8].

Using a geometric interpretation of $\mathcal{M}_D$ as the space of sections of a certain line bundle on a variety, Magyar [19] showed that χ_D is given by the formula

$$\chi_D = \omega_1^{k_1} \dots \omega_n^{k_n} \pi_{i_1}(\omega_{i_1}^{m_1} \pi_{i_2}(\omega_{i_2}^{m_2}(\dots \pi_{i_\ell}(\omega_{i_\ell}^{m_\ell}) \dots))).$$

Here, $\omega_i = x_1 \dots x_i$ is a fundamental weight, $\pi_i = \partial_i x_i$ is a Demazure operator, and

$$\mathbf{k}(D) = (k_1, \dots, k_n), \qquad \mathbf{i}(D) = (i_1, \dots, i_\ell), \qquad \mathbf{m}(D) = (m_1, \dots, m_\ell)$$

is combinatorial data associated to the *orthodontic sequence* of D, which builds a %-avoiding diagram from "smaller" %-avoiding diagrams.

Manuscript received 3rd January 2026, accepted 20th February 2026.

Keywords. Grothendieck polynomials, Lascoux polynomials.

1.2. Doubled orthodontia. The double Grothendieck polynomials, denoted $\mathfrak{G}_w(x_1,\dots,x_n;y_1,\dots,y_n)$ and introduced by Lascoux [16], are distinguished representatives of Schubert varieties in the equivariant K-theory of the flag variety. Schubert polynomials can be obtained from double Grothendieck polynomials by setting $y_i \mapsto 0$ (corresponding to forgetting equivariance) and taking the lowest degree part (corresponding to taking the associated graded in K-theory). There is considerable interest in extending our understanding of Schubert polynomials to double Grothendieck polynomials and their various specializations [7, 10, 26, 28, 15, 4, 3, 22, 8].

There is no known K-theoretic analogue of $\mathcal{M}_D$. Despite this, we can combinatorially extend Magyar's formula to double Grothendieck polynomials:

Theorem 1.1. *Let D be a %-avoiding diagram with double orthodontic sequence $\mathbf{K},\mathbf{i},\mathbf{j},\mathbf{M}$. Define*

$$\mathscr{G}_D(\mathbf{x};\mathbf{y}) := \overline{\omega}_1^{K_1}\overline{\omega}_2^{K_2}\dots\overline{\omega}_n^{K_n}\overline{\pi}_{i_1,j_1}(\overline{\omega}_{i_1}^{M_1}\overline{\pi}_{i_2,j_2}(\overline{\omega}_{i_2}^{M_2}\dots\overline{\pi}_{i_\ell,j_\ell}(\overline{\omega}_{i_\ell}^{M_\ell})\dots)). \tag{1}$$

When $D = D(w)$ is the Rothe diagram of a permutation, then $\mathscr{G}_D(\mathbf{x};\mathbf{y}) = \mathfrak{G}_w(\mathbf{x};\mathbf{y})$.

The operators $\overline{\omega}_i^M$ and $\overline{\pi}_{i,j}$ are defined in Equations (2) and (3).

Double Schubert polynomials $\mathfrak{S}_w(\mathbf{x};\mathbf{y})$ are equal to the lowest degree part of $\mathfrak{G}_w(\mathbf{x};-\mathbf{y})$, and Theorem 1.1 implies a similar formula for double Schubert polynomials (Corollary 3.12). Theorem 1.1 also extends our previous joint work with Mészáros [20, Thm 1.1], which gave a similar formula for ordinary Grothendieck polynomials.

As is the case with Magyar's formula for χ_D, and in contrast to the usual degree-*decreasing* recurrence defining Grothendieck polynomials via divided difference operators, the formula (1) is degree-*increasing.* This makes the formula (1) well suited for some combinatorial applications.

1.3. Lascoux positivity. When $D = D(\alpha)$ is the skyline diagram of a composition $\alpha \in \mathbb{N}^n$, the dual character χ_D is the *key polynomial* $\kappa_\alpha(x_1,\dots,x_n)$ [23]. Key polynomials are minimal among the dual characters χ_D of %-avoiding diagrams: every dual character, and in particular every Schubert polynomial, is a nonnegative sum of key polynomials [24].

The Lascoux polynomials $\mathfrak{L}_\alpha(x_1,\dots,x_n)$, indexed by compositions $\alpha \in \mathbb{N}^n$, are inhomogeneous polynomials which often play the same role to the key polynomials as Grothendieck polynomials do to Schubert polynomials. One inhomogeneous extension of the key positivity [24] of Schubert polynomials is the Grothendieck-to-Lascoux expansion [25, 27]. One application of our formula (1) is another inhomogeneous extension of this key positivity:

Theorem 1.2. *Let $D \subseteq [n]\times[m]$ be a diagram whose columns are ordered by inclusion. Let $\mathscr{S}_D(\mathbf{x};\mathbf{y})$ be the lowest degree part of $\mathscr{G}_D(\mathbf{x};\mathbf{y})$. Then, the polynomial*

$$x_1^m\dots x_n^m\mathscr{S}_D(x_n^{-1},\dots,x_1^{-1};-1,\dots,-1)$$

is a graded nonnegative sum of Lascoux polynomials $\mathfrak{L}_\alpha(x_1,\dots,x_n)$.

Theorem 1.2 can be modified to incorporate β-Lascoux polynomials: the β-Lascoux expansion of $x_1^m\dots x_n^m\mathscr{S}_D(x_n^{-1},\dots,x_1^{-1};\beta,\dots,\beta)$ is $\mathbb{Z}_{\geqslant 0}[\beta]$-nonnegative.

Theorem 1.2 and Corollary 3.12 together imply the following corollary.

Corollary 1.3. *Let $w \in S_n$ be a vexillary permutation and denote the corresponding double Schubert polynomial by $\mathfrak{S}_w(x_1,\dots,x_n;y_1,\dots,y_n)$. Then the polynomial*

$$x_1^n\dots x_n^n\mathfrak{S}_w(x_n^{-1},\dots,x_1^{-1};1,\dots,1)$$

is a graded nonnegative sum of Lascoux polynomials $\mathfrak{L}_\alpha(x_1,\dots,x_n)$.

We conjecture that the extra assumption on the diagram D in Theorem 1.2 is unnecessary.

Conjecture 1.4. *For any %-avoiding diagram $D \subseteq [n] \times [m]$, the polynomial*

$$x_1^m \dots x_n^m \mathscr{S}_D(x_n^{-1}, \dots, x_1^{-1}; -1, \dots, -1)$$

is a graded nonnegative sum of Lascoux polynomials $\mathfrak{L}_\alpha(x_1, \dots, x_n)$. In particular, the same holds for

$$x_1^n \dots x_n^n \mathfrak{G}_w(x_n^{-1}, \dots, x_1^{-1}; 1, \dots, 1),$$

for any $w \in S_n$.

In Proposition 4.4, we show that Conjecture 1.4 would follow from the following conjecture of independent interest:

Conjecture 1.5. *The product $x_1 \dots x_i(1 - x_{i+1}) \dots (1 - x_n)\mathfrak{L}_\alpha$, for any $\alpha \in \mathbb{N}^n$ and $i \in [n]$, is a graded nonnegative linear combination of Lascoux polynomials.*

1.4. Outline of the paper. Our proof of Theorem 1.1 is based on an improvement on the ideas in [20]: we induct on a partial order on S_n called the *orthodontic sort order* (Definition 2.16), and a key step is to give a precise description of the first few terms of the orthodontic sequences of *sorted permutations* (Theorem 3.9). Although the argument is narratively similar to that in [20], we need different methods.

Our proof of Theorem 1.2 examines the orthodontic sequence of diagrams D whose columns are totally ordered by inclusion (Corollary 4.6) and studying the behavior of the operators ω_i^M and $\pi_{i,j}$ when specializing $y_j \mapsto -1$ and intertwining with the operator $r_{m,n}$ of [29, Defn 1.1] defined on polynomials by

$$f(x_1, \dots, x_n) \mapsto x_1^m \dots x_n^m f(x_n^{-1}, \dots, x_1^{-1}).$$

2. Background

2.1. Conventions. We write permutations $w \in S_n$ in one-line notation. For $j \in [n-1]$, the notation s_j denotes the adjacent transposition in S_n which swaps j and $j+1$. For $w \in S_n$, let $\ell(w)$ denote the length of w. Permutations act on the right: ws_j is equal to w with $w(j)$ and $w(j+1)$ swapped.

2.2. Difference operators and families of polynomials. For $i \in [n-1]$, define the *divided difference operator* $\partial_i \colon \mathbb{R}[x_1, \dots, x_n] \to \mathbb{R}[x_1, \dots, x_n]$ by the formula

$$\partial_i(f) := \frac{f - s_i f}{x_i - x_{i+1}}.$$

The *isobaric divided difference operators* $\overline{\partial}_i$, *Demazure operators* π_i, and *Demazure–Lascoux operators* $\overline{\pi}_i$ are defined on $\mathbb{R}[x_1, \dots, x_n]$ by the formulas

$$\begin{aligned}
\overline{\partial}_i(f) &:= \partial_i((1 - x_{i+1})f),\\
\pi_i(f) &:= \partial_i(x_i f),\\
\overline{\pi}_i(f) &:= \overline{\partial}_i(x_i f).
\end{aligned}$$

Lemma 2.1. *Let φ_i denote any of ∂_i, $\overline{\partial}_i$, π_i, or $\overline{\pi}_i$. Then*

$$\varphi_i\varphi_{i+1}\varphi_i = \varphi_{i+1}\varphi_i\varphi_{i+1} \qquad \text{and} \qquad \varphi_i\varphi_j = \varphi_j\varphi_i \text{ if } |i-j| \geqslant 2.$$

The *double Grothendieck polynomial* $\mathfrak{G}_w(\mathbf{x}; \mathbf{y})$ of $w \in S_n$ is defined recursively on the weak Bruhat order, starting from the longest permutation $w_0 \in S_n$. The polynomial $\mathfrak{G}_w(\mathbf{x}; \mathbf{y})$ is defined by

$$\mathfrak{G}_w(\mathbf{x}; \mathbf{y}) = \begin{cases} \prod_{i+j \leqslant n}(x_i + y_j - x_i y_j) & \text{if } w = w_0,\\ \overline{\partial}_i \mathfrak{G}_{ws_i}(\mathbf{x}; \mathbf{y}) & \text{if } \ell(w) < \ell(ws_i). \end{cases}$$

Lemma 2.1 guarantees that double Grothendieck polynomials are well-defined. The *double Schubert polynomial* $\mathfrak{S}_w(\mathbf{x};\mathbf{y})$ is defined to be the lowest degree part of $\mathfrak{G}_w(\mathbf{x};-\mathbf{y})$. In particular,

$$\mathfrak{S}_w(\mathbf{x};\mathbf{y}) = \begin{cases} \prod_{i+j\leqslant n}(x_i - y_j) & \text{if } w = w_0, \\ \partial_i \mathfrak{S}_{ws_i}(\mathbf{x};\mathbf{y}) & \text{if } \ell(w) < \ell(ws_i). \end{cases}$$

The ordinary *Grothendieck polynomials* $\mathfrak{G}_w(\mathbf{x})$ and ordinary *Schubert polynomials* $\mathfrak{S}_w(\mathbf{x})$ are defined to be the specializations $\mathfrak{G}_w(\mathbf{x};\mathbf{0})$ and $\mathfrak{S}_w(\mathbf{x};\mathbf{0})$ of their doubled counterparts. In particular,

$$\mathfrak{G}_w(\mathbf{x}) = \begin{cases} x_1^n x_2^{n-1}\dots x_n & \text{if } w = w_0, \\ \overline{\partial}_i \mathfrak{G}_{ws_i}(\mathbf{x}) & \text{if } \ell(w) < \ell(ws_i), \end{cases} \qquad \text{and}$$

$$\mathfrak{S}_w(\mathbf{x}) = \begin{cases} x_1^n x_2^{n-1}\dots x_n & \text{if } w = w_0, \\ \partial_i \mathfrak{S}_{ws_i}(\mathbf{x}) & \text{if } \ell(w) < \ell(ws_i). \end{cases}$$

The symmetric group S_n acts on compositions via permuting coordinates. The *Lascoux polynomial* $\mathfrak{L}_\alpha$ of a composition $\alpha \in \mathbb{N}^n$ is defined recursively by

$$\mathfrak{L}_\alpha(\mathbf{x}) = \begin{cases} x_1^{\alpha_1}\dots x_n^{\alpha_n} & \text{if } \alpha_1 \geqslant \dots \geqslant \alpha_n \\ \overline{\pi}_i \mathfrak{L}_{\alpha\cdot s_i}(\mathbf{x}) & \text{if } \alpha_i < \alpha_{i+1}. \end{cases}$$

Lemma 2.1 guarantees that Lascoux polynomials are well-defined. The *key polynomial* $\kappa_\alpha(\mathbf{x})$ is defined to be the lowest degree part of $\mathfrak{L}_\alpha(\mathbf{x})$. In particular,

$$\kappa_\alpha(\mathbf{x}) = \begin{cases} x_1^{\alpha_1}\dots x_n^{\alpha_n} & \text{if } \alpha_1 \geqslant \dots \geqslant \alpha_n \\ \pi_i \kappa_{\alpha\cdot s_i}(\mathbf{x}) & \text{if } \alpha_i < \alpha_{i+1}. \end{cases}$$

2.3. Pipe dreams. A *pipe dream* is a filling of a triangular grid $\{(i,j)\in[n]\times[n]\colon i+j\leqslant n\}$ with crossing tiles $\boxplus$ and bumping tiles . By placing half-bumping tiles at $\{(i,j)\in[n]\times[n]\colon i+j=n+1\}$, a pipe dream forms a network of n pipes running from the top edge of the grid to the left edge (see Figure 1).

For any pipe dream P, there is an associated permutation $\partial(P)\in S_n$ given by labeling the pipes 1 through n along the top edge, tracing the pipes and ignoring any crossings between any pair of pipes which have already crossed, i.e. replacing redundant crossing tiles with bump tiles; reading the labels of the pipes along the left edge from top to bottom gives a string of numbers $(\partial(P))(1),(\partial(P))(2),\dots,(\partial(P))(n)$ that defines $\partial(P)\in S_n$. (The permutation $\partial(P)$ is the *Demazure product* of the transpositions $s_i\in S_n$ corresponding to antidiagonals on which the crosses sit, reading right to left, starting from the top row; cf. [10, Ex 5.1], [28, §6.1]. Our convention agrees with the original definition [1]: the pipe at the i^{th} row is connected to the $(\partial(P))(i)^{\text{th}}$ column.)

Identify P with the set $\{(i,j)\colon P \text{ has } \boxplus \text{ at } (i,j)\}$ of crosses of P, and denote by $P^{(i)}$ the pipe of P entering in the i^{th} column.

Example 2.2. All five pipe dreams P such that $\partial(P)=1423$ are displayed in Figure 1. Redundant crossing tiles $\boxplus$ are highlighted in red.

Theorem 2.3 ([28, Thm 6.1]; see also [10, Cor 5.4], [7, Thm 2.3]). *The double Grothendieck polynomial of $w\in S_n$ is given by the formula*

$$\mathfrak{G}_w(\mathbf{x};\mathbf{y}) = \sum_{P\in\mathrm{PD}(w)} \prod_{(i,j)\in P} (-1)^{|P|-\ell(w)}(x_i + y_j - x_i y_j).$$

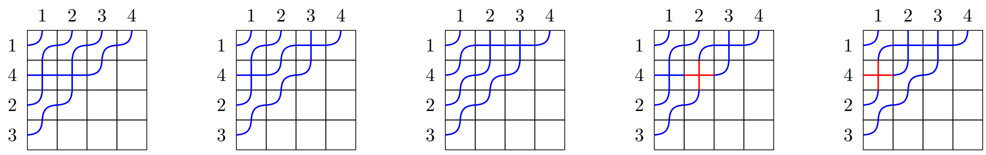

FIGURE 1. The five pipe dreams in PD(1423).

2.4. ORTHODONTIC SEQUENCE. A *diagram* is defined to be a subset $D \subseteq [n] \times [m]$. View $D = (D_1, \ldots, D_m)$ as a subset of an $n \times m$ grid, where $D_j := \{i \in [n] \colon (i,j) \in D\}$ encodes the j^{th} column: an element $i \in D_j$ corresponds to a box in row i and column j.

DEFINITION 2.4. *Let $w \in S_n$. The Rothe diagram $D(w)$ is defined to be*

$$D(w) = \{(i,j) \in [n] \times [n] \colon i < w^{-1}(j) \text{ and } j < w(i)\}.$$

EXAMPLE 2.5. The Rothe diagram $D(31542)$ consists of the purple squares in Figure 2.

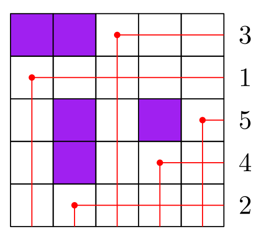

FIGURE 2. The Rothe diagram for $w = 31542$.

DEFINITION 2.6 ([24]). *A diagram D is called* %-avoiding *when $i_2 \in D_{j_1}$ and $i_1 \in D_{j_2}$ for $i_1 < i_2$ and $j_1 < j_2$ implies $i_1 \in D_{j_1}$ or $i_2 \in D_{j_2}$.*

Equivalently, a diagram is %-avoiding if it does not have a pair of rows and a pair of columns to which its restriction looks like the configuration in Figure 3.

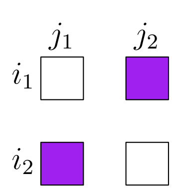

FIGURE 3. A forbidden configuration in a %-avoiding diagram.

Given a %-avoiding diagram D, we describe an algorithm to compute its *double orthodontic sequence*

$$\mathbf{K}(D) = (K_1, \ldots, K_n), \ \ \mathbf{i}(D) = (i_1, \ldots, i_\ell), \ \ \mathbf{j}(D) = (j_1, \ldots, j_\ell), \ \ \mathbf{M}(D) = (M_1, \ldots, M_\ell).$$

Begin by setting $K_i := \{j \colon D_j = [i]\}$, and let D_- denote the diagram obtained by replacing any such columns with the empty column. If D_- is empty, then $\mathbf{i} = \mathbf{j} = \mathbf{M}$ is the empty vector and the algorithm terminates.

An *orthodontic step* applied to D_- outputs a diagram $(D_-)''$ which we now describe. A *missing tooth* of a column $C \subseteq [n]$ is an integer $i \in [n]$ so that $i \notin C$ and $i+1 \in C$; any nonempty column of D_- has a missing tooth. Set i_1 to be the smallest missing tooth of the leftmost nonempty column $(D_-)_j$ of D_-. Set j_1 to be $j - \#\{a \leqslant i_1 : a \notin (D_-)_j\}$. Now swap rows i_1 and i_1+1 to get a diagram $(D_-)'$. Any column of $(D_-)'$ which is a standard interval is necessarily equal to $[i_1]$; set $M_1 = \{j : (D_-)'_j = [i_1]\}$. Let $(D_-)''$ denote the diagram obtained from $(D_-)'$ by removing all columns equal to $[i_1]$.

LEMMA 2.7 ([24, Prop 12], see also [19, pg. 12]). *Any %-avoiding diagram D will be the empty diagram after sufficiently many orthodontic steps.*

Repeatedly apply orthodontic steps to the resulting diagrams $(D_-)''$, keeping track of the data $\mathbf{i}, \mathbf{j}, \mathbf{M}$ at each step, until the diagram is empty. These diagrams form the double orthodontic sequence of D.

EXAMPLE 2.8. The orthodontic sequence for the Rothe diagram $D(31542)$ is shown in Figure 4.

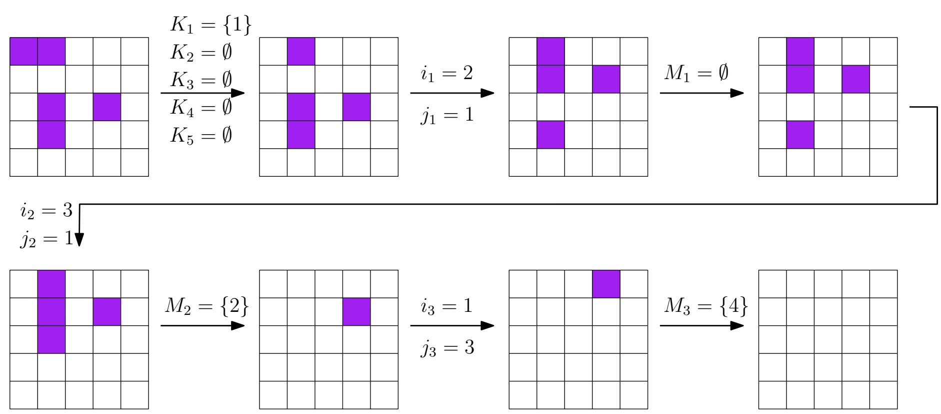

FIGURE 4. The double orthodontic sequence for $w = 31542$.

For a polynomial $f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$, define the operators

$$\pi_{i,j} := \partial_i((x_i + y_j)f), \qquad \overline{\pi}_{i,j} := \overline{\partial}_i((x_i + y_j - x_i y_j)f). \tag{2}$$

For $i \in [n]$ and $M \subseteq [n]$, define the quantities

$$\omega_i^M := \prod_{\substack{j \in [i] \\ m \in M}} (x_j + y_m), \qquad \overline{\omega}_i^M := \prod_{\substack{j \in [i] \\ m \in M}} (x_j + y_m - x_j y_m). \tag{3}$$

DEFINITION 2.9. *Let D be a %-avoiding diagram with double orthodontic sequence* $\mathbf{K}, \mathbf{i}, \mathbf{j}, \mathbf{M}$. *Define*

$$\mathscr{G}_D(\mathbf{x}; \mathbf{y}) := \overline{\omega}_1^{K_1} \overline{\omega}_2^{K_2} \dots \overline{\omega}_n^{K_n} \overline{\pi}_{i_1, j_1}(\omega_{i_1}^{M_1} \overline{\pi}_{i_2, j_2}(\omega_{i_2}^{M_2} \dots \overline{\pi}_{i_\ell, j_\ell}(\omega_{i_\ell}^{M_\ell}) \dots)),$$

and write

$$\mathscr{S}_D(\mathbf{x}; \mathbf{y}) := \omega_1^{K_1} \omega_2^{K_2} \dots \omega_n^{K_n} \pi_{i_1, j_1}(\omega_{i_1}^{M_1} \pi_{i_2, j_2}(\omega_{i_2}^{M_2} \dots \pi_{i_\ell, j_\ell}(\omega_{i_\ell}^{M_\ell}) \dots)).$$

REMARK 2.10. It can be shown by inducting down the orthodontic sequence that the monomial $\prod_{(i,j) \in D} x_i$ appears in $\mathscr{S}_D$ with coefficient 1. As $\mathscr{S}_D$ is nonzero, it follows that $\mathscr{S}_D$ is the lowest degree part of $\mathscr{G}_D$.

REMARK 2.11. The polynomial $\mathscr{G}_D(\mathbf{x}; \mathbf{y})$ is not invariant under permuting columns, in contrast to previous work [19, 20] on the orthodontia algorithm. This is by design: the Rothe diagram $D(2413)$ can be obtained from the Rothe diagram $D(132)$

by permuting the columns and adding a column equal to $\{1,2\}$, and although the ordinary Grothendieck polynomials satisfy $\mathfrak{G}_{2413}(\mathbf{x}) = x_1x_2\mathfrak{G}_{132}(\mathbf{x})$, there does not exist a polynomial $g(\mathbf{x};\mathbf{y})$ so that $\mathfrak{G}_{2413}(\mathbf{x};\mathbf{y}) = g(\mathbf{x};\mathbf{y}) \cdot \mathfrak{G}_{132}(\mathbf{x};\mathbf{y})$.

2.5. Orthodontic sort order on permutations. Following [20], we recall some basic facts about sorted permutations and the sorting projection.

In what follows, a *standard interval* is a set of the form $[j]$ for some $j \geqslant 0$. Recall that a permutation $w \in S_n$ is called *dominant* if it is 132-avoiding. A permutation w is dominant if and only if its Rothe diagram $D(w) = (D(w)_1, \ldots, D(w)_n)$ consists only of standard intervals, which necessarily satisfy $\#D(w)_1 \geqslant \cdots \geqslant \#D(w)_n$.

Definition 2.12. *Fix a permutation $w \in S_n$. The* primary column data *of w, denoted $(h, C, \alpha, i_1, \beta)$, is defined as follows.*

If w is not dominant, the diagram $D(w)$ has a column which is not a standard interval. Set h to be the smallest integer such that the column $D(w)_{h+1}$ is not a standard interval. Set C to be the column $D(w)_{h+1} \subsetneq [n]$. Set α to be the largest integer such that $[\alpha] \subseteq C$. Set i_1 to be the smallest missing tooth of C. Lastly, set $\beta = i_1 - \alpha$, the size of the "uppermost gap" of C.

If w is dominant, set $h = n$, $C = \varnothing$, $\alpha = 0$, $i_1 = n$, and $\beta = n$.

Lemma 2.13 ([20, Lem 3.4]). *Any permutation w restricts to a bijection $[\alpha+1, i_1] \to [h-\beta+1, h]$. The corresponding permutation $\sigma \in S_\beta$ is dominant.*

See Figure 5 for an example.

For $w \in S_n$, write $\sigma(w)$ for the dominant permutation obtained by restricting w to $[\alpha+1, i_1]$.

Definition 2.14. *The permutation w is* sorted *if $\sigma(w)$ is the identity. The* sorting *of w, denoted w_{sort}, is the permutation obtained from w by reordering $w(\alpha+1), \ldots, w(i_1)$ to be in increasing order.*

Example 2.15. Let $w = 68342751$. The primary column data of w is

$$h = 4,\ C = \{1,2,6\},\ \alpha = 2,\ i_1 = 5,\ \beta = 3.$$

Note that w restricts to a bijection $\{3,4,5\} \to \{2,3,4\}$; the corresponding permutation $\sigma(w) \in S_3$ is $\sigma(w) = 231$. The sorting of w is $w_{\mathrm{sort}} = 68234751$. The Rothe diagrams of w and w_{sort} are displayed in Figure 5.

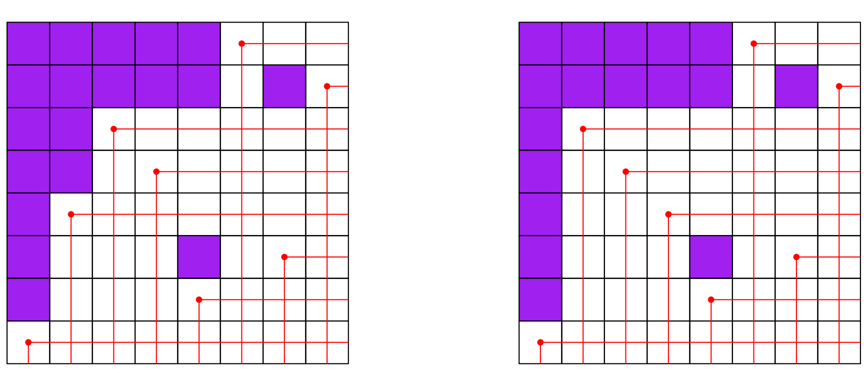

Figure 5. The Rothe diagram of $w = 68342751$ on the left, and of its sorting $w_{\mathrm{sort}} = 68234751$ on the right. In this example, $\sigma(w) = 231$.

Definition 2.16. *The* orthodontic sort order $\leqslant_{\mathrm{os}}$ *is the reflexive and transitive closure of the relations*

$$(\dagger)\qquad w_{\mathrm{sort}} \preceq w$$

$$(\ddagger)\qquad ws_{i_1}\dots s_{\alpha+1} \preceq w \text{ whenever } w \text{ is nonidentity and sorted.}$$

Proposition 2.17 ([20, Prop 5.6]). *The relation $\leqslant_{\mathrm{os}}$ is a partial order on S_n and the identity permutation is the minimum element.*

3. Orthodontia and double Grothendieck polynomials

We establish Theorem 1.1 by studying the relationship between $\mathfrak{G}_w$ and $\mathfrak{G}_{w_{\mathrm{sort}}}$ (Proposition 3.7), the relationship between $\mathscr{G}_{D(w)}$ and $\mathscr{G}_{D(ws_{i_1}\dots s_\alpha)}$ (Theorem 3.9), and then inducting on the orthodontic sort order (Definition 2.16).

The following example motivates Proposition 3.2 and Corollary 3.3.

Example 3.1. Let $w = 68342751$, as in Example 2.15, so that $w_{\mathrm{sort}} = 68234751$. Observe that $D(w_{\mathrm{sort}}) = D(w) \smallsetminus \{(2,3),(2,4)\}$ is obtained from $D(w)$ by removing bottom-aligned portions of standard interval columns. Proposition 3.2 asserts that this phenomenon holds in general.

Write $\mathbf{K}(w) = (K_1,\dots,K_n)$ and $\mathbf{K}(w_{\mathrm{sort}}) = (K'_1,\dots,K'_n)$. Then:

$$K_1 = \varnothing,\quad K_2 = \{3,4\},\quad K_3 = \varnothing,\quad K_4 = \{2\},\quad K_5 = \varnothing,\quad K_6 = \varnothing,\quad K_7 = \{1\}$$

and

$$K'_1 = \varnothing,\quad K'_2 = \{2,3,4\},\quad K'_3 = \varnothing,\quad K'_4 = \varnothing,\quad K'_5 = \varnothing,\quad K'_6 = \varnothing,\quad K'_7 = \{1\}.$$

The fact that $K'_4 = K_4 \smallsetminus \{2\}$ and $K'_2 = K_2 \cup \{2\}$ can be viewed as a consequence of the fact that $D(\sigma(w)) = D(231)$ has Rothe diagram consisting of one standard interval column of size two.

As $D(w_{\mathrm{sort}})$ can be obtained from $D(w)$ by removing bottom-aligned portions of standard interval columns, the two diagrams agree after removing standard interval columns. It follows that the $\mathbf{i},\mathbf{j},\mathbf{M}$ orthodontic sequences of w and w_{sort} agree: $\mathbf{i}(w) = \mathbf{i}(w_{\mathrm{sort}})$, $\mathbf{j}(w) = \mathbf{j}(w_{\mathrm{sort}})$, and $\mathbf{M}(w) = \mathbf{M}(w_{\mathrm{sort}})$. This phenomenon holds more generally, and is stated precisely in Corollary 3.3.

Proposition 3.2. *Let $(h, C, \alpha, i_1, \beta)$ denote the primary column data of $w \in S_n$. Set $\lambda^\sigma_j := \#D(\sigma(w))_j$. The Rothe diagram $D(w_{\mathrm{sort}})$ can be obtained from $D(w)$ by removing bottom-aligned portions of standard-interval columns, and*

$$(4)\qquad D(w) \smallsetminus D(w_{\mathrm{sort}}) = \{(\alpha + a, h - \beta + b)\colon 1 \leqslant a \leqslant \lambda^\sigma_b \text{ and } 1 \leqslant b \leqslant \beta\}.$$

Proof of Proposition 3.2. As w_{sort} is obtained from w by reordering numbers $\{w^{-1}(j)\colon j \in [h-\beta+1, h]\}$, the columns $D(w_{\mathrm{sort}})_j$ and $D(w)_j$ agree whenever $j \notin [h-\beta+1, h]$. By definition of the primary column data and of the sorting of a permutation, $D(w)_{h-\beta+b} = [\alpha + \lambda^\sigma_b]$ and $D(w_{\mathrm{sort}})_{h-\beta+b} = [\alpha]$. ☐

Corollary 3.3 (cf. [20, Prop 4.4]). *Let $(h, C, \alpha, i_1, \beta)$ denote the primary column data of $w \in S_n$. Set*

$$\mathcal{S}_a := \left\{ h - \beta + j\colon \begin{array}{c} j \in [\beta] \\ \#D(\sigma(w))_j = a \end{array} \right\}.$$

Write $\mathbf{K}(w) = (K_1,\dots,K_n)$ and $\mathbf{K}(w_{\mathrm{sort}}) = (K'_1,\dots,K'_n)$. Then

$$\mathbf{i}(w) = \mathbf{i}(w_{\mathrm{sort}}),\qquad \mathbf{j}(w) = \mathbf{j}(w_{\mathrm{sort}}),\qquad \mathbf{M}(w) = \mathbf{M}(w_{\mathrm{sort}}),$$

and

$$K_j = \begin{cases} K'_j & \text{if } j \leqslant \alpha - 1, \\ K'_j \smallsetminus \{h-\beta+1, \dots, h\} \cup \mathcal{S}_0 & \text{if } j = \alpha, \\ K'_j \cup \mathcal{S}_a & \text{if } j = \alpha + a, \text{ where } a \in [\beta], \\ K'_j & \text{if } j \geqslant i_1 + 1. \end{cases}$$

In particular,

$$\mathscr{G}_{D(w)} = \left(\prod_{(a,b) \in D(w) \smallsetminus D(w_{\mathrm{sort}})} (x_a + y_b - x_a y_b) \right) \cdot \mathscr{G}_{D(w_{\mathrm{sort}})}.$$

Proof. Proposition 3.2 implies that the diagrams $D(w)$ and $D(w_{\mathrm{sort}})$ agree after removing their standard interval columns. It follows that the orthodontic sequences $\mathbf{i}, \mathbf{j}, \mathbf{M}$ of w and w_{sort} agree. The formulas for K_j in terms of K'_j hold by (4). The relationship between $\mathscr{G}_{D(w)}$ and $\mathscr{G}_{D(w_{\mathrm{sort}})}$ follows from the construction (1) of $\mathscr{G}_D$. $\square$

Lemma 3.4. *Let $(h, C, \alpha, i_1, \beta)$ denote the primary column data of $w \in S_n$. For $j \in [h]$, set*

$$\lambda_j := \#D(w)_j \quad \text{and} \quad \nu_j := \#\{i \leqslant j \colon \#D(w)_i \geqslant w^{-1}(j)\}.$$

Fix any $P \in \mathrm{PD}(w)$. Then for every $j \in [h]$, the pipe $P^{(j)}$ begins by traversing λ_j crossing tiles vertically, ends by traversing ν_j tiles horizontally, and otherwise traverses only bump tiles. In particular, every tile in

$$C_w := \left\{ (i,j) \colon \begin{matrix} 1 \leqslant i \leqslant \lambda_j, \\ 1 \leqslant j \leqslant h \end{matrix} \right\}$$

is a cross, and every tile in

$$E_w := \left\{ (i,j) \colon \begin{matrix} h - \beta + 1 \leqslant i \leqslant h, \\ \alpha + 1 \leqslant j \leqslant i_1, \\ i + j \leqslant h + \alpha + 1 \end{matrix} \right\} \smallsetminus C_w$$

is an elbow.

Example 3.5. Figure 6 gives an example of a typical pipe dream, with the structure guaranteed by Lemma 3.4 highlighted.

Proof of Lemma 3.4. Induct on j. The pipe $P^{(1)}$ necessarily traverses λ_1 cross tiles vertically before traversing a bump tile to exit at the $w^{-1}(1)^{\text{th}}$ row. Now assume that, for all $k < j$, the pipes $P^{(k)}$ traverse λ_k cross tiles vertically, then traverse bump tiles, before ending by traversing ν_k tiles horizontally.

As $\lambda_1 \geqslant \cdots \geqslant \lambda_j$, the inductive hypothesis guarantees that all tiles

$$\left\{ (i,k) \colon \begin{matrix} 1 \leqslant i \leqslant \lambda_j \\ 1 \leqslant k \leqslant j - 1 \end{matrix} \right\}$$

are cross tiles. Thus if $P^{(j)}$ traverses a bump tile after traversing $a < \lambda_j$ cross tiles vertically, pipe $P^{(j)}$ is forced to travel horizontally until exiting at the $(a+1)^{\text{th}}$ row. As $a + 1 \leqslant \lambda_j < w^{-1}(j)$, this is a contradiction.

As λ_i is weakly decreasing, $\#D(w)_i \geqslant w^{-1}(j)$ if and only if $i \in [\nu_j]$. The inductive hypothesis guarantees that the pipe exiting at row $w^{-1}(j)$ traverses ν_j crossing tiles horizontally before traversing any bump tiles.

It remains to show that $P^{(j)}$ does not traverse any other cross tiles. To this end, observe that any such cross tile (a,b) satisfies $a > \lambda_j$ and $b > \nu_j$. It follows that any such cross tile cannot involve $P^{(w^{-1}(k))}$ for $k \leqslant \lambda_j$ as the paths of these pipes are contained in $\{(a,b) \colon a \leqslant k\}$, and cannot involve $P^{(k)}$ for $k \leqslant \nu_j$ as the paths of these

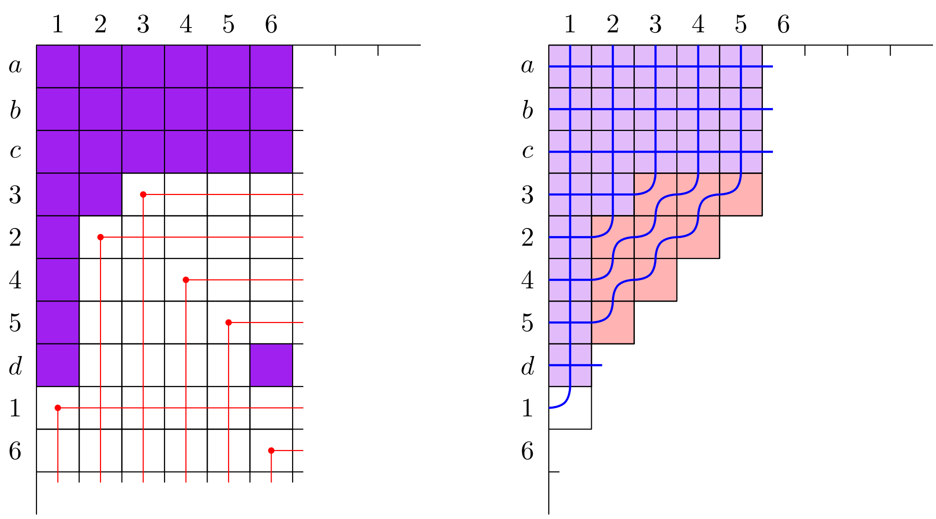

Figure 6. Left: The Rothe diagram of a permutation $w = abc3245d16...$, for $a,b,c,d > 6$. Right: A typical pipe dream $P \in \mathrm{PD}(w)$. The boxes in C_w are shaded purple and the boxes in E_w are shaded red.

pipes are contained in $\{(a,b)\colon b \leqslant \nu_j\}$. In other words, any such cross tile involves $P^{(j)}$ and $P^{(k)}$ for $k \notin w^{-1}([\lambda_j])$ and $k \notin [\nu_j]$.

Then, the first part of the result follows from the fact that the inversions of w involving j are

$$\{(k,j)\colon k \in w^{-1}([\lambda_j]) \text{ or } k \in [\nu_j]\}.$$

The rest of the lemma follows from the fact that the set of pipes entering the top edge of the triangular grid

$$\left\{(i,j)\colon \begin{array}{c} h-\beta+1 \leqslant i \leqslant h, \\ \alpha+1 \leqslant j \leqslant i_1, \\ i+j \leqslant h+\alpha+1 \end{array}\right\},$$

namely the set $\{P^{(j)}\colon h-\beta+1 \leqslant j \leqslant h\}$, is equal to the set of pipes exiting the left edge, hence restricts to a pipe dream for $\sigma(w)$. $\square$

Corollary 3.6. *Let $(h, C, \alpha, i_1, \beta)$ denote the primary column data of $w \in S_n$, and let $\mathcal{S} := \{h-\beta+1, \ldots, h\}$. Fix $P \in \mathrm{PD}(w)$.*

- *For $i,j \in \mathcal{S}$ and $k \notin \mathcal{S}$, the pipes $P^{(i)}$ and $P^{(k)}$ cross if and only if $P^{(j)}$ and $P^{(k)}$ cross.*
- *For $i \in \mathcal{S}$ and $j \in [n]$, the pipes $P^{(i)}$ and $P^{(j)}$ cross at most once.*
- *For $i,j \in \mathcal{S}$, the pipes $P^{(i)}$ and $P^{(j)}$ can cross only at $C_w \smallsetminus C_{w_{\mathrm{sort}}}$, and conversely every tile in $C_w \smallsetminus C_{w_{\mathrm{sort}}}$ is a crossing of pipes $P^{(i)}$ and $P^{(j)}$ for $i,j \in \mathcal{S}$.*

Proof. For any $i \in \mathcal{S}$, Lemma 3.4 guarantees that pipe $P^{(i)}$ begins by traversing α many cross tiles vertically, crossing pipes $P^{(k)}$ for $k \in w^{-1}([\alpha])$, and ends by traversing $h-\beta$ many cross tiles horizontally, crossing pipes $P^{(k)}$ for $k \in [h-\beta]$. Lemma 3.4 also guarantees that the tiles in $C_w \cup E_w \smallsetminus C_{w_{\mathrm{sort}}}$ involve only the pipes $P^{(i)}$ for $i \in \mathcal{S}$. All three parts of the claim follow.

$\square$

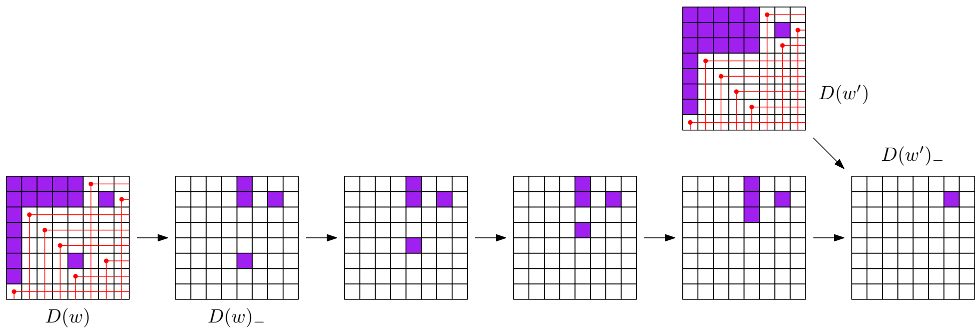

FIGURE 7. Left: Rothe diagram $D(w)$ for $w = 68234751$. Middle: first few orthodontic steps for $D(w)$. Top: Rothe diagram of $D(w')$ for $w' = ws_5s_4s_3 = 68723451$.

PROPOSITION 3.7 (cf. [20, Prop 3.9]). *Let $w \in S_n$ and set $\lambda_i := \#D(\sigma(w))_i$. Then*

$$\mathfrak{G}_w = \left(\prod_{(a,b)\in D(w)\smallsetminus D(w_{\mathrm{sort}})} (x_a + y_b - x_a y_b) \right) \cdot \mathfrak{G}_{w_{\mathrm{sort}}}.$$

Proof. Fix a pipe dream $P \in \mathrm{PD}(w)$. By Proposition 3.2 and Lemma 3.4, the tiles in $D(w) \smallsetminus D(w_{\mathrm{sort}})$ are crosses. Let $F(P)$ be the pipe dream obtained from P by replacing the crosses in $D(w) \smallsetminus D(w_{\mathrm{sort}})$ with elbows. By uncrossing all pipes $P^{(i)}$ and $P^{(j)}$ with $i, j \in \mathcal{S}$, Corollary 3.6 guarantees that the Demazure word $\partial(F(P))$ of $F(P)$ is w_{sort}.

For any pipe dream $P \in \mathrm{PD}(w_{\mathrm{sort}})$, let $G(P)$ be the pipe dream obtained from P by replacing the elbows in $D(w) \smallsetminus D(w_{\mathrm{sort}}) \subseteq E_{w_{\mathrm{sort}}}$ by crosses. Corollary 3.6 guarantees that the Demazure word $\partial(G(P))$ is w. As $FG\colon \mathrm{PD}(w_{\mathrm{sort}}) \to \mathrm{PD}(w_{\mathrm{sort}})$ and $GF\colon \mathrm{PD}(w) \to \mathrm{PD}(w)$ are the identity, the map F is a bijection.

The claim follows from the fact that $F\colon \mathrm{PD}(w) \to \mathrm{PD}(w_{\mathrm{sort}})$ scales the weight of a pipe dream by the monomial

$$\prod_{(a,b)\in D(w)\smallsetminus D(w_{\mathrm{sort}})} (x_a + y_b - x_a y_b). \qquad \square$$

The following example motivates Theorem 3.9, which gives a precise description of the first few terms of the orthodontic sequences of sorted permutations.

EXAMPLE 3.8 (cf. [20, Ex 4.1]). Consider the sorted permutation $w = 68234751$, with Rothe diagram depicted in the left of Figure 7. The Rothe diagram of $w' := ws_5s_4s_3$ "almost" appears in the double orthodontic sequence for $D(w)$, as shown in the top of Figure 7. Part (6) of Theorem 3.9 makes this connection precise.

The primary column data $(h, C, \alpha, i_1, \beta)$ of w is given by $h = 4$, $C = \{1, 2, 6\}$, $\alpha = 2$, $i_1 = 5$, and $\beta = 3$. From that the uppermost gap has size $\beta = 3$ and the first missing tooth $i_1 = 5$ occurs in column $h + 1 = 5$, it follows the next missing teeth are $i_2 = i_1 - 1 = 4$ and $i_3 = i_2 - 1 = 3$ and which occur in the fifth column. This phenomenon is captured as part (1) of Theorem 3.9. Direct computation gives $j_1 = 5 - 3 = 2$, $j_2 = j_1 + 1 = 3$, and $j_3 = j_2 + 1 = 4$; this is part (2) of Theorem 3.9. None of these row-swap orthodontic moves introduce standard interval columns; this is part (5) of Theorem 3.9.

As w is sorted, the three columns immediately to the left of column $h + 1 = 5$ (i.e., columns $2, 3, 4$) are standard intervals equal to $[\alpha] = [2]$. This is part (3) of

Theorem 3.9. Furthermore, there are no standard interval columns of size k for any $k \in [\alpha+1, i_1] = \{3,4,5\}$. This is part (4) of Theorem 3.9.

Theorem 3.9 (cf. [20, Thm 4.2]). *Let $w \in S_n$ be a nonidentity sorted permutation, and suppose w has orthodontic sequence*

$$\mathbf{i} = (i_1, \dots, i_\ell), \qquad \mathbf{j} = (j_1, \dots, j_\ell), \qquad \mathbf{K} = (K_1, \dots, K_n), \qquad \mathbf{M} = (M_1, \dots, M_\ell).$$

Let $(h, C, \alpha, i_1, \beta)$ be the primary column data of w. Write $\mathcal{S} := \{h-\beta+1, \dots, h\}$. Then:

1. *For $k \in [\beta]$, we have $i_k = i_1 - k + 1$.*
2. *For $k \in [\beta]$, we have $j_k = h - \beta + k$.*
3. *If $\alpha > 0$, then $K_\alpha \supseteq \mathcal{S}$.*
4. *For $k \in [\alpha+1, i_1]$, we have $K_k = \varnothing$.*
5. *For $k \in [\beta - 1]$, we have $M_k = \varnothing$.*
6. *The permutation $w' := ws_{i_1} \dots s_{\alpha+1}$ has orthodontic sequence*

$$\mathbf{i}(w') = (i_{\beta+1}, \dots, i_\ell), \qquad \mathbf{j}(w') = (j_{\beta+1}, \dots, j_\ell), \qquad \mathbf{M}(w') = (M_{\beta+1}, \dots, M_\ell),$$

and

$$\mathbf{K}(w') = \begin{cases} (K_1, \dots, K_{\alpha-1}, K_\alpha \smallsetminus \mathcal{S}, \mathcal{S} \sqcup M_\beta, K_{\alpha+2}, \dots, K_n) & \text{if } \alpha > 0 \\ (\mathcal{S} \sqcup M_\beta, K_2, \dots, K_n) & \text{if } \alpha = 0. \end{cases}$$

In particular,

$$(\star) \qquad \mathscr{G}_{D(w)} = \overline{\pi}_{i_1, h+1-\beta} \dots \overline{\pi}_{\alpha+1, h} \left(\prod_{s \in \mathcal{S}} (x_{\alpha+1} + y_s - x_{\alpha+1} y_s)^{-1} \cdot \mathscr{G}_{D(w')} \right).$$

Proof. By definition, $C = D(w)_{h+1}$ contains $[\alpha] \cup \{i_1 + 1\}$ and does not contain any of $\alpha+1, \dots, i_1$. Thus, the leftmost missing teeth of the diagrams in the orthodontic sequence of $D(w)$ are equal to $(i_1, i_1 - 1, \dots, \alpha+1)$, and all occur at column $h+1$, with corresponding sequence of uppermost gaps $(\beta, \beta-1, \dots, 1)$. Parts (1) and (2) follow.

As w is sorted, the columns $D(w_{\mathrm{sort}})_{h-\beta+b}$ are equal to $[\alpha]$, and part (3) follows. Parts (4) and (5) are proven as [20, Thm 4.2, (iii), (iv)] respectively.

In order to relate the orthodontic sequences of w and w', consider the diagrams

$$E = (\underbrace{\varnothing, \dots, \varnothing}_{h \text{ many}}, D(w)_{h+1}, \dots, D(w)_n),$$
$$E' = (\underbrace{\varnothing, \dots, \varnothing}_{h \text{ many}}, D(w')_{h+1}, \dots, D(w')_n).$$

Note that $E' = E \cdot s_{i_1} \dots s_{\alpha+1}$ and that $M_\beta = \{j \colon E'_j = [\alpha+1]\}$. As the first h columns of $D(w)$ and $D(w')$ are standard intervals, it follows that $D(w')_-$ occurs in the orthodontic sequence of diagrams for $D(w)$ and furthermore that

$$\mathbf{i}(w') = (i_{\beta+1}, \dots, i_\ell), \qquad \mathbf{j}(w') = (j_{\beta+1}, \dots, j_\ell), \qquad \mathbf{M}(w') = (M_{\beta+1}, \dots, M_\ell).$$

From the facts that

- $D(w)_j = D(w')_j$ for $j \leqslant h - \beta$,
- $D(w)_j \cup \{\alpha+1\} = D(w')_j$ for $h - \beta + 1 \leqslant j \leqslant h$, and
- $M_\beta = \{j \colon E'_j = [\alpha+1]\}$,

the formula for $\mathbf{K}(w')$ in terms of $\mathbf{K}(w)$ follows. This gives part (6).

Finally, the equation $(\star)$ is a consequence of part (4) of the claim via the string of equalities

$$\begin{aligned}\mathscr{G}_{D(w)} &= \overline{\omega}_1^{K_1}\dots\overline{\omega}_n^{K_n}\overline{\pi}_{i_1,h+1-\beta}\dots\overline{\pi}_{\alpha+1,h}(\mathscr{G}_{D(w')_-})\\ &\overset{(4)}{=} \overline{\pi}_{i_1,h+1-\beta}\dots\overline{\pi}_{\alpha+1,h}\left(\overline{\omega}_1^{K_1}\dots\overline{\omega}_n^{K_n}\mathscr{G}_{D(w')_-}\right)\\ &= \overline{\pi}_{i_1,h+1-\beta}\dots\overline{\pi}_{\alpha+1,h}\left(\prod_{s\in\mathcal{S}}(x_{\alpha+1}+y_s-x_{\alpha+1}y_s)^{-1}\cdot\mathscr{G}_{D(w')}\right). \qquad \square\end{aligned}$$

In order to relate the formula from Theorem 3.9 to Grothendieck polynomials, the following lemmas will be useful.

Lemma 3.10. *Let $\overline{\Delta}_a := \overline{\partial}_{i+a}\circ\cdots\circ\overline{\partial}_i$. For $a=-1$, set $\overline{\Delta}_a := 1$ to be the identity operator. Given $g\in\mathbb{C}[\mathbf{x},\mathbf{y}]$ and $\ell\geqslant 1$ such that $\partial_{i+\ell}(g)=0$, the equality*

$$\overline{\pi}_{i+\ell,j_\ell}\circ\overline{\Delta}_{\ell-1}(g)=\overline{\partial}_{i+\ell}\circ\overline{\pi}_{i+\ell-1,j_\ell}\circ\overline{\Delta}_{\ell-2}(g)$$

holds. In particular, for any $\ell\geqslant 0$,

$$\overline{\pi}_{i+\ell,j_\ell}\circ\overline{\Delta}_{\ell-1}(g)=\overline{\Delta}_\ell((x_i+y_{j_\ell}-x_iy_{j_\ell})g)$$

whenever $\partial_{i+1}(g)=\cdots=\partial_{i+\ell}(g)=0$.

Proof. Work in the ring of endomorphisms of $\mathbb{C}[\mathbf{x},\mathbf{y}]$. For $h\in\mathbb{C}[\mathbf{x},\mathbf{y}]$, write m_h for the "multiplication by h" operator, given by $f\mapsto fh$. As

$$\partial_i\circ m_{x_i}=(m_{x_{i+1}}\circ\partial_i)+1 \qquad\text{and}\qquad \partial_i\circ m_{y_j}=m_{y_j}\circ\partial_i \text{ for all } j,$$

observe that

$$\begin{aligned}m_{x_{i+1}-y_j+x_{i+1}y_j}\circ\partial_i &= (m_{x_{i+1}}\circ m_{1+y_j}-m_{y_j})\circ\partial_i\\ &= (\partial_i\circ m_{x_i}-1)\circ m_{1+y_j}-\partial_i\circ m_{y_j}\\ &= \partial_i\circ(m_{x_i-y_j+x_iy_j})-m_{1+y_j}.\end{aligned}\tag{$\diamondsuit$}$$

Hence

$$\begin{aligned}\overline{\pi}_{i+\ell,j_\ell}\circ\overline{\Delta}_{\ell-1} &= \overline{\partial}_{i+\ell}\circ m_{x_{i+\ell}+y_{j_\ell}-x_{i+\ell}y_{j_\ell}}\circ\partial_{i+\ell-1}\circ m_{1-x_{i+\ell}}\circ\overline{\Delta}_{\ell-2}\\ &\overset{(\diamondsuit)}{=} \overline{\partial}_{i+\ell}\circ(\partial_{i+\ell-1}\circ m_{x_{i+\ell-1}+y_{j_\ell}-x_{i+\ell-1}y_{j_\ell}}-m_{1+y_{j_\ell}})\circ m_{1-x_{i+\ell}}\circ\overline{\Delta}_{\ell-2}\\ &= \overline{\partial}_{i+\ell}\circ\overline{\pi}_{i+\ell-1,j_\ell}\circ\overline{\Delta}_{\ell-2}-\overline{\partial}_{i+\ell}\circ m_{1+y_{j_\ell}}\circ m_{1-x_{i+\ell}}\circ\overline{\Delta}_{\ell-2}.\end{aligned}\tag{$*$}$$

Note that $\overline{\partial}_{i+\ell}$ and $m_{1-x_{i+\ell}}$ commute with $\overline{\Delta}_{\ell-2}$, so that

$$\begin{aligned}\overline{\partial}_{i+\ell}\circ m_{1+y_{j_\ell}}\circ m_{1-x_{i+\ell}}\circ\overline{\Delta}_{\ell-2}(g) &= m_{1+y_{j_\ell}}\circ\overline{\Delta}_{\ell-2}\circ\overline{\partial}_{i+\ell}\circ m_{1-x_{i+\ell}}(g)\\ &= m_{1+y_{j_\ell}}\circ\overline{\Delta}_{\ell-2}\circ\underbrace{\partial_{i+\ell}((1-x_{i+\ell+1})(1-x_{i+\ell})g)}_{=0}.\end{aligned}\tag{$**$}$$

Applying the equality $(*)$ of operators to the function g, and using $(**)$, the claimed equality

$$\overline{\pi}_{i+\ell,j_\ell}\circ\overline{\Delta}_{\ell-1}(g)=\overline{\partial}_{i+\ell}\circ\overline{\pi}_{i+\ell-1,j_\ell}\circ\overline{\Delta}_{\ell-2}(g)$$

follows.

The second part of the lemma, for $\ell=0$, is nothing but the identity

$$\overline{\pi}_{i,j_0}(g)=\overline{\partial}_i((x_i-y_{j_0})g),\tag{5}$$

and in general it follows from repeated application of the first part of the lemma combined with the identity (5). $\square$

Lemma 3.11 (cf. [20, Lem 5.11]). *Let g be a polynomial with*

$$\partial_{i+1}(g) = \cdots = \partial_{i+k}(g) = 0.$$

Then

$$\overline{\pi}_{i+k,j_k}\overline{\pi}_{i+k-1,j_{k-1}}\dots\overline{\pi}_{i,j_0}(g) = \overline{\partial}_{i+k}\dots\overline{\partial}_i\left(\prod_{a=0}^{k}(x_i + y_{j_a} - x_iy_{j_a})\cdot g\right).$$

Proof. Induct on k as follows. The base case $k=0$ is the identity (5). For $k>0$, the inductive hypothesis guarantees that

$$\pi_{i+k-1,j_{k-1}}\dots\overline{\pi}_{i,j_0}(g) = \overline{\Delta}_{k-1}\left(\prod_{a=0}^{k-1}(x_i + y_{j_a} - x_iy_{j_a})\cdot g\right).$$

Then, because

$$\partial_{i+1}\left(\prod_{a=0}^{k-1}(x_i + y_{j_a} - x_iy_{j_a})g\right) = \cdots = \partial_{i+k}\left(\prod_{a=0}^{k-1}(x_i + y_{j_a} - x_iy_{j_a})g\right) = 0,$$

Lemma 3.10 implies that

$$\overline{\pi}_{i+k,j_k}\circ\overline{\Delta}_{k-1}\left(\prod_{a=0}^{k-1}(x_i + y_{j_a} - x_iy_{j_a})\cdot g\right) = \overline{\Delta}_k\left(\prod_{a=0}^{k}(x_i + y_{j_a} - x_iy_{j_a})\cdot g\right). \qquad \square$$

Theorem 1.1. *Let D be a %-avoiding diagram with double orthodontic sequence $\mathbf{K}, \mathbf{i}, \mathbf{j}, \mathbf{M}$. Define*

$$\mathscr{G}_D(\mathbf{x};\mathbf{y}) := \overline{\omega}_1^{K_1}\overline{\omega}_2^{K_2}\dots\overline{\omega}_n^{K_n}\overline{\pi}_{i_1,j_1}(\overline{\omega}_{i_1}^{M_1}\overline{\pi}_{i_2,j_2}(\overline{\omega}_{i_2}^{M_2}\dots\overline{\pi}_{i_\ell,j_\ell}(\overline{\omega}_{i_\ell}^{M_\ell})\dots)). \tag{1}$$

When $D = D(w)$ is the Rothe diagram of a permutation, then $\mathscr{G}_D(\mathbf{x};\mathbf{y}) = \mathfrak{G}_w(\mathbf{x};\mathbf{y})$.

Proof of Theorem 1.1. Induct on the orthodontic sort order (Definition 2.16). In the base case $w = \mathrm{id}$, both $\mathscr{G}_{D(\mathrm{id})} = \mathscr{G}_\varnothing$ and $\mathfrak{G}_{\mathrm{id}}$ are equal to 1.

Assume that w is not sorted and that $\mathscr{G}_{D(w_{\mathrm{sort}})} = \mathfrak{G}_{w_{\mathrm{sort}}}$. Corollary 3.3 and Proposition 3.7 imply that

$$\begin{aligned}\mathscr{G}_{D(w)} &= \left(\prod_{(a,b)\in D(w)\smallsetminus D(w_{\mathrm{sort}})}(x_a + y_b - x_ay_b)\right)\mathscr{G}_{D(w_{\mathrm{sort}})}\\ &= \left(\prod_{(a,b)\in D(w)\smallsetminus D(w_{\mathrm{sort}})}(x_a + y_b - x_ay_b)\right)\mathfrak{G}_{w_{\mathrm{sort}}}\\ &= \mathfrak{G}_w.\end{aligned}$$

Now assume that w is sorted and that $\mathscr{G}_{D(ws_{i_1}\dots s_\alpha)} = \mathfrak{G}_{ws_{i_1}\dots s_\alpha}$. Writing $w' := ws_{i_1}\dots s_\alpha$ and $\mathcal{S} := \{h-\beta+1,\dots,h\}$, Theorem 3.9 and Lemma 3.11 imply that

$$\begin{aligned}\mathscr{G}_{D(w)} &= \overline{\pi}_{i_1,h+1-\beta}\dots\overline{\pi}_{\alpha+1,h}\left(\prod_{s\in\mathcal{S}}(x_{\alpha+1} + y_s - x_{\alpha+1}y_s)^{-1}\cdot\mathscr{G}_{D(w')}\right)\\ &= \overline{\partial}_{i_1}\dots\overline{\partial}_\alpha\left(\mathscr{G}_{D(w')}\right)\\ &= \overline{\partial}_{i_1}\dots\overline{\partial}_\alpha\left(\mathfrak{G}_{w'}\right)\\ &= \mathfrak{G}_w. \qquad \square\end{aligned}$$

Corollary 3.12. *If $D = D(w)$ is a Rothe diagram, then $\mathscr{S}_D(\mathbf{x};\mathbf{y}) = \mathfrak{S}_w(\mathbf{x};-\mathbf{y})$.*

Proof. By definition, $\mathfrak{S}_w(\mathbf{x};\mathbf{y})$ is the lowest degree part of $\mathfrak{G}_w(\mathbf{x};-\mathbf{y})$. Remark 2.10 guarantees that $\mathscr{S}_D(\mathbf{x};\mathbf{y})$ is the lowest degree part of $\mathscr{G}_D(\mathbf{x};\mathbf{y})$. The result now follows from Theorem 1.1. ☐

4. Lascoux positivity

We prove Theorem 1.2 by reducing the problem to the special case where every column is a standard interval (cf. Lemma 4.3 and Corollary 4.6). This case can be checked explicitly (Proposition 4.11).

Recall the operator $r_{m,n}$ introduced in [29] that acts on n variable polynomial via

$$f(x_1,\dots,x_n)\mapsto x_1^m\dots x_n^m f(x_n^{-1},\dots,x_1^{-1}).$$

Lemma 4.1. *For $i\in[n]$ and $M\subseteq[n]$, and $f\in\mathbb{C}[\mathbf{x}]$ satisfying $\deg_{x_i}(f)\leqslant m$, the equalities*

$$\omega_i^M|_{y_j\mapsto -1}=(x_1-1)^{|M|}\dots(x_i-1)^{|M|} \tag{6}$$

$$r_{m+1,n}\left((x_1-1)\dots(x_i-1)f\right)=x_1\dots x_{n-i}(1-x_{n-i+1})\dots(1-x_n)r_{m,n}(f) \tag{7}$$

$$\pi_{i,j}(f)|_{y_j\mapsto -1}=\partial_i((x_i-1)(f|_{y_j\mapsto 1})) \tag{8}$$

$$r_{m,n}(\partial_i((x_i-1)f))=\overline{\pi}_{n-i}(r_{m,n}(f)) \tag{9}$$

hold.

Proof. Equations (6) and (8) are immediate from the definition of ω_i^M and $\pi_{i,j}$. Equations (7) and (9) can be proven by a manual check when f is a monomial and applying linearity. (Equation (9) also follows from [29, Lem 3.4]). ☐

Let φ_i denote the operator

$$\varphi_i\colon f\mapsto x_1\dots x_i(1-x_{i+1})\dots(1-x_n)f.$$

Corollary 4.2. *The polynomial $r_{m,n}(\mathscr{S}_D(\mathbf{x};-\mathbf{1}))$ can be obtained from the polynomial $1\in\mathbb{C}[\mathbf{x}]$ by repeated application of operators of the form $f\mapsto\overline{\pi}_i(f)$ and φ_i.*

Proof. By the orthodontic formula (1), along with Equations (6) and (8), $\mathscr{S}_D(\mathbf{x};-\mathbf{1})$ can be obtained from the polynomial $1\in\mathbb{C}[\mathbf{x}]$ by repeated application of operators of the form

$$f\mapsto\partial_i(x_i-1)f\qquad\text{and}\qquad f\mapsto(x_1-1)\dots(x_i-1)f.$$

Using Equations (7) and (9) to commute the operator $r_{m,n}$ past the two operators above, it follows that $r_{m,n}(\mathscr{S}_D(\mathbf{x};-\mathbf{1}))$ can be obtained by repeated application of the operators $\overline{\pi}_i$ and φ_{n-i}. ☐

Lemma 4.3. *The equality*

$$\overline{\pi}_i(\mathfrak{L}_\alpha)=\begin{cases}\mathfrak{L}_\alpha & \text{if } \alpha_i>\alpha_{i+1}\\ \mathfrak{L}_{\alpha\cdot s_i} & \text{if } \alpha_i<\alpha_{i+1}\end{cases}$$

holds; in particular, $\overline{\pi}_i$ preserves Lascoux positivity.

Proof. When $\alpha_i>\alpha_{i+1}$, the definition of Lascoux polynomials guarantees that $\mathfrak{L}_\alpha=\overline{\pi}_i\mathfrak{L}_{\alpha\cdot s_i}$. The claim then follows from the fact that $\overline{\pi}_i$ is idempotent. When $\alpha_i<\alpha_{i+1}$, the claim follows from the definition of $\mathfrak{L}_{\alpha\cdot s_i}$. ☐

Proposition 4.4. *Conjecture 1.5 implies Conjecture 1.4.*

Proof. By Corollary 4.2, it suffices to show that the operators $\overline{\pi}_i$ and φ_i preserve Lascoux positivity. This follows from Lemma 4.3 and Conjecture 1.5, respectively. ☐

Lemma 4.5. *Assume that the columns of D are ordered by inclusion, and write $\mathbf{K}, \mathbf{i}, \mathbf{j}, \mathbf{M}$ for the orthodontic sequence. Then:*

(1) *Every diagram appearing in the orthodontic sequence of D also has all columns ordered by inclusion.*
(2) *If $K_k \neq \varnothing$, then $i_j \neq k$ for all j.*
(3) *If $M_k \neq \varnothing$ for some k, then $i_j \neq i_k$ for all $j > k$.*

Proof. Part 1 of the lemma follows from the fact that orthodontic moves either remove a column or swap two rows; both of these moves preserve the inclusion property of the columns.

If $K_k \neq \varnothing$, then D has a column equal to $[k]$. As the columns of D are ordered by inclusion, every other column of D is either contained in $[k]$ or is equal to $[k] \sqcup S$ for some S; in particular, no empty box in the k^{th} row has a square below it in its column. As orthodontic moves remove columns or move boxes up, no empty box in the k^{th} row of any diagram D' appearing in the orthodontic sequence of D has a square below it in its column; in particular, k can never be a missing tooth of D'. Part 2 of the lemma follows.

Similarly, if $M_k \neq \varnothing$ for some k, then there is a diagram D' in the orthodontic sequence of D which has a column equal to $[i_k]$. Arguing as above, it follows that i_k can never be a missing tooth of a diagram in the orthodontic sequence of D'. Part 3 of the lemma follows. $\square$

Corollary 4.6. *Let D be a diagram whose columns are ordered by inclusion, and let $\mathbf{K}, \mathbf{i}, \mathbf{j}, \mathbf{M}$ denote the orthodontic sequence of D. Then,*

$$\mathscr{S}_D(\mathbf{x};\mathbf{y}) = \pi_{i_1,j_1}(\dots\pi_{i_\ell,j_\ell}(\omega_1^{K_1}\dots\omega_n^{K_n}\cdot\omega_{i_1}^{M_1}\dots\omega_{i_\ell}^{M_\ell})\dots).$$

Proof. Because ω_i^J is symmetric with respect to $x_1,\dots,x_i$ and also with respect to $x_{i+1},\dots,x_n$, it follows that ω_i^J commutes with $\pi_{i',j}$ whenever $i' \neq i$. Lemma 4.5 guarantees that the ω_i^J in the orthodontic formula (1) can be commuted past the $\pi_{i',j}$ occurring to the right. The claim follows. $\square$

Let $C_{n,k,\ell}$ be the set of $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$ with $\alpha_1 = \dots = \alpha_k = \ell$ and $\alpha_j \leqslant \ell$ for all j.

Lemma 4.7. *Let $\alpha \in C_{n,k,\ell}$ and $i \leqslant k$. Then*

$$\mathfrak{L}_\alpha(x_1,\dots,x_n) = x_1^\ell\dots x_i^\ell\mathfrak{L}_{\alpha(i)}(x_{i+1},\dots,x_n), \qquad \text{where } \alpha(i) := (\alpha_{i+1},\dots,\alpha_n).$$

Proof. Write $\lambda := \operatorname{sort}(\alpha)$ for the partition obtained by reordering α so that the components are weakly decreasing. Then

$$\mathfrak{L}_\alpha = \overline{\pi}_{i_1}\dots\overline{\pi}_{i_\ell}(\mathbf{x}^\lambda),$$

where $i_j > k$ for all j. As $\lambda_1 = \dots = \lambda_k = \ell$,

$$\mathfrak{L}_\alpha = x_1^\ell\dots x_i^\ell\overline{\pi}_{i_1}\dots\overline{\pi}_{i_\ell}(\mathbf{x}^{\lambda(i)}), \qquad \lambda(i) := (\lambda_{i+1},\dots,\lambda_n)$$

for any $i \leqslant k$. The claim follows from the fact that

$$\overline{\pi}_{i_1}\dots\overline{\pi}_{i_\ell}(\mathbf{x}^{\lambda(i)}) = \mathfrak{L}_{\alpha(i)}(x_{i+1},\dots,x_n). \qquad \square$$

Lemma 4.8. *The set of Lascoux polynomials $\{\mathfrak{L}_\alpha \colon \alpha \in C_{n,k,\ell}\}$ forms a basis for the vector space $V_{k,\ell}$ of polynomials of the form*

$$x_1^\ell\dots x_k^\ell\cdot f(x_{k+1},\dots,x_n), \qquad \deg_{x_i}(f) \leqslant \ell.$$

Proof. The set of polynomials $\mathfrak{L}_\alpha(x_{k+1},\dots,x_n)$ with $\alpha_i \leqslant \ell$ forms a basis for the set of polynomials $\mathbb{C}[x_{k+1},\dots,x_n]$ such that $\deg_{x_i}(f) \leqslant \ell$ for all ℓ. Lemma 4.7 implies that $\{\mathfrak{L}_\alpha \colon \alpha \in C_{n,k,\ell}\}$ spans $V_{k,\ell}$. Since Lascoux polynomials are linearly independent, the claim follows. $\square$

Recall that the stable Grothendieck polynomial $G_w(x_1,\dots,x_n)$ of a permutation $w \in S_m$ is defined to be the limit $\lim_{N\to\infty} \mathfrak{G}_{1^N\times w}(x_1,\dots,x_n,0,0,\dots)$.

Example 4.9. For $w = 21$, the stable Grothendieck polynomial $G_{21}(x_1,\dots,x_n)$ is equal to

$$G_{21}(x_1,\dots,x_n) = \sum_{i=1}^{n} (-1)^{i+1}\mathbf{e}_i(\mathbf{x}),$$

where $\mathbf{e}_i$ is the i^{th} elementary symmetric polynomial. (More generally, for permutations with a unique descent, $G_w(x_1,\dots,x_n)$ can be computed e.g. using [18, Thm 2.2].)

In particular,

$$(1-x_1)\dots(1-x_n) = 1 - G_{21}(x_1,\dots,x_n).$$

The key ingredient for the Lascoux positivity in Theorem 1.2 is the following result by Orelowitz and Yu.

Theorem 4.10 ([21, Thm 1.3]). *The product $\mathfrak{L}_\alpha(x_1,\dots,x_n)\cdot G_w(x_1,\dots,x_n)$ is a graded nonnegative sum of Lascoux polynomials:*

$$\mathfrak{L}_\alpha(x_1,\dots,x_n)\cdot G_w(x_1,\dots,x_n) = \sum_\beta c_\beta(-1)^{|\beta|-\ell(w)-|\alpha|}\mathfrak{L}_\beta(x_1,\dots,x_n), \qquad c_\beta \geqslant 0.$$

Orelowitz and Yu prove more in [21, Thm 1.3]: they express the product as a sum over certain tableaux, each of which contribute a Lascoux polynomial in the expansion.

Proposition 4.11. *Assume that $\varphi_i^{a_i}\varphi_{i+1}^{a_{i+1}}\dots\varphi_n^{a_n}$ is a graded nonnegative sum of Lascoux polynomials for some $a_i, a_{i+1},\dots,a_n \geqslant 0$. Then*

$$\varphi_i\cdot\varphi_i^{a_i}\varphi_{i+1}^{a_{i+1}}\dots\varphi_n^{a_n}$$

is again a graded nonnegative sum of Lascoux polynomials.

Proof. For $i = n$, the result is trivial.

Assume that $i \leqslant n-1$ and write $f := \varphi_i^{a_i}\varphi_{i+1}^{a_{i+1}}\dots\varphi_n^{a_n}$. Write $\ell := a_i + \dots + a_n$ and observe that $f \in V_{i,\ell}$. Lemma 4.8 implies that the Lascoux expansion of f, graded nonnegative by assumption, reads

$$f = \sum_{\alpha\in C_{n,i,\ell}} (-1)^{|\alpha|-\ell}c_\alpha\mathfrak{L}_\alpha(x_1,\dots,x_n), \qquad c_\alpha \geqslant 0.$$

By Lemma 4.7, the polynomial f expands as

$$f = \sum_{\alpha\in C_{n,i,\ell}} (-1)^{|\alpha|-\ell}c_\alpha x_1^\ell\dots x_i^\ell\cdot\mathfrak{L}_{\alpha(i)}(x_{i+1},\dots,x_n), \quad \text{where } \alpha(i) := (\alpha_{i+1},\dots,\alpha_n).$$

As $\varphi_i = x_1\dots x_i(1-G_{21}(x_{i+1},\dots,x_n))$ (cf. Example 4.9), the polynomial $\varphi_i f$ expands as

(10)

$$\varphi_i f = \sum_{\alpha\in C_{n,i,\ell}} (-1)^{|\alpha|-\ell}c_\alpha x_1^{\ell+1}\dots x_i^{\ell+1}\cdot(1-G_{21}(x_{i+1},\dots,x_n))\cdot\mathfrak{L}_{\alpha(i)}(x_{i+1},\dots,x_n).$$

Theorem 4.10 implies that

$$G_{21}(x_{i+1},\dots,x_n)\mathfrak{L}_{\alpha(i)}(x_{i+1},\dots,x_n) = \sum_{\beta(i)} c_{\alpha(i),\beta(i)}(-1)^{|\beta(i)|-1-|\alpha(i)|}\mathfrak{L}_{\beta(i)}(x_{i+1},\dots,x_n).$$

As $\deg_{x_j}(\mathfrak{L}_{\alpha(i)}) \leqslant \ell$ and $\deg_{x_j}(G_{21}(x_{i+1},\dots,x_n)) = 1$ for $i+1 \leqslant j \leqslant n$, it follows that $\deg_{x_j}(\mathfrak{L}_{\beta(i)}) \leqslant \ell+1$ for $i+1 \leqslant j \leqslant n$. Writing $\beta(i) := (\beta_{i+1},\dots,n)$, Lemma 4.7 implies that

$$x_1^{\ell+1}\dots x_i^{\ell+1}\mathfrak{L}_{\beta(i)}(x_{i+1},\dots,x_n) = \mathfrak{L}_\beta(x_1,\dots,x_n), \qquad \beta := (\underbrace{\ell,\dots,\ell}_{i\text{ many}},\beta_{i+1},\dots,\beta_n).$$

It follows that each summand in Equation (10) has a graded nonnegative Lascoux expansion. $\square$

Theorem 1.2. *Let $D \subseteq [n]\times[m]$ be a diagram whose columns are ordered by inclusion. Let $\mathscr{S}_D(\mathbf{x};\mathbf{y})$ be the lowest degree part of $\mathscr{G}_D(\mathbf{x};\mathbf{y})$. Then, the polynomial*

$$x_1^m\dots x_n^m\mathscr{S}_D(x_n^{-1},\dots,x_1^{-1};-1,\dots,-1)$$

is a graded nonnegative sum of Lascoux polynomials $\mathfrak{L}_\alpha(x_1,\dots,x_n)$.

Proof. Corollary 4.6 asserts

$$\mathscr{S}_D(\mathbf{x};\mathbf{y}) = \pi_{i_1,j_1}(\dots\pi_{i_\ell,j_\ell}(\omega_1^{K_1}\dots\omega_n^{K_n}\cdot\omega_{i_1}^{M_1}\dots\omega_{i_\ell}^{M_\ell})\dots).$$

Equations (6)–(9), along with the fact that the φ_i commute with each other, imply that

$$r_{m,n}(\mathscr{S}_D(\mathbf{x};-\mathbf{1})) = \overline{\pi}_{n-i_1}\dots\overline{\pi}_{n-i_\ell}(\varphi_{k_1}\dots\varphi_{k_m}(1))$$

for $k_1 \geqslant \dots \geqslant k_m$. Proposition 4.11 implies that $\varphi_{k_1}\dots\varphi_{k_m}(1)$ is Lascoux positive, and the result follows from Lemma 4.3. $\square$

Corollary 1.3. *Let $w \in S_n$ be a vexillary permutation and denote the corresponding double Schubert polynomial by $\mathfrak{S}_w(x_1,\dots,x_n;y_1,\dots,y_n)$. Then the polynomial*

$$x_1^n\dots x_n^n\mathfrak{S}_w(x_n^{-1},\dots,x_1^{-1};1,\dots,1)$$

is a graded nonnegative sum of Lascoux polynomials $\mathfrak{L}_\alpha(x_1,\dots,x_n)$.

Proof. By Corollary 3.12, the double Schubert polynomial $\mathfrak{S}_w(\mathbf{x};\mathbf{1})$ is equal to $\mathscr{S}_{D(w)}(\mathbf{x};-\mathbf{1})$. The result follows from Theorem 1.2. $\square$

Example 4.12. Replacing $w \in S_n$ with $w(n+1) \in S_{n+1}$ amounts to replacing

$$f := x_1^n\dots x_n^n\mathfrak{S}_w(x_n^{-1},\dots,x_1^{-1};\mathbf{1}) \quad \text{with} \quad x_1^{n+1}x_2\dots x_nx_{n+1}f(x_2,\dots,x_{n+1}).$$

This operation preserves Lascoux positivity by Lemma 4.7. For example, if $w = 321$, then

$$\begin{aligned}\mathfrak{S}_w(\mathbf{x};\mathbf{y}) &= (x_1-y_1)(x_2-y_1)(x_1-y_2)\\ &= x_1^2x_2 - x_1^2y_1 - x_1x_2y_1 - x_1x_2y_2 + x_1y_1^2 + x_1y_1y_2 + x_2y_1y_2 - y_1^2y_2,\end{aligned}$$

which gives

$$\begin{aligned}x_1^3x_2^3x_3^3\mathfrak{S}_w(x_3^{-1},x_2^{-1},x_1^{-1};\mathbf{1}) &= x_1^3x_2^2x_3 - x_1^3x_2^3x_3 - x_1^3x_2^2x_3^2 - x_1^3x_2^2x_3^2\\ &\quad + x_1^3x_2^3x_3^2 + x_1^3x_2^3x_3^2 + x_1^3x_2^2x_3^3 - x_1^3x_2^3x_3^3\\ &= (\mathfrak{L}_{321}) - (2\mathfrak{L}_{322} + \mathfrak{L}_{331}) + (\mathfrak{L}_{323} + \mathfrak{L}_{332}).\end{aligned}$$

Taking $w = 3214$ gives

$$\begin{aligned}x_1^4x_2^4x_3^4x_4^4\mathfrak{S}_w(x_4^{-1},x_3^{-1},x_2^{-1},x_1^{-1};\mathbf{1}) &= x_1^4x_2^4x_3^3x_4^2 - x_1^4x_2^4x_3^4x_4^2 - x_1^4x_2^4x_3^3x_4^3 - x_1^4x_2^4x_3^3x_4^3\\ &\quad + x_1^4x_2^4x_3^4x_4^3 + x_1^4x_2^4x_3^4x_4^3 + x_1^4x_2^4x_3^3x_4^4 - x_1^4x_2^4x_3^4x_4^4\\ &= (\mathfrak{L}_{4432}) - (2\mathfrak{L}_{4433} + \mathfrak{L}_{4442}) + (\mathfrak{L}_{4434} + \mathfrak{L}_{4443}).\end{aligned}$$

Acknowledgements. We thank David Anderson, Sergey Fomin, Peter Magyar, Karola Mészáros, Alexander Yong, and Tianyi Yu for many helpful conversations and useful comments which improved our exposition.

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Linus Setiabrata, Massachusetts Institute of Technology, Department of Mathematics, Cambridge, MA 02139
E-mail : setia@mit.edu

Avery St. Dizier, Michigan State University, Department of Mathematics, East Lansing, MI 48824
E-mail : stdizier@msu.edu