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Polarizations of Artin monomial ideals

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Abstract We show that any polarization of an Artin monomial ideal defines a triangulated ball. This settles a conjecture of A. Almousa, H. Lohne and the first author [3].

Geometrically, polarizations of ideals strictly containing $(x_1^{a_1}, \dots, x_n^{a_n})$ define full-dimensional triangulated balls on the sphere which is the *join* of boundaries of simplices of dimensions $a_1 - 1, \cdots, a_n - 1$. We prove that every full-dimensional Cohen-Macaulay sub-complex of this joined sphere is of this kind, and these balls are constructible.

Such a triangulated ball has a dual cell complex which is a sub-complex of the *product* of simplices of dimensions $a_1 - 1, \cdots, a_n - 1$. We prove that this cell complex gives a cellular minimal free resolution of the Alexander dual ideal of the triangulated ball. When the product of simplices is a hypercube, using these dual cell complexes we classify in a range of examples all polarizations of the Artin monomial ideal.

We also show that the squeezed balls of G. Kalai [32] derive from polarizations of Artin monomial ideals.

1. Introduction

A polarization of an Artin monomial ideal is a squarefree monomial ideal and so corresponds to a simplicial complex via the Stanley-Reisner (SR) correspondence. We prove a fundamental theorem on the combinatorial geometry of monomial ideals:

$\boxed{\text{For any polarization of an } \textit{Artin} \text{ monomial ideal, the associated simplicial complex has a topological realization which is a } \textit{ball} \text{ (save when the Artin monomial ideal is a complete intersection, then the polarization gives a sphere).}}$

This proves a conjecture by A. Almousa, H. Lohne, and the first author [3, Conjecture 3.2]. We give further a detailed geometric analysis and understanding of polarizations of Artin monomial ideals in $k[x_1, \dots, x_n]$. For a finite set V, let $\mathbb{R}^V$ be the real vector space with basis the elements of V, and let $\Delta(V)$ be the geometric simplex in $\mathbb{R}^V$ with vertices V. For finite sets $A_1, A_2, \dots, A_n$ the product polytope

$$\Pi(A_\bullet) = \Delta(A_1) \times \Delta(A_2) \times \cdots \times \Delta(A_n)$$

(if each cardinality $|A_i| = 2$ this is a hypercube), is dual (polar) to the simplicial polytope which is the join of boundaries

$$\partial\Delta(A_\bullet) = \partial\Delta(A_1) * \partial\Delta(A_2) * \cdots * \partial\Delta(A_n),$$

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(this is a hyperoctahedron when each $|A_i| = 2$). It is known that polarizations of Artin monomial ideals are precisely the full-dimensional Cohen-Macaulay sub-complexes of $\partial\Delta(A_\bullet)$. We show that such sub-complexes are balls, by proving that they are *constructible* simplicial complexes.

These sub-complexes have dual cell sub-complexes of the product polytope $\Pi(A_\bullet)$, which have the property we call *restriction-acyclic*: every restriction to a sub-cell $\Pi(R_\bullet)$ of $\Pi(A_\bullet)$ (where $R_i \subseteq A_i$) has vanishing homologies. In many example cases this enables simple analysis and classification of all polarizations of an Artin monomial ideal, Section 7.

We proceed to recall the notions of separation and polarization, with an example. We further give more detailed descriptions of the above.

1.1. Separation and polarization. For a field k and a set V, let $k[V]$ be the polynomial ring with the elements of V as variables.

Definition 1.1. *Let $V' \xrightarrow{p} V$ be a surjection of finite sets with the cardinality of V' one more than that of V. Let v_1 and v_2 be the two distinct elements of V' which map to a single element v in V. Let J be a monomial ideal in the polynomial ring $k[V]$ and J' a monomial ideal in $k[V']$. Then J' is a simple separation of J if:*

i. *The monomial ideal J is the image of J' by the map $k[V'] \to k[V]$.*
ii. *Both the variables x_{v_1} and x_{v_2} occur in some minimal generators of J' (usually in distinct generators, but in same generator is allowed).*
iii. *The variable difference $x_{v_1} - x_{v_2}$ is a non-zero divisor in the quotient ring $k[V']/J'$.*

The minimal generators of J and J' will then be in bijection. More generally, if $V' \xrightarrow{p} V$ is a surjection of finite sets and $J \subseteq k[V]$ and $J' \subseteq k[V']$ are monomial ideals such that J' is obtained by a succession of simple separations of J, J' is a proper separation of J. If further $V' \subseteq V''$ is an inclusion, the extension ideal $J'' = (J') \subseteq k[V'']$ is a separation of the starting ideal J.

Note that J'' is obtained from J by a sequence of simple separations where we do not require ii. to hold.

Definition 1.2. *Let $J \subseteq k[V]$ be a monomial ideal and $V' \to V$ be a surjection of finite sets. An ideal $J' \subseteq k[V']$ is a polarization of J if J' is squarefree and a separation of J. It is a proper polarization if it is a squarefree proper separation of J.*

Example 1.3. Consider the monomial ideal $\mathfrak{j} = (x_1, x_2, x_3)^2 \subseteq k[x_1, x_2, x_3]$ (with k a field). This is Artin as it contains a power of each variable. The ideal has generators:

$$\mathfrak{j} : x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2.$$

Up to isomorphism this ideal has two polarizations J_1 (the standard polarization) and J_2 (the box polarization) with generators:

$$J_1 : x_{10}x_{11}, x_{10}x_{20}, x_{20}x_{21}, x_{10}x_{30}, x_{20}x_{30}, x_{30}x_{31},$$
$$J_2 : x_{10}x_{11}, x_{10}x_{21}, x_{20}x_{21}, x_{10}x_{31}, x_{20}x_{31}, x_{30}x_{31}.$$

These are polarizations because the quotient ring $k[x_1, x_2, x_3]/\mathfrak{j}$ is obtained from $k[x_{10}, x_{11}, x_{20}, x_{21}, x_{30}, x_{31}]/J_i$ for $i = 1, 2$ by dividing out by the regular sequence of variable differences:

$$x_{10} - x_{11}, x_{20} - x_{21}, x_{30} - x_{31}.$$

The corresponding simplicial complexes of the two ideals are given in Figure 1. Topologically both are discs of dimension two, i.e. two-dimensional balls.

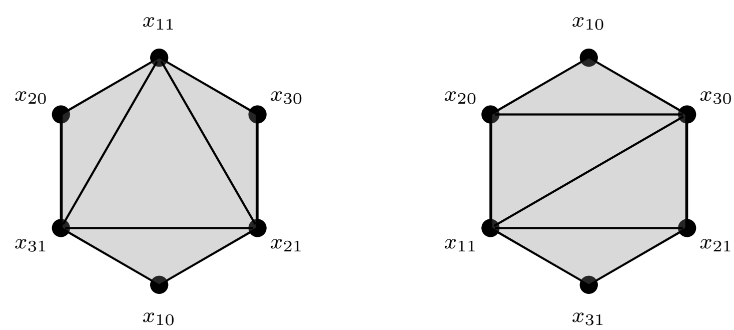

Figure 1. Simplicial complexes of J_1 and J_2

1.2. Geometric setting. For a set V consider the (abstract) combinatorial simplex $\Delta(V)$ on V. It is the simplicial complex on V consisting of all subsets of V. Its geometric realization is topologically a ball. Its boundary is the simplicial complex $\partial\Delta(V)$ whose maximal faces are all sets $V\backslash\{v\}$ for $v \in V$. The geometric realization of this boundary is topologically a sphere $S^{|V|-2}$ where $|V|$ is the cardinality of V.

Given non-empty disjoint finite sets $A_1, A_2, \ldots, A_n$ consider the simplicial join

$$\partial\Delta(A_\bullet) = \partial\Delta(A_1) * \partial\Delta(A_2) * \cdots * \partial\Delta(A_n).$$

Its topological realization is the topological joins of the topological realizations of the $\partial\Delta(A_i)$, and so is a sphere of dimension $\sum_{i=1}^n |A_i| - 1 - n$. In particular, if all A_i have cardinality two, $\partial\Delta(A_\bullet)$ is the boundary of an n-dimensional hyperoctahedron. Let A be the (disjoint) union of the A_i. We note that the facets of $\partial\Delta(A_\bullet)$ are indexed by elements

$$\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \cdots \times A_n. \tag{1}$$

To such an element corresponds the facet $A\backslash\{a_1, \ldots, a_n\}$.

We relate these geometric objects to Artin monomial ideals. Let $k[A]$ be the polynomial ring with variables from A. Denote by $m(A_i)$ the monomial $\prod_{a\in A_i} a$, and by α_i the cardinality of A_i. In the polynomial ring $k[x_1, \ldots, x_n]$ we have the complete intersection Artin monomial ideal:

$$\mathfrak{c} = (x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_n^{\alpha_n}).$$

It has (up to isomorphism) a unique polarization, the squarefree complete intersection ideal

$$C = (m(A_1), m(A_2), \ldots, m(A_n)),$$

which is the Stanley-Reisner ideal of the simplicial sphere $\partial\Delta(A_\bullet)$.

Our main result is:

Theorem 1.4 (See Theorem 4.3). *Let $\mathfrak{j}$ be an Artin monomial ideal in $k[x_1, \ldots, x_n]$ strictly containing the complete intersection $\mathfrak{c}$ (but the $x_i^{\alpha_i}$ need not be minimal generators of $\mathfrak{j}$). Let $A \to \{x_1, x_2, \ldots, x_n\}$ be a map with fiber A_i over x_i of cardinality α_i.*

Any polarization $J \subseteq k[A]$ of $\mathfrak{j}$ defines, via the SR-correspondence, a full dimensional simplicial ball on the simplicial sphere $\partial\Delta(A_\bullet)$. Moreover, any proper polarization of $\mathfrak{j}$ is, when followed by a suitable extension, isomorphic to such a J.

By [3, Remark 2.5] the polarization J in $k[A]$ is proper precisely when each $x_i^{\alpha_i}$ is a minimal generator for $\mathfrak{j}$. Also, if $\mathfrak{j} = \mathfrak{c}$, the polarization is not a ball, but the sphere $\partial\Delta(A_\bullet)$.

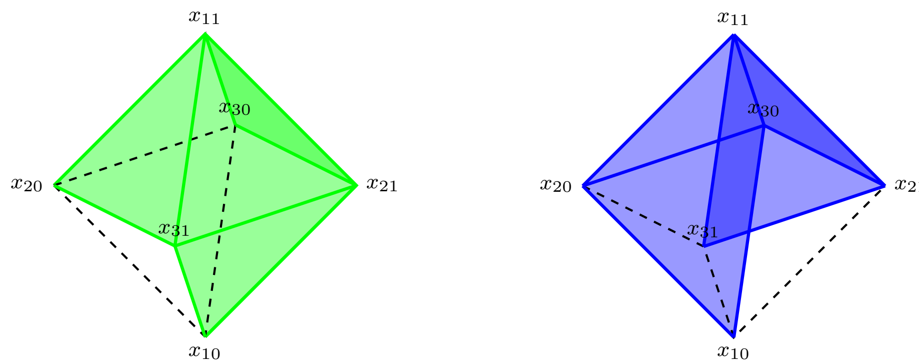

Figure 2. Triangulated balls defined by J_1 and J_2

Example 1.5. We continue Example 1.3. In Figure 2 we have the embeddings on the octahedron of the two hexagon triangulations in Figure 1.

The essential element in proving Theorem 4.3 is to show the following:

Theorem 1.6 (See Theorem 4.4). *Any full-dimensional Cohen-Macaulay proper sub-complex on the simplicial sphere $\partial\Delta(A_\bullet)$ is a constructible triangulated ball.*

In fact, such sub-complexes X are precisely those that via the Stanley-Reisner correspondence come from polarizations of Artin monomial ideals. As noted above (see (1)), the facets of X are then indexed by a subset $T \subseteq A_1 \times \cdots \times A_n$. Thus we may write $X = X(T)$, and its Stanley-Reisner ideal $J = J(T)$.

It is noteworthy that by [25] there are such sub-complexes X of $\partial\Delta(A_\bullet)$ which are *not shellable*, see Proposition 8.7. But by the above, they are constructible.

1.3. Dual geometric setting.

A standard fact (see Subsection 2.4) is that J is CM iff its Alexander dual I is an ideal with linear resolution. We write $I(T)$ for the Alexander dual of $J(T)$. Considering each A_i as a color class, each $\mathbf{a} \in A_1 \times \cdots \times A_n$ gives a *rainbow monomial* $m(\mathbf{a}) = \prod_{i=1}^n a_i$. The ideal $I(T)$ is generated by the $m(\mathbf{a})$ for $\mathbf{a} \in T$. The subset $T \subseteq A_1 \times \cdots \times A_n$ identifies with a subset of the vertices of the product polytope

$$\Pi(A_\bullet) = \Delta(A_1) \times \cdots \times \Delta(A_n).$$

The vertices of $\Pi(A_\bullet)$ are naturally labelled by rainbow monomials. The subset T induces a *saturated* cell sub-complex $C(T)$ of $\Pi(A_\bullet)$ consisting of all sub-cells $\Pi(R_\bullet)$ of $\Pi(A_\bullet)$ (for non-empty subsets $R_i \subseteq A_i$) such that the product set $R_1 \times \cdots \times R_n \subseteq T$. We prove that when $I(T)$ has a linear resolution, it admits a cellular minimal free resolution supported by the cell sub-complex $C(T)$, Theorem 5.5. As consequence, we get a homological criterion on T, Corollary 5.7, for T to correspond to a polarization of an Artin monomial ideal. The maximal cells of $C(T)$ correspond bijectively to the components in the unique irreducible monomial decomposition of the Artin ideal $\mathfrak{j}$. Analysing then the possible sub cell-complexes $C(T)$ corresponding to this decomposition, enables in many example cases a full understanding of the possible polarizations of the Artin monomial ideal $\mathfrak{j}$, Section 7.

Example 1.7.

$$\mathfrak{j} = (x^2, y^2, z^2, w^2) + (xz, xw, yz, yw, zw).$$

The simplicial complex associated to the square-free part here is given in Figure 3.

The associated cell complex $C(T)$ will then consist of a square and two edges (in general a $(d-1)$-simplex is turned into a d-hypercube, when each $\alpha_i = 2$). The cell

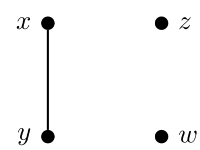

FIGURE 3. Simplicial complex of ideal (xz, xw, yz, yw, zw)

complex must be acyclic. This gives, up to isomorphism, the four possibilities for $C(T)$ given in Figure 4.

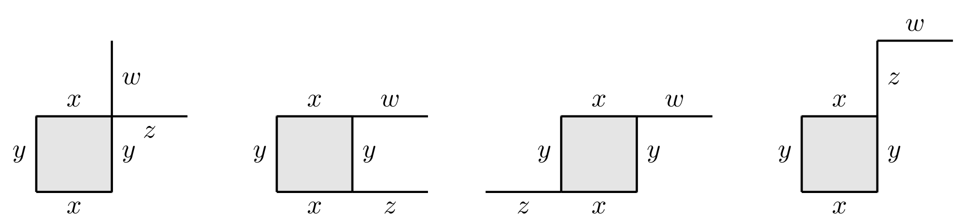

FIGURE 4. Acyclic cell complexes of one square and two line segments

These correspond to the four, up to isomorphism, possible polarizations of the Artin ideal j. They can be worked out to be:

$$J_1 = (x_0x_1, y_0y_1, z_0z_1, w_0w_1) + (x_0z_0, x_0w_0, y_0z_0, y_0w_0, z_0w_0)$$
$$J_2 = (x_0x_1, y_0y_1, z_0z_1, w_0w_1) + (x_0z_0, x_0w_0, y_1z_0, y_0w_0, z_0w_0)$$
$$J_3 = (x_0x_1, y_0y_1, z_0z_1, w_0w_1) + (x_1z_0, x_0w_0, y_1z_0, y_0w_0, z_0w_0)$$
$$J_4 = (x_0x_1, y_0y_1, z_0z_1, w_0w_1) + (x_0z_0, x_0w_0, y_0z_0, y_0w_0, z_1w_0)$$

This is explained in more detail in Example 7.6.

Let us inform on a line of research leading to our results. In [9], T.Bier constructed simplicial spheres from a simplicial complex X on a set V. Essentially, the idea is that X and its Alexander dual should glue together to give a simplicial sphere, as in the topological setting of Alexander duality, [27], see [35, Sec.5.6] for an argument.

In [11] Björner, Pfaffenholz, Sjöstrand and Ziegler generalized this from down-sets in Boolean posets (that is, simplicial complexes) to down-sets in any poset. In addition they proved (by a counting argument) that Bier spheres were so numerous that most of them could not be realized as convex polytopes. Bier spheres were considerably generalized by D'Alì, Nematbakhsh and the first author in [14]: Given any poset P let $\mathcal{I}$ be a finite down-set in the set of order-preserving maps $\mathrm{Hom}(P, \mathbb{N})$. This gives rise to the so called letterplace ideals $L(\mathcal{I}; P)$, see Subsection 11.3 or [21], which are SR-ideals of triangulated balls and whose boundaries are simplicial spheres, again with a simply defined SR-ideal. Our main result here is a further considerable generalization of letterplace ideals. In particular since even the double specialization to Bier spheres is so numerous that they do not in general have convex realizations (though recently shown to be star-shaped [31]), the same holds for the simplicial spheres given here.

The organization of this article is as follows.

Part I, Sections 2-4: Polarizations and triangulated balls. We give background and prove the main Theorem 4.3.

Section 2 recalls basics on simplicial complexes, Stanley-Reisner rings, Alexander duals, joins, and constructible monomial ideals. Section 3 gives the geometric setting, the join of boundaries $\partial\Delta(A_\bullet)$. The facets of these joins correspond bijectively to elements $(a_1, \ldots, a_n)$ of $A_1 \times \cdots \times A_n$. This gives a *rainbow monomial* $a_1 \cdots a_n$. Subsets T of $A_1 \times \cdots \times A_n$ then generate a *rainbow ideal* $I(T)$. The Alexander dual ideals of polarizations are rainbow ideals with linear resolution. In Section 4 we give the main result, Theorem 4.3, that polarizations of Artin monomial ideals give triangulated balls. The argument is by showing that rainbow ideals with linear resolution are constructible ideals.

Part II, Sections 5-7: Dual cell complexes. We study the dual cell complex $C(T)$, which is a sub-complex of the product polytope $\Pi(A_\bullet)$.

Section 5 defines $C(T)$. When $I(T)$ has linear resolution, we show that it admits a cellular minimal free resolution supported by $C(T)$. We derive the homological criterion for T to correspond to a polarization. Such T are called *restriction-acyclic*. Section 6 shows that the maximal cells in $C(T)$ are in bijection with the components in the unique minimal irreducible decomposition of the Artin monomial ideal $\mathfrak{j}$.

How can we find all polarizations of the Artin ideal $\mathfrak{j}$? Section 7 addresses this when $\mathfrak{j}$ contains the second power x_i^2 of all variables x_i. Such $\mathfrak{j}$ are in bijection to simplicial complexes on vertices $\{1, 2, \ldots, n\}$. The product polytope $\Pi(A_\bullet)$ is then a hypercube. We show in a range of examples how we may classify the polarizations of such $\mathfrak{j}$.

Part III, Sections 8-10: Linkage and complement triangulations. We study the complement set T^c, the complement triangulation of $X(T)$, and relations to linkage theory.

Section 8 shows that T corresponds to a polarization iff T^c does. The Alexander duals of $I(T)$ and $I(T^c)$ are then linked Cohen-Macaulay ideals. They define complement triangulated balls $X(T)$ and $X(T^c)$ on the join polytope $\partial\Delta(A_\bullet)$. Section 9 describes the boundaries of these triangulated balls. These are triangulated spheres whose Stanley-Reisner ideals are then Gorenstein ideals. Section 10 shows how the linkage descends to linkage of Artin monomial ideals.

Part IV, Sections 11-12: Squeezed spheres and further problems. Section 11 is somewhat independent from the rest. We show that the squeezed balls and spheres of G. Kalai [32] have Stanley-Reisner ideals which come from letterplace ideals, these ideals being polarizations of Artin monomial ideals. The last Section 12 sketches open research directions. In Subsection 12.2 we indicate how most simplicial balls are not of the type we construct here. However, as we see in examples, they might be *derived* from the simplicial balls here by joining variables (as we show in Section 11 for squeezed balls and spheres).

2. Preliminaries

We recall basics on simplicial complexes, the Stanley-Reisner correspondence, and Alexander duality. For introduction to these, see [28, 39] or [13, Sec.II.5].

2.1. Simplicial complexes and squarefree monomial ideals. Let V be a finite set. A simplicial complex on V is a family X of subsets of V such that if $F \in X$ and $G \subseteq F$, then also $G \in X$. The F are called the *faces* of X and the dimension of F is the cardinality $|F| - 1$. Maximal faces are called *facets*. If all facets have the same dimension, X is said to be *pure*. If X is the full power set of V, then X is called the *simplex* on V and is denoted $\Delta(V)$. The *boundary* ∂X of a pure simplicial complex X is the smallest simplicial complex containing all faces of X which are on only one facet, and are properly contained in this facet. A simplicial complex X has a natural topological realization $|X|$ in the space $\mathbb{R}^V$, [42].

Let k be a field and $k[V]$ the polynomial ring with elements of V as variables. If $R \subseteq V$ denote by $m(R) = \prod_{v \in R} v$ the associated monomial. Such a monomial is said to be *squarefree*, and if an ideal is generated by such monomials, it is also said to be *squarefree*.

Given a simplicial complex X we have its associated Stanley-Reisner (SR) ideal $I = I_X$ which is the ideal generated by the monomials $m(R)$ such that R is *not* in X. The quotient ring $k[V]/I$ is the Stanley-Reisner ring of X.

Simplicial complexes form a lattice with join $X \cup Y$, the union of the families X and Y, and meet $X \cap Y$. The squarefree monomial ideals also form a lattice with join the sum of ideals, and meet the intersection of ideals.

The Stanley-Reisner correspondence gives a one-to-one correspondence between the lattice of simplicial complexes on V and the lattice of squarefree monomial ideals in $k[V]$, with order relation reversed. Hence the join $X \cup Y$ corresponds to the meet of ideals $I_X \cap I_Y$, and the meet $X \cap Y$ corresponds to the sum of ideals $I_X + I_Y$.

2.2. Alexander duality. The Alexander dual monomial ideal of $I(= I_X)$ is the monomial ideal $J = I^\vee$ generated by all monomials m which have a non-trivial common divisor with every monomial in I. This is an involutive correspondence. The squarefree monomials in J are the $m(F^c)$ where the F are faces of X, and $F^c = V \backslash F$ its complement. In particular the minimal generators of J are the monomials $m(F^c)$ of the complements of facets F of X, see [28, Lemma 1.5.3].

Alexander duality gives a bijection between lattices of ideals with order relation reversed. Hence the Alexander dual of the join $I + L$ is the meet $I^\vee \cap L^\vee$, and the dual of $I \cap L$ is $I^\vee + L^\vee$.

2.3. External join of simplicial complexes. Let X be a simplicial complex on V, and Y a simplicial complex on a disjoint vertex set W. The (external) join of X and Y is the simplicial complex $X * Y$ on $V \cup W$ given by:

$$X * Y = \{A \cup B \mid A \in X, B \in Y\}.$$

(This is not to be confused with the (internal) join in the lattice of simplicial complexes on a fixed set V, which is simply the union of X and Y. Henceforth join will always mean external join.) The SR-ideal of the join $X * Y$ is the ideal $(I_X) + (I_Y)$ generated by I_X and I_Y in $k[V \cup W]$.

The topological realization $|X * Y|$ of the join, is obtained by connecting points in $|X|$ and $|Y|$ by lines. So it is the set $|X| \times [0, 1] \times |Y|$ modulo identifying pairs $(x, 0, y)$ and $(x, 0, y')$ and identifying pairs $(x, 1, y)$ and $(x', 1, y)$. The join of two spheres S^n and S^m of dimensions n and m, is the sphere S^{n+m+1}. If $|Y|$ is two points (corresponding to the simplicial complex $\{\{v_1\}, \{v_2\}, \varnothing\}$, which is topologically the sphere S^0), the join $X * Y$ has topological realization the suspension $S|X|$. The suspension of S^n is S^{n+1}.

2.4. Constructible squarefree monomial ideals. A simplicial complex of dimension d is *constructible* if either

- X is a simplex of dimension d, or
- there is a decomposition $X = Y \cup Z$ where Y and Z are constructible of dimension d and $Y \cap Z$ is constructible of dimension $d - 1$.

The following implications are well known:

vertex decomposable $\Rightarrow$ shellable $\Rightarrow$ constructible $\Rightarrow$ Cohen-Macaulay/any field.

Let I be the Alexander dual ideal of the Stanley-Reisner ideal I_X. Translating the above to conditions on the ideal I we get the following. A squarefree monomial ideal I is *e-constructible* (notion introduced in [46]) if:

- $I = (m)$ is generated by a monomial of degree e, or

- there is a decomposition $I = J + K$ such that J and K are e-constructible, and $J \cap K$ is $(e+1)$-constructible.

Remark 2.1. If I corresponds to the simplicial complex Y via the SR-correspondence, one should note that I being constructible does *not* correspond to Y being constructible. Rather I being constructible corresponds to the Alexander dual simplicial complex $Y^\vee$ being constuctible (where $Y^\vee$ has the Alexander dual ideal $I^\vee$ as Stanley-Reisner ideal).

The Eagon-Reiner theorem, [17], states that a squarefree monomial ideal I has a linear resolution if and only if the Alexander dual $I^\vee$ is a Cohen-Macaulay ideal, see standard textbooks [28, Thm. 8.1.9], [39, Thm. 5.56]. As a consequence constructible ideals have linear resolutions. In general for ideals there are implications:

vertex splittable $\Rightarrow$ linear quotients $\Rightarrow$ constructible $\Rightarrow$ linear resolution/any field.

The notion of vertex splittable is introduced in [41], and linear quotients in [29].

The statement of the following lemma is analogous to the idea of constructibility. We state it as we need it later for an important example in Proposition 8.7.

Lemma 2.2. *Let X and Y be Cohen-Macaulay (CM) simplicial complexes of dimension d on the same vertex set. Suppose the intersection $X \cap Y$ has dimension $< d$. Then the union $X \cup Y$ is CM iff the intersection $X \cap Y$ is CM of dimension $d-1$.*

Proof. A simplicial complex X of dimension d is CM iff the local cohomology groups $H^i_{\mathfrak{m}}(k[X])$ vanish for $i \leqslant d$, and is non-vanishing for $i = d+1$, [28, A.7]. We show that there is an isomorphism

$$H^p_{\mathfrak{m}}(k[X \cap Y]) \cong H^{p+1}_{\mathfrak{m}}(k[X \cup Y]), \quad p \leqslant d-1, \tag{2}$$

which will prove the claim, since for e the dimension of $X \cap Y$, the cohomology group $H^{e+1}_{\mathfrak{m}}(k[X \cap Y])$ is non-vanishing.

Consider the exact sequences:

$$0 \to K_1 \to k[X \cup Y] \to k[X] \to 0, \quad 0 \to K_2 \to k[Y] \to k[X \cap Y] \to 0. \tag{3}$$

Here

$$K_1 = \frac{I_X}{I_{X \cup Y}} = \frac{I_X}{I_X \cap I_Y} \cong \frac{I_X + I_Y}{I_Y} = \frac{I_{X \cap Y}}{I_Y} = K_2.$$

Whence taking long exact cohomology sequences of the two short exact sequences (3), we get (2). □

3. Polarizations and their Alexander duals

In polarizations of Artin ideals $\mathfrak{j}$ in $k[x_1, \ldots, x_n]$, the variables in the polynomial ring of the polarization J, come in classes $A_1, A_2, \ldots, A_n$. The variables in A_i map to x_i and we call the $A_1, \ldots, A_n$ color classes. This polarization J defines a full-dimensional sub-complex of the polytope whose boundary is a join of simplices.

We show that the Alexander dual of J is a *rainbow ideal*: It is generated by monomials which have one variable from each color class. These ideals enables us in the next section to prove our main result.

3.1. Joins of boundaries of simplices. Recall that $\Delta(V)$ is the simplex on the finite set V. The SR-ideal generated by the single monomial $m(V) = \prod_{v \in V} v$ defines the boundary $\partial\Delta(V)$ of $\Delta(V)$, which is topologically a sphere $S^{|V|-2}$ where $|V|$ is the cardinality of V.

Let $A_1, A_2, \ldots, A_n$ be disjoint finite sets, and $A = \sqcup_{i=1}^n A_i$ their union. The complete intersection ideal

$$C = (m(A_1), m(A_2), \ldots, m(A_n)) \subseteq k[A]$$

is the SR-ideal of the join of the boundaries of simplices

$$\partial\Delta(A_\bullet) = \partial\Delta(A_1) * \partial\Delta(A_2) * \cdots * \partial\Delta(A_n), \tag{4}$$

which topologically is a join of spheres, and so also a sphere. Its dimension is $\sum_{i=1}^n |A_i| - 1 - n$. If each cardinality $|A_i| = 2$, then each boundary $\partial\Delta(A_i)$ is two points, and we are taking iterated suspensions. This gives a sphere of dimension $n-1$: For $n = 2$, (the boundary of) the square, for $n = 3$ (the boundary of) the octahedron, and in general (the boundary of) the hyperoctahedron.

3.2. Rainbow ideals. Denote $\mathbf{A} = A_1 \times A_2 \times \cdots \times A_n$. For $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ in $\mathbf{A}$, denote by $[\mathbf{a}]$ the set $\{a_1, a_2, \ldots, a_n\}$. The faces of $\partial\Delta(A_\bullet)$ are all subsets $S \subseteq A$ such that no $A_i \subseteq S$. In other words it is all S such that each $A_i \backslash S$ is non-empty. Equivalently we may find $\mathbf{a} \in \mathbf{A}$ with $[\mathbf{a}] \cap S = \varnothing$. The facets of $\partial\Delta(A_\bullet)$ are then precisely the complement sets $[\mathbf{a}]^c$ for $\mathbf{a} \in \mathbf{A}$.

The following is of course true for any triangulated sphere, but the argument is explicit and informative in our setting.

Lemma 3.1. *Any codimension one face of $\partial\Delta(A_\bullet)$ is on exactly two facets.*

Proof. A codimension one face of $\partial\Delta(A_\bullet)$ is $F = [\mathbf{a}]^c \backslash \{a'\}$ for some $a' \in A$. So there is an i such that $a' \in A_i$, and write $a'_i := a'$. Let $\mathbf{a}' = (a_1, \ldots, a'_i, \ldots, a_n)$. Then F is precisely on the two facets $[\mathbf{a}]^c$ and $[\mathbf{a}']^c$. □

For $\mathbf{a} \in \mathbf{A}$ let $m(\mathbf{a})$ be the monomial $\prod_{i=1}^n a_i$. Thinking of A_i as a color class, we call such a monomial a *rainbow monomial*. The Alexander dual ideal B of the complete intersection $C = (m(A_1), \ldots, m(A_n))$ is generated by the complements of the facets of $\partial\Delta(A_\bullet)$, and so it is the monomial ideal generated by all the rainbow monomials. Letting (A_i) be the "color ideal" generated by the elements of A_i (considered as variables), then B is the product of these color ideals:

$$B = (m(\mathbf{a}) \,|\, \mathbf{a} \in \mathbf{A}) = \prod_{i=1}^n (A_i).$$

Definition 3.2. *For a subset $T \subseteq \mathbf{A}$, denote by $I(T)$ the ideal generated by $m(\mathbf{a})$ for $\mathbf{a} \in T$. Such an ideal is called a rainbow ideal. For the complement set $T^c = \mathbf{A} \backslash T$, we usually write $L(T)$ for $I(T^c)$, the complement rainbow ideal.*

Remark 3.3. The ideal $B = I(\mathbf{A})$ is the squarefree monomial ideal defining the Cox irrelevant ideal for the toric variety which is the multiprojective space

$$\mathbb{P}^{|A_1|-1} \times \cdots \times \mathbb{P}^{|A_n|-1}.$$

In this setting [7] introduces *virtual resolutions*, i.e. resolutions with homology annihilated by powers of B. Combinatorial aspects of this, in particular Stanley-Reisner rings that are virtually Cohen-Macaulay, are studied in [8].

Remark 3.4. For $T \subseteq \mathbf{A}$ the family of subsets $[\mathbf{a}] \subseteq A$ where $\mathbf{a} \in T$ are precisely the collection of edges of n-partite n-uniform hypergraphs, also called n-partite n-uniform clutters, [45]. They are also called multipartite uniform hypergraphs.

In [44] a class of d-partite d-uniform hypergraphs, Ferrer's hypergraphs are studied. The associated rainbow ideals are shown to have a linear resolutions, and an explicit form as a cellular resolution by the "complex of boxes" is given. These ideals are again a special case of co-letterplace ideals [21], see Subsection 12.1, which also have linear resolutions of an explicit form given in [14]. [45] uses the terminology d-partite d-uniform clutters, and explicitly describes the first linear strand of rainbow ideals $I(T)$.

Remark 3.5. d-partite d-uniform hypergraphs have been a topic of interest in combinatorics and combinatorial geometry. In [23] they look at the associated embedding by generic points in $\mathbb{R}^d$, relating it to the colored Tverberg's theorem [35]. Matchings in such hypergraphs are studied in [16, 6]. Mengerian r-uniform hypergraphs are shown to be r-partite in [5], but the converse implication does not hold. From a commutative algebra perspective, [2] relates this to symbolic powers of ideals, and identify the subclass of tripartite tri-uniform hypergraphs for which $I^{(2)} = I^2$.

3.3. Polarizations. Recall from the introduction, Definitions 1.1 and 1.2 of separations and polarizations of monomial ideals.

Let $\mathfrak{j}$ be an Artin monomial ideal in $k[x_1, \ldots, x_n]$, containing the Artin complete intersection

$$\mathfrak{c} = (x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}).$$

We do not require the powers $x_i^{\alpha_i}$ to be minimal generators of the ideal $\mathfrak{j}$. Let $A_1, \ldots, A_n$ be disjoint sets with A_i of cardinality α_i, and $A = \cup_i A_i$. We have a map of polynomial rings $k[A] \to k[x_1, \ldots, x_n]$ induced by the map $A \to \{x_1, x_2, \ldots, x_n\}$ sending elements of A_i to the variable x_i. By [3, Remark 2.5] any proper polarization J' of an Artin monomial ideal containing the complete intersection $\mathfrak{c}$ is, after possibly extending J', isomorphic to a polarization J contained in $k[A]$.

We also recall the following, which is essentially [3, Prop.3.6]. Due to its importance here, we include the proof for the benefit of the reader.

Proposition 3.6. *Let $C = (m(A_1), \ldots, m(A_n))$ be the complete intersection. Polarizations $J \subseteq k[A]$ of Artin monomial ideals containing the complete intersection $\mathfrak{c}$ are precisely the Cohen-Macaulay squarefree ideals $J \subseteq k[A]$ of height n containing C.*

Such ideals J are in turn precisely the Alexander duals of rainbow monomial ideals in $k[A]$ with linear resolution.

Proof. i. Assume $J \subseteq k[A]$ is a polarization of an Artin monomial ideal containing $\mathfrak{c}$. Then J contains C, as $m(A_i)$ must be the monomial mapping to $x_i^{\alpha_i}$. Since C is cut down to an Artin monomial ideal, J is also cut down by the regular sequence (for $k[A]/J$) to an Artin monomial ideal. An Artin ideal is Cohen-Macaulay, and so the same holds for J by [13, Thm. 2.1.3.a].

ii. Conversely assume J is Cohen-Macaulay and squarefree of height n containing C. The sequence of variable differences cuts C down to an Artin monomial ideal, and so also cuts J down to an Artin monomial ideal $\mathfrak{j}$. Thus starting from $k[A]/J$, which has dimension $|A| - n$, at each step when we divide out by a variable difference, dimension is cut by one. By [13, Thm. 2.1.2.c] the sequence is regular for $k[A]/J$ and so J is a polarization of $\mathfrak{j}$.

By the Eagon-Reiner theorem, see Subsection 2.4, J is Cohen-Macaulay iff its Alexander dual I has a linear resolution. Let X be the simplicial complex associated to J. That J is unmixed of height n is equivalent to the facets of X being facets of $\partial\Delta(A_\bullet)$, and so $I = I(T)$ for some $T \subseteq \mathbf{A}$. ☐

Example 3.7. The standard polarization of $(x_1^2, x_1x_2^2, x_2^3)$ is

$$(x_{10}x_{11}, x_{10}x_{20}x_{21}, x_{20}x_{21}x_{22}).$$

(Standard polarization means that each power x_i^r is replaced by the product $x_{i0}x_{i1} \cdots x_{i,r-1}$.) It corresponds to the simplicial complex to the left in Figure 5. To the right it is drawn in blue on $\partial\Delta(A_1) * \partial\Delta(A_2)$ where $|A_1| = 2$ and $|A_2| = 3$. It

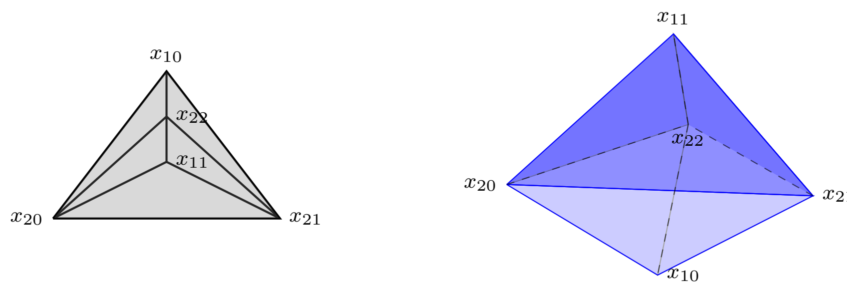

Figure 5. Triangulated ball of Example 3.7

consists of five of the six facets. Only the lower front facet, with vertices x_{20}, x_{21} and x_{10}, is not in the simplicial complex.

Example 3.8. The ideal J_2 of Example 1.3 is a polarization of $(x_1, x_2, x_3)^2$ containing the complete intersection (x_1^2, x_2^2, x_3^2). We can let

$$A_1 = \{x_{10}, x_{11}\}, \quad A_2 = \{x_{20}, x_{21}\}, \quad A_3 = \{x_{30}, x_{31}\}.$$

The Alexander dual ideal of J_2 is the rainbow ideal I_2 with generators:

$$x_{10}x_{20}x_{30}, x_{10}x_{20}x_{31}, x_{10}x_{21}x_{31}, x_{11}x_{21}x_{31}.$$

We may identify each of A_1, A_2, A_3 with $\{0, 1\}$, and then the ideal I_2 is $I(T)$ where

$$T = \{000, 001, 011, 111\}.$$

Remark 3.9. The polarizations of powers of graded maximal ideals $(x_1, \dots, x_n)^m$ are classified by Almousa, Lohne and the first author in [3]. In particular polarizations of $(x_1, \dots, x_n)^2$ are in bijections with trees with n edges. These results in [3] are partially extended to strongly stable ideals generated in a single degree in Hao, Luo and Saint Rain [26, Prop.4.2, Thm. 6.1]. Note that such strongly stable ideals are not Artin unless they are a power of the graded maximal ideal. [4] gives resolutions of polarizations of $(x_1, \dots, x_n)^m$, connecting the combinatorics of [3] with that of Hook tableaux.

4. Polarizations define triangulated balls

The previous section shows that polarizations of Artin monomial ideals in $k[x_1, \dots, x_n]$ containing the complete intersection $\mathfrak{c} = (x_1^{\alpha_1}, \dots, x_n^{\alpha_n})$ correspond, by Alexander duality, to rainbow ideals in $k[A]$ with linear resolutions, for A a union of color classes $A_1, \dots, A_n$ and cardinalities $|A_i| = \alpha_i$. Rainbow ideals are $I(T)$ for a subset $T \subseteq \mathbf{A} = A_1 \times \cdots \times A_n$.

We investigate ideals of type

$$I = b_1 I_1 + b_2 I_2 + \cdots + b_m I_m$$

where the I_j do not involve the b-variables, and the situation when such ideals have a linear resolution. We apply this to show that rainbow ideals have linear resolutions if and only if they are constructible. As a consequence we get our main theorem, that polarizations of Artin monomial ideals define triangulated balls by the Stanley-Reisner correspondence.

4.1. The main result. The following is the key result leading to our main statement.

Theorem 4.1. *For a subset $T \subseteq A_1 \times \cdots \times A_n$, the rainbow ideal $I(T)$ has n-linear resolution if and only if it is n-constructible.*

By Subsection 2.4, if $I(T)$ is constructible, then $I(T)$ has a linear resolution. The challenge is to show the other direction. But let us first give the consequence which is our main result. First recall the following, stated in the overview article [10, Thm. 11.4] by A. Björner:

Theorem 4.2. *Let X be a pure constructible simplicial complex such that every codimension one face is on one or two facets. Then the topological realization $|X|$, is a ball or a sphere (the latter when every codimension one face is on exactly two facets).*

Using this, Theorem 4.1 implies Conjecture 3.2 in [3].

Theorem 4.3 (Main Theorem). *Let $\mathfrak{j}$ be an Artin monomial ideal in $k[x_1, \ldots, x_n]$ strictly containing the complete intersection $\mathfrak{c}$ (but the $x_i^{\alpha_i}$ need not be minimal generators of $\mathfrak{j}$). Let $A \to \{x_1, x_2, \ldots, x_n\}$ have fiber A_i over x_i of cardinality α_i.*

Any polarization $J \subseteq k[A]$ of $\mathfrak{j}$ defines, via the SR-correspondence, a full dimensional simplicial ball on the simplicial sphere $\partial\Delta(A_\bullet)$. Moreover, any proper polarization of $\mathfrak{j}$ is, when followed by a suitable extension, isomorphic to such a J.

Proof. By [3, Remark 2.5], any proper polarization of $\mathfrak{j}$ will have at most α_i variables mapping to x_i. Hence it is, after possible extension, isomorphic to a polarization $J \subseteq k[A]$. By Proposition 3.6 such a polarization is of the form $J = J(T)$, and is Cohen-Macaulay. Denote the associated simplicial complex by $X(T)$. The Alexander dual $I(T)$ of $J(T)$ has a linear resolution. By Theorem 4.1, $I(T)$ is constructible and so $X(T)$ is constructible. By Lemma 3.1 every codimension one face of $X(T)$ is on at most two facets. Theorem 4.2 above gives that $X(T)$ is topologically a ball. □

We also get the following.

Theorem 4.4. *Any full-dimensional Cohen-Macaulay proper sub-complex on the simplicial sphere $\partial\Delta(A_\bullet)$ is a constructible ball. It is obtained as in Theorem 4.3 above.*

Proof. This follows by Proposition 3.6 and Theorem 4.1. □

At first we tried to show that such a sub-complex is shellable, which failed. In fact, as pointed out by an anonymous referee, such a sub-complex may not be shellable, see Proposition 8.7.

4.2. Ideals with linear resolutions and constructability. We proceed to prove Theorem 4.1. First we give a general fact. Let $k[B \cup X]$ be a polynomial ring, and $\mathbb{Z}^B$ the free abelian group with basis e_b for $b \in B$. We consider the polynomial ring to be $\mathbb{Z}^B \oplus \mathbb{Z}$ graded, where the variables $b \in B$ have degree e_b and the variables in X have degree $e = (\mathbf{0}, 1)$.

The following is a crucial fact: If p is a homogeneous polynomial in $k[B \cup X]$ (for the grading by $\mathbb{Z}^B \oplus \mathbb{Z}$), of degree $(\mathbf{r}, s)$, then p is a product $(\prod_{b \in B} b^{r_b}) \cdot p'$ where p' is a homogeneous polynomial in $k[X]$ of degree s.

Let M be a $\mathbb{Z}^B \oplus \mathbb{Z}$-graded $k[B \cup X]$-module. If $m \in M$ is homogeneous of degree $(\mathbf{r}, s)$, write $\deg_B(m) = \mathbf{r}$. For $\mathbf{r} \in \mathbb{Z}^B$, denote $M_{\leqslant \mathbf{r}}$ be the submodule of M generated by the homogeneous elements $m \in M$ with $\deg_B(m) \leqslant \mathbf{r}$. Note that if $F = \oplus_{j \in J} S(-\mathbf{u}_j)$ is a free module, then $F_{\leqslant \mathbf{r}} = \oplus_{\deg_B(\mathbf{u}_j) \leqslant \mathbf{r}} S(-\mathbf{u}_j)$.

Lemma 4.5. *Let $\phi : N \to M$ be a morphism of $\mathbb{Z}^B \oplus \mathbb{Z}$-graded $k[B \cup X]$-modules, with M a torsion free module. Let $\mathbf{r} \in \mathbb{Z}^B$. If K is the kernel of ϕ, then $K_{\leqslant \mathbf{r}}$ is the kernel of $\phi_{\leqslant r} : N_{\leqslant \mathbf{r}} \to M_{\leqslant \mathbf{r}}$.*

Proof. Clearly $K_{\leqslant \mathbf{r}}$ is contained in the kernel of $\phi_{\leqslant \mathbf{r}}$. Suppose n is homogeneous and in the kernel of $\phi_{\leqslant \mathbf{r}}$. We may write $n = \sum p_i n_i$, with p_i,n_i homogeneous in respectively $k[B \cup X]$ and N, and with $\deg_B(n_i) \leqslant \mathbf{r}$. Let $\deg_B(n) = \mathbf{q}$, and consider the monomial $u = \prod_{q_b > r_b} b^{q_b - r_b}$. Since each $\deg_B(n_i) \leqslant \mathbf{r}$, we have each $\deg_B(p_i) \geqslant \deg(u)$. But then p_i is divisible by u. Thus we may write $n = u \cdot n'$ with $\deg_B(n') \leqslant \mathbf{r}$.

Then $0 = \phi(n) = u\phi(n')$ and since M is torsion free, this gives $\phi(n') = 0$, and so $n' \in \ker \phi = K$. Thus $n \in K_{\leqslant \mathbf{r}}$. □

This gives the following (a generalization of [24, Thm. 2.1]).

Corollary 4.6. *Let M be a $\mathbb{Z}^B \oplus \mathbb{Z}$-graded torsion free $k[B \cup X]$-module, and $F_\bullet$ a minimal free resolution of M. Let $\mathbf{r} \in \mathbb{Z}^B$. Then $F_{\bullet, \leqslant \mathbf{r}}$ is a minimal free resolution of $M_{\leqslant \mathbf{r}}$.*

We say $F_\bullet$ is a *d-linear resolution*, if $F_i = \oplus_{j \in J} S(-\mathbf{u}_{ij})$ and for each $\deg(u_{ij}) = \mathbf{q} \oplus s$, its total degee $s + \sum_{b \in B} q_b$ equals $d + i$.

Corollary 4.7. *If $F_\bullet$ is a d-linear resolution of M, then $F_{\bullet, \leqslant \mathbf{r}}$ is a d-linear resolution of $M_{\leqslant \mathbf{r}}$.*

In the following we let $B = \{b_1, b_2, \dots, b_m\}$ have cardinality m. We write $\mathbb{Z}^B = \mathbb{Z}^m$ and $e_i = e_{b_i}$ for $i = 1, \dots, m$, and $e_{m+1} = \mathbf{0} \oplus 1$.

Proposition 4.8. *Let $I_1, \dots, I_m$ be ideals in $k[X]$. Consider the product ideals $(b_j)(I_j) \subseteq k[B \cup X]$, which we simply denote as $b_j I_j$. Also for $R \subseteq \{1, 2, \cdots, m\}$ denote*

$$I_R = \cap_{j \in R} I_j \subseteq k[X].$$

If the ideal

$$I = b_1 I_1 + b_2 I_2 + \cdots + b_m I_m$$

has $(d+1)$-linear resolution, then each I_R has d-linear resolution.

Proof. First we claim: For each $R \subseteq \{1, 2, \dots, m\}$ the sub-sum $\sum_{j \in R} b_j I_j$ has $(d+1)$-linear resolution. Let $\mathbf{r} = \sum_{j \in R} e_j$. Then $I_{\leqslant \mathbf{r}}$ is $\sum_{j \in R} b_j I_j$, and by Corollary 4.7 it has a linear resolution. As a special case each $b_j I_j$ has $(d+1)$-linear resolution, and so each I_j has d-linear resolution.

We now use induction on m. If $m = 1$ the statement holds. Let $m = 2$. We have an exact sequence

$$0 \to b_1 I_1 \cap b_2 I_2 \to b_1 I_1 \oplus b_2 I_2 \to b_1 I_1 + b_2 I_2 \to 0,$$

and a long exact Tor-sequence:

$$\begin{aligned} \cdots \to &\mathrm{Tor}_{p+1}(b_1 I_1 + b_2 I_2, k)_e \to \mathrm{Tor}_p(b_1 b_2 (I_1 \cap I_2), k)_e \to \mathrm{Tor}_p(b_1 I_1, k)_e \oplus \mathrm{Tor}_p(b_2 I_2, k)_e \\ &\to \mathrm{Tor}_p(b_1 I_1 + b_2 I_2, k)_e \to \cdots . \end{aligned}$$

Since $I = b_1 I_1 + b_2 I_2$ has $(d+1)$-linear resolution, the Tor-groups to the left vanish for $e - (p+1) \geqslant d + 2$ or equivalently for $e - p \geqslant d + 3$. Similarly, to the right above the Tor-groups vanish for $e - p \geqslant d + 2$. Thus

$$\mathrm{Tor}_p(b_1 b_2 (I_1 \cap I_2), k)_e = 0, \quad e - p \geqslant d + 3.$$

As $I_1 \cap I_2$ is generated in degree $\geqslant d$ it must have a d-linear resolution.

Assume now $m = 3$. From

$$0 \to (b_1 I_1 + b_2 I_2) \cap b_3 I_3 \to (b_1 I_1 + b_2 I_2) \oplus b_3 I_3 \to b_1 I_1 + b_2 I_2 + b_3 I_3 \to 0,$$

the same argument gives that $(b_1 I_1 + b_2 I_2) \cap (I_3)$ has $(d+1)$-linear resolution. Let $b_1 i_1 + b_2 i_2$ be a generator for this ideal in degree $d + 1$. It is also in (I_3), but since

I_3 is generated in degree d and its generators do not involve the variables b_1 and b_2, both i_1, i_2 are in (I_3). Therefore

$$(b_1 I_1 + b_2 I_2) \cap (I_3) = b_1 I_1 \cap I_3 + b_2 I_2 \cap I_3.$$

By the $m = 2$ case, both $I_1 \cap I_3$ and $I_2 \cap I_3$ have d-linear resolution, as well as their intersection $I_1 \cap I_2 \cap I_3$.

For larger m we split as $(b_1 I_1 + \cdots + b_{m-1} I_{m-1}) + b_m I_m$, and proceed in the same way as above. $\square$

The converse of Proposition 4.8 is not true in general, but for monomial ideals it is. We now proceed to do this. The key is the following:

Lemma 4.9. *Let I and $J_1, \ldots, J_m$ be monomial ideals. Then*

$$I \cap (J_1 + \cdots + J_m) = I \cap J_1 + \cdots + I \cap J_m.$$

Proof. It is enough to prove this for $m = 2$, as the general case easily follows by induction. Suppose $f \in I \cap (J_1 + J_2)$. Then every monomial in f is in I and in $J_1 + J_2$. Then the monomial is in either J_1 or J_2. Whence $f = f_1 + f_2$ where f_1 consists of monomials in $I \cap J_1$ and f_2 of monomials in $I \cap J_2$. $\square$

Proposition 4.10. *Let $I_1, \ldots, I_m$ be monomial ideals in $k[X]$ such that for every $R \subseteq [m]$, the ideal $I_R = \cap_{j \in R} I_j$ has d-linear resolution. Then the ideal $I = \sum_{j=1}^{n} b_j I_j$ in $k[B \cup X]$ has $(d+1)$-linear resolution.*

Proof. Again we use induction on m. Write

$$I = (b_1 I_1 + \cdots + b_{m-1} I_{m-1}) + b_m I_m = K_{m-1} + b_m I_m. \tag{5}$$

By induction we may assume K_{m-1} has $(d+1)$-linear resolution. Note also that I_m has d-linear resolution. We have a short exact sequence:

$$0 \to K_{m-1} \cap b_m I_m \xrightarrow{f} K_{m-1} \oplus b_m I_m \to I \to 0. \tag{6}$$

For the first term:

$$\begin{aligned} K_{m-1} \cap b_m I_m &= (b_1 I_1 + \cdots + b_{m-1} I_{m-1}) \cap b_m I_m \\ &= b_m (b_1 I_1 \cap I_m + \cdots + b_{m-1} I_{m-1} \cap I_m) \end{aligned}$$

By induction the last ideal above has $(d+2)$-linear resolution. So $K_{m-1} \cap b_m I_m$ has $(d+2)$-linear resolution. The mapping cone of f in the exact sequence (6) then becomes a $(d+1)$-linear resolution of I. $\square$

In fact, almost the same argument as above may be applied for the notion of constructability.

Proposition 4.11. *Let $I_1, \ldots, I_m$ be monomial ideals in $k[X]$. If for every $R \subseteq [m]$, I_R is d-constructible, then*

$$I = b_1 I_1 + \cdots + b_m I_m$$

is $(d+1)$-constructible.

Proof. We write $I = K_{m-1} + b_m I_m$ as in (5). Here $b_m I_m$ is $(d+1)$-constructible and by induction on n, K_{m-1} is also $(d+1)$-constructible. Further their intersection is:

$$\begin{aligned} K_{m-1} \cap b_m I_m &= (b_1 I_1 + \cdots + b_{m-1} I_{m-1}) \cap b_m I_m \\ &= b_m (b_1 I_1 \cap I_m + \cdots + b_{m-1} I_{m-1} \cap I_m), \end{aligned}$$

and by induction, this is $(d+2)$-constructible. Thus I is $(d+1)$-constructible. $\square$

Proof of Theorem 4.1. When we have one color class, $n = 1$, I is generated by variables and so is constructible. Let $n \geqslant 2$, and $A_1 = \{a_1, a_2, \dots, a_m\}$. We may then write

$$I = a_1 I_1 + \cdots + a_m I_m$$

where the $I_j = I(T_j)$ for various $T_j \subseteq A_2 \times \cdots \times A_n$. Since I has n-linear resolution, for every $R \subseteq [m]$, $I_R = \cap_{j \in R} I_j$ has $(n-1)$-linear resolution. As all the I_j are monomial ideals generated in degree $(n-1)$, we must have $I_R = I(T_R)$ where $T_R = \cap_{j \in R} T_j$. By induction on n, each of these are $(n-1)$-constructible. Then by Proposition 4.11, I is n-constructible. $\square$

5. Cell complexes and minimal free resolutions of rainbow ideals

The subset $T \subseteq A_1 \times \cdots \times A_n$ determines a canonical cell complex $C(T)$. This cell complex has a canonical monomial labelling. By classical constructions [39, Section 4] we obtain a cellular complex $\tilde{C}_\bullet(T)$. In our case this is a linear complex. The beautiful thing is that when $I(T)$ has linear resolution, the resolution is precisely this cellular complex, Theorem 5.5.

In particular, this gives an explicit homological criterion on T, Corollary 5.7, for when $J(T)$ is a polarization of an Artin monomial ideal.

5.1. The cell complex and free cellular complex associated to T.

A finite set A gives rise to the geometric simplex spanned by the elements of A:

$$\Delta(A) = \left\{ \sum_{a \in A} \gamma_a \cdot a \,|\, 0 \leqslant \gamma_a \leqslant 1, \sum_{a \in A} \gamma_a = 1 \right\} \subseteq \mathbb{R}^A.$$

Given finite non-empty sets $A_1, A_2, \dots, A_n$, we get a polytope:

$$\Pi(A_\bullet) = \Delta(A_1) \times \Delta(A_2) \times \cdots \times \Delta(A_n) \subseteq \times_{i=1}^n \mathbb{R}^{A_i} = \mathbb{R}^A.$$

Its dual (or polar) polytope is the join

$$\partial\Delta(A_\bullet) = \partial\Delta(A_1) * \partial\Delta(A_2) * \cdots * \partial\Delta(A_n).$$

The set of vertices of $\Pi(A_\bullet)$ is the set $A_1 \times \cdots \times A_n$.

Notation 5.1. When $R_i \subseteq A_i$ for $i = 1, \dots, n$ are *non-empty* subsets, we write $R_\bullet \subseteq^* A_\bullet$.

Definition 5.2. *The polytope $\Pi(A_\bullet)$ gives rise to a cell complex $C(A_\bullet)$, actually a polyhedral cell complex, whose cells are the $\Pi(R_\bullet)$ with $R_\bullet \subseteq^* A_\bullet$. The dimension of such a cell is $\sum_i |R_i| - n$. A subset $T \subseteq A_1 \times A_2 \times \cdots \times A_n$ induces a cell subcomplex $C(T)$ of $C(A_\bullet)$ consisting of all $\Pi(R_\bullet)$ such $R_1 \times R_2 \times \cdots \times R_n \subseteq T$. We say $C(T)$ is the saturated cell sub-complex of $C(A_\bullet)$ associated to T.*

The cell complex $C(T)$ gives rise to two important algebraic complexes.

The (augmented) chain complex:

$$\tilde{C}_\bullet(T) : C_{-1}(T) \leftarrow C_0(T) \leftarrow C_1(T) \leftarrow \cdots \leftarrow C_p(T) \leftarrow \cdots , \tag{7}$$

where $C_p(T)$ is the k-vector space with basis all cells of $C(T)$ of dimension p (and $C_{-1}(T)$ is the one-dimensional vector space with basis the empty cell), and standard differentials (see (8) below). The homology groups are the reduced homology groups. Note that $\tilde{H}_{-1}(\tilde{C}_\bullet(T)) \neq 0$ iff T is empty.

Let S be the polynomial ring $k[A]$ whose variables are the elements of $A = A_1 \cup \cdots \cup A_n$. A vertex $\mathbf{a} = (a_1, a_2, \dots, a_n)$ of $\Pi(A_\bullet)$ may be labelled by the monomial

$m(\mathbf{a}) = a_1 a_2 \cdots a_n$. The cell $\Pi(R_\bullet)$ is then labelled by the least common multiple of the monomials on its vertices. This is simply

$$m(R_\bullet) = m(R_1)m(R_2)\cdots m(R_n), \text{ where } m(R_i) = \prod_{r \in R_i} r,$$

which is a monomial of degree $\dim \Pi(R_\bullet) + n$.

The (augmented) free cellular complex: For $T \subseteq A_1 \times \cdots \times A_n$, this induces a labelling by monomials of the cell sub-complex $C(T)$. By [39, Section 4] it then gives a *free cellular complex*

$$\tilde{\mathcal{C}}_\bullet(T) : \mathcal{C}_{-1}(T) \xleftarrow{d_0} \mathcal{C}_0(T) \xleftarrow{d_1} \cdots \leftarrow \mathcal{C}_p(T) \leftarrow \cdots,$$

where $\mathcal{C}_p(T)$ is the free S-modules with generators all cells $\Pi(R_\bullet)$ in $C(T)$ of dimension p (and $\mathcal{C}_{-1}(T) = S$). The image of d_0 is $I(T)$. We may put a total order on each A_i. Together with the ordered sequence $A_1, A_2, \ldots, A_n$, this gives a total order on the union $A = \cup_1^n A_i$. Then $R_\bullet \subseteq^* A_\bullet$ inherits a total order on $R = \cup_1^n R_i$. For $b \in R$ denote by $\mathrm{pos}(b)$ its position in this total order. For $p \geqslant 1$ the differential d^p is then given by

$$\Pi(R_\bullet) \mapsto \sum_{\substack{i=1,\ldots,n \\ b \in R_i, |R_i| \geqslant 2}} (-1)^{\mathrm{pos}(b)} b \cdot \Pi(R_1, \ldots, R_i \backslash \{b\}, \ldots, R_n). \tag{8}$$

Example 5.3. Let $A_1 = \{a_1, b_1\}$ and $A_2 = \{a_2, b_2\}$ and $T = A_1 \times A_2$. The product polytope $\Pi(A_\bullet)$ is the square labelled as in Figure 6.

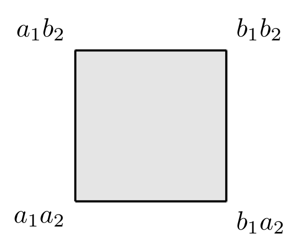

Figure 6. The square $\Pi(A_\bullet)$, with monomial labels on vertices

The augmented chain complex is

$$k \leftarrow k^4 \leftarrow k^4 \leftarrow k,$$

which has no homology, it is acyclic. Let $S = k[a_1, b_1, a_2, b_2]$. The rainbow ideal is $I(T) = (a_1a_2, a_1b_2, b_1a_2, b_1b_2)$. The augmented free cellular complex is

$$S \xleftarrow{d} S(-2)^4 \leftarrow S(-3)^4 \leftarrow S(-4).$$

The image of d is $I(T)$, and in this case, omitting the first term, the complex is a minimal free resolution of $I(T)$. In general, if T is the full $A_1 \times \cdots \times A_n$, the free cellular complex is always a minimal free resolution of $I(T)$. Finding out for which T this free cellular complex *is* a resolution, Theorem 5.5, is an aspect of the main result of our article.

The generators of $\mathcal{C}_p(T)$ for $p \geqslant 0$ have degrees $p + n$, so $\tilde{\mathcal{C}}_{\geqslant 0}(T)$ is a *linear* free cellular complex. The free cellular complex $\tilde{\mathcal{C}}_\bullet(T)$ is $\mathbb{N}^A$-graded. A family of subsets $R_\bullet \subseteq^* A_\bullet$ gives a restriction $T_{|R} = T \cap (R_1 \times \cdots \times R_n)$. It also gives a squarefree degree $\deg(R_\bullet) \in \mathbb{N}^A$. The free cellular complex $\tilde{\mathcal{C}}_\bullet(T)_{|\deg(R_\bullet)}$ in this specific degree identifies as the augmented chain complex $\tilde{C}_\bullet(T_{|R_\bullet})$ (see (7)).

Fact 1. *Let $R_\bullet \subseteq^* A_\bullet$. The homology of the augmented free cellular complex $\tilde{\mathcal{C}}_\bullet(T)$ in degree* $\deg(R_\bullet) \in \mathbb{N}^A$ *is the reduced homology*

$$H_i(\tilde{\mathcal{C}}_\bullet(T))_{\deg(R_\bullet)} = H_i(\tilde{C}_\bullet(T_{|R_\bullet})), \ i \geqslant -1.$$

This is standard and rather immediate from the above, see [39, Section 4].

Proposition 5.4. *For $p \geqslant 0$, the homology group $H_p(\tilde{\mathcal{C}}_\bullet(T))$ can only be nonzero in degrees $\geqslant p+n+2$.*

Proof. In short, this is a consequence of $C(T)$ being a saturated cell complex. All generators of $\mathcal{C}_p(T)$ are in degree $p+n$. Thus $\ker d_p$ is generated in degrees $\geqslant p+n+1$. Let u be a generator of $\ker d_p$ of degree $p+n+1$. The free cellular complex is $\mathbb{N}^A$-graded, and so we may assume u is homogeneous for this grading. This degree is also squarefree as it is a minimal generator. Its degree is then $\deg(R_\bullet)$ for some $R_\bullet \subseteq^* A_\bullet$. Consider the free cellular complex $\tilde{\mathcal{C}}_\bullet(R_\bullet)$. We have subcomplexes

$$\tilde{\mathcal{C}}_\bullet(T_{|R_\bullet}) \subseteq \tilde{\mathcal{C}}_\bullet(R_\bullet), \quad \tilde{\mathcal{C}}_\bullet(T_{|R_\bullet}) \subseteq \tilde{\mathcal{C}}_\bullet(T).$$

The latter is an isomorphism in degree $\deg(R_\bullet)$. Thus if $\ker d_p$ has a generator u in degree $\deg(R_\bullet)$, due to $\ker d_p^{\tilde{\mathcal{C}}_\bullet(T_{|R})} \hookrightarrow \ker d_p^{\tilde{\mathcal{C}}_\bullet(R_\bullet)}$ we get a generator of the latter kernel (in degree $\deg(R_\bullet)$), of total degree $p+1+n$. But $\ker d_p^{\tilde{\mathcal{C}}_\bullet(R_\bullet)} = \operatorname{im} d_{p+1}^{\tilde{\mathcal{C}}_\bullet(R_\bullet)}$ and $\mathcal{C}_{p+1}(R_\bullet)$ is the free rank one S-module $S\cdot\Pi(R_\bullet)$, with the generator $\Pi(R_\bullet)$ of degree $p+1+n$. This maps to a sum involving every face of $\Pi(R_\bullet)$ of dimension p. But due to the commutativity of the inclusion maps and differentials of $\tilde{\mathcal{C}}_\bullet(T_{|R_\bullet}) \hookrightarrow \tilde{\mathcal{C}}_\bullet(R_\bullet)$, the degree $p+n$ part $\mathcal{C}_p(T_{|R_\bullet})$ must have each of the generators corresponding to a face of $\Pi(R_\bullet)$ of dimension p. In particular all vertices of $\Pi(R_\bullet)$ must be in T. Since $C(T)$ is saturated, the full cell $\Pi(R_\bullet)$ is in $C(T)$. Then the generator u of $\ker d_p$ is the image of a cell in $\mathcal{C}_{p+1}(T)$. ☐

5.2. The criterion on T to correspond to a polarization. We now get a completely explicit description of the resolution of $I(T)$, when its resolution is linear.

Theorem 5.5. *Let $T \subseteq A_1 \times \cdots \times A_n$. If $I(T)$ has linear resolution, the free cellular complex $\tilde{\mathcal{C}}_{\geqslant 0}(T)$ is its minimal free resolution. (And $\tilde{\mathcal{C}}_{\geqslant 0}(T)$ is a minimal free resolution precisely in this case.)*

Proof. Let $\mathcal{F}_\bullet$ be a minimal free resolution of $I(T)$. As $I(T)$ is $\mathbb{N}^A$-graded, the same holds for $\mathcal{F}_\bullet$. Since $\mathcal{F}_\bullet$ is a resolution, and $\mathcal{C}_\bullet(T)$ consists of free modules, there is a morphism $\phi : \mathcal{C}_\bullet(T) \to \mathcal{F}_\bullet$ lifting the identity map on $I(T)$. Both $\mathcal{F}_0$ and $\mathcal{C}_0(T)$ have free bases mapping bijectively to the minimal generators of $I(T)$, the monomials $m(\mathbf{a})$ for $\mathbf{a} \in T$. So there is an isomorphism $\mathcal{F}_0 \cong \mathcal{C}_0(T)$.

Assume ϕ_i is an isomorphism for $i \leqslant p$. Consider the diagram:

$$\begin{tikzcd}
\mathcal{C}_p(T) \arrow[d, "\phi_p"] & \mathcal{C}_{p+1}(T) \arrow[l, "d^{\mathcal{C}}_{p+1}"'] \arrow[d, "\phi_{p+1}"] \\
\mathcal{F}_p & \mathcal{F}_{p+1}. \arrow[l, "d^{\mathcal{F}}_{p+1}"']
\end{tikzcd}$$

Since ϕ_p is injective, and $d^{\mathcal{C}}_{p+1}$ is injective on the vector space of generators, which has degree $p+n+1$, the right vertical map ϕ_{p+1} is also injective on the vector space of generators. The map ϕ_{p+1} is then injective since both its domain an codomain are free with generators in degree $p+1+n$.

Let Q_{p+1} be the cokernel of the right vertical map. It is an S-module generated in degree $p+n+1$. By the long exact cohomology sequence of the short exact sequences of complexes

$$0 \to \mathcal{C}_{\leqslant p+1}(T) \to \mathcal{F}_{\leqslant p+1} \to Q_{p+1}[-p-1] \to 0$$

(where $[-p-1]$ denotes that Q_{p+1} is in homological degree $p+1$) we see that Q_{p+1} maps surjectively to the homology $H_p(\mathcal{C}_\bullet(T))$. So the latter module is generated in degree $p+n+1$. But it has no homology in this degree, by Proposition 5.4. Hence this homology module vanishes. Then

$$H_{p+1}(\mathcal{F}_{\leqslant p+1}) \to H_{p+1}(Q_{p+1}[-p-1]) = Q_{p+1}$$

is surjective, where the left module is in fact the kernel $\ker d^{\mathcal{F}}_{p+1}$. Since this kernel is generated in degree $p+n+2$, this again implies that $Q_{p+1} = 0$. Whence ϕ_{p+1} is an isomorphism. □

Remark 5.6. The above result has a more general framework. K. Yanagawa [51] shows that when a squarefree monomial ideal I_Δ has a linear resolution, it may be constructed directly from the canonical module $\omega_{\Delta^\vee}$ of the (Cohen-Macaulay) Stanley-Reisner ring of the Alexander dual simplicial complex $\Delta^\vee$. In fact given a squarefree module M, there is a linear complex $F_\bullet(M)$ with terms given by $F_i(M) = \oplus_{|\mathbf{r}|=i} M_{\mathbf{1}-\mathbf{r}} \otimes S(-\mathbf{r})$ (where $\mathbf{r}$ has entries 0 or 1 and $\mathbf{1} = (1,1,\ldots,1)$). The linear resolution of I_Δ is $F_\bullet(\omega_{k[\Delta^\vee]})$.

When Δ is a full-dimensional triangulated ball on a simplicial polytope P (as is the case here), by [13, Thm. 5.7.2] the canonical module is given as $\omega_{k[\Delta^\vee]} = I_\Sigma / I_\Delta$ where Σ is the boundary of Δ, and the latter identifies as I_{Δ^c}/I_P where Δ^c is the complement simplicial complex on P. One works out that $F_\bullet(I_{\Delta^c}/I_P)$ identifies as the dual of a free cellular complex supported on the relative simplicial complex (P, Δ^c). Letting $P^\vee$ be the dual polytope (in our case this is the product polytope $\Pi(A_\bullet)$), the linear resolution identifies again as a subcomplex of the free cellular complex associated to $P^\vee$.

The following is the general homological criterion on a subset $T \subseteq A_1 \times \cdots \times A_n$ for when $I(T)$ has a linear resolution.

Corollary 5.7. *$I(T)$ has a linear resolution if and only if all homologies of all restrictions vanish:*

$$H_i(\tilde{C}_\bullet(T_{|R_\bullet})) = 0 \tag{9}$$

for all $i \geqslant 0$ and $R_\bullet \subseteq^* A_\bullet$.

Proof. Use Fact 1. □

Definition 5.8. *A subset $T \subseteq A_1 \times \cdots \times A_n$ is acyclic if the vanishing* (9) *holds for all $i \geqslant 0$ and $R_\bullet \subseteq^* A_\bullet$. We say the cell complex $C(T)$ is restriction-acyclic or r-acyclic.*

Remark 5.9. Note that $C(T)$ may be acyclic (all homologies vanish), but not be restriction-acyclic. Thus being r-acyclic is not an intrinsic property of $C(T)$, but depends on how it is embedded in $\Pi(A_\bullet)$, see Remark 7.7. Section 7 has many examples of r-acyclic cell complexes.

Corollary 5.10. *$J(T)$ is a polarization of some Artin monomial ideal in $k[x_1,\ldots,x_n]$ containing $\mathfrak{c} = (x_1^{\alpha_1},\ldots,x_n^{\alpha_n})$ iff T is acyclic.*

5.3. Inductive construction of sets T. When T is acyclic, the constructability established in Section 4 suggests T may be built up inductively. Here we detail this.

The subset $T \subseteq A_1 \times \cdots \times A_n$ gives the saturated cell sub-complex $X = C(T)$ of $\Pi(A_\bullet)$. Vertices of the latter are $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$. For $a \in A_1$ let:

- $T_a = \{\mathbf{a} \in T \,|\, a_1 = a\}$,
- $T_{-a} = \{\mathbf{a} \in T \,|\, a_1 \neq a\}$,
- $T_{/a} = \{\mathbf{a} \in T \,|\, a_1 \neq a$ and both $\mathbf{a}$ and $(a, a_2, \ldots, a_n) \in T\}$,
- $T^a = T_a \cup T_{/a}$.

Furthermore we have the associated cell sub-complexes of X:

- $X_a = C(T_a)$,
- $X_{-a} = C(T_{-a})$,
- $X_{/a} = C(T_{/a})$,
- $X^a = C(T^a)$.

Note that $X_{/a}$ consists of the cells in X_{-a} which may be extended to a cell with a vertex in T_a. Also, X^a consists of the sub-cells of those cells of X which have some vertex in T_a. Note further that

$$X = X^a \cup X_{-a}, \quad X_{/a} = X^a \cap X_{-a}. \tag{10}$$

Lemma 5.11. *X_a and X^a are homotopy equivalent.*

Proof. There is an inclusion $X_a \subseteq X^a$. For any set B and $a \in B$, there is a retract of the simplex $\Delta(B)$ on $\Delta(\{a\})$ given by

$$\sum_{b \in B} \alpha_b \cdot b \mapsto (\sum_{b \in B} \alpha_b \cdot b)(1-t) + ta.$$

By letting $B = A_1$ we get a retract of $\Pi(A_\bullet)$ on $\Delta(\{a\}) \times \Delta(A_2) \times \cdots \times \Delta(A_n)$, and this restricts to a retract of X^a on X_a. $\square$

The following shows how we can inductively build up acyclic sets T. When $R_\bullet \subseteq^* A_\bullet$, for simplicity denote $X_{|\Pi(R_\bullet)}$ by $X_{|R}$.

Proposition 5.12. *Let $T \subseteq A_1 \times \cdots \times A_n$ and $a \in A_1$. A saturated cell sub-complex $X = C(T)$ is r-acyclic iff*

- *X_a, X_{-a} and $X_{/a}$ are r-acyclic,*
- *For every $R_\bullet \subseteq^* A_\bullet$, whenever $X_{a|R}$ and $X_{-a|R}$ are non-empty, then $X_{/a|R}$ is non-empty.*

In this situation, X^a is also r-acyclic.

Proof. By (10) we have a Mayer-Vietoris sequence (using reduced homology):

$$\tilde{H}_{i+1}(X_{|R}) \to \tilde{H}_i(X_{/a|R}) \to \tilde{H}_i(X^a_{|R}) \oplus \tilde{H}_i(X_{-a|R}) \to \tilde{H}_i(X_{|R}) \to \tilde{H}_{i-1}(X_{/a|R}) \to \cdots.$$

Assume X is r-acyclic. Since X_a and X_{-a} are restrictions, they are r-acyclic. Furthermore, X^a is homotopy equivalent to X_a, and similarly $X^a_{|R}$ is homotopy equivalent to $X_{a|R}$. Thus

$$\tilde{H}_i(X^a_{|R}), \quad \tilde{H}_i(X_{-a|R}), \quad \tilde{H}_i(X_{|R})$$

all vanish for $i \geqslant 0$. Thus $X_{/a\,|R}$ is r-acyclic. Also, when $X_{a|R}$ and $X_{-a|R}$ are non-empty, the sequence above, when $i = -1$, gives that $\tilde{H}_{-1}(X_{/a|R})$ vanishes, and so $X_{/a|R}$ is non-empty.

Conversely assume X_a, X_{-a} and $X_{/a}$ are all r-acyclic. The sequence above gives that all $\tilde{H}_i(X_{|R})$ vanish for $i \geqslant 0$, except possibly in the following case: $i = 0$ and $X_{/a|R}$ is empty and $X_{|R}$ is not empty. But in this case at least one of $X_{-a|R}$ or $X_{a|R}$ is empty, which implies the vanishing of $\tilde{H}_0(X_{|R})$. $\square$

6. Minimal irreducible decomposition and maximal cells

Given an Artin monomial ideal $\mathfrak{j}$. The question we address here is what does T and $C(T)$ look like for polarizations $J(T)$ of $\mathfrak{j}$? How can we make various polarizations of $\mathfrak{j}$ by working with the geometry of the cell complex $C(T)$?

We show there is a one-to-one correspondence between the maximal faces of $C(T)$ and the components of the minimal irreducible decompositions of the Artin monomial ideal $\mathfrak{j}$. This is what guides us to find the possible T's which gives polarizations $J(T)$ of $\mathfrak{j}$, see Section 7.

6.1. Maximal cells in $C(T)$. We first need some essential structural properties of the maximal cells in $C(T)$ for an acyclic $T \subseteq A_1 \times \cdots \times A_n$.

Lemma 6.1. *Let $\Pi(R_\bullet)$ be a maximal cell in $C(T)$. Suppose R_1 has at least two elements, and that $a \in R_1$ is one of them. Let $\overline{R}_\bullet$ be $R_\bullet$ but with R_1 replaced by $R_1 \backslash \{a\}$. Then $\Pi(\overline{R}_\bullet)$ is a maximal cell in $C(T_{/a})$. In particular, since $T_{/a} \subseteq T_{-a}$, there is a maximal cell in $C(T_{-a})$ containing $\Pi(\overline{R}_\bullet)$.*

Proof. Let $\Pi(R'_\bullet)$ be a maximal cell in $C(T_{/a})$ with $R'_i \supseteq R_i$ for $i \geqslant 2$, and $R'_1 \supseteq R_1 \backslash \{a\}$. By definition of $T_{/a}$, adding a to R'_1 and keeping the rest of the R'_i unchanged, gives a new $R''_\bullet$ and cell $\Pi(R''_\bullet)$. This cell must be maximal in $C(T)$ as $\Pi(R'_\bullet)$ is maximal in $C(T_{/a})$. But then $\Pi(R''_\bullet)$ must be $\Pi(R_\bullet)$, and so we must have each $R'_i = \overline{R}_i$. □

Proposition 6.2. *Let $\Pi(R_\bullet)$ and $\Pi(S_\bullet)$ be distinct maximal cells in an r-acyclic cell complex $C(T)$. Then there are indices i and j such that we have strict inclusions*

$$R_i \subset S_i, \quad S_j \subset R_j. \tag{11}$$

Proof. Since $C(T_{-a})$ is r-acyclic when $C(T)$ is, we may by successive restrictions for $a \in A_i \backslash (R_i \cup S_i)$, assume that $R_i \cup S_i = A_i$ for each i.

We want to show we cannot have the negation of (11). So assume the negation. Write $R_i \not\sim S_i$ if none of these two is included in the other. We may then assume that there is $0 \leqslant p \leqslant n$ such that $R_i \supseteq S_i$ for $i = 1, \ldots, p$ and $R_i \not\sim S_i$ for $i = p+1, \ldots, n$.

Our proof is by induction on n and the size of the A_i. As an induction base we note two cases: i) If $n = 1$ they cannot both be maximal. ii) We also cannot have all $|R_i| = 1$, since then $\Pi(R_\bullet)$ is a point and would not be connected to the rest, contradicting that $C(T)$ is r-acyclic.

Case $p \geqslant 1$: So $R_1 \supseteq S_1$. Assume $R_1 = S_1$. Let $a \in S_1$. If the cardinalities $|R_1| = |S_1|$ is $\geqslant 2$, we may by Lemma 6.1 get smaller maximal sets in $C(T_{/a})$. Since $T_{/a}$ is acyclic, this is against induction. If $R_1 = S_1$ is of cardinality 1, we may simply drop the first factor and reduce n.

So assume R_1 is strictly bigger than S_1. Let $a \in R_1 \backslash S_1$. We then get a maximal $\Pi(R'_\bullet)$ in $C(T_{-a})$ with $R'_i \supseteq R_i$ for each $i \geqslant 2$. Also $R'_1 \supseteq S_1$. Again since T_{-a} is acyclic, we get a contradiction by induction, since we have no strict $R'_i \subset S_i$.

Case $p = 0$: So we have $R_i \not\sim S_i$ for each i. If some $R_i \cap S_i$ is non-empty (then both $|R_i|, |S_i| \geqslant 2$), we may use Lemma 6.1, and get a contradiction by induction. So we may assume every $R_i \cap S_i$ is empty.

At least one of the R_i has more than one element, say R_1. Let $a \in R_1$. By restricting away from a, we get a maximal $\Pi(R'_\bullet)$ in $C(T_{-a})$ with $R'_i \supseteq R_i$ for each $i \geqslant 2$, and so none $R'_i \subseteq S_i$ for $i \geqslant 2$. We cannot have $R'_1 \subseteq S_1$, since R_1 and S_1 had empty intersection, and $|R_1| \geqslant 2$. By induction we get a contradiction. □

6.2. Regular linear subspaces. Let V be a finite dimensional k-vector space and $\ell \subseteq V$ a linear subspace.

Definition 6.3. *For a polynomial ring $S = k[V]$ let $I \subseteq S$ be an ideal. If ℓ has a basis which is a regular sequence for S/I, we call ℓ a regular linear space for S/I (or for I).*

Lemma 6.4. *Let ℓ be a linear subspace of V and $L = (\ell) \subseteq S$ the ideal generated by ℓ. Then ℓ is a regular linear space for S/I iff $\operatorname{Tor}_i(S/I, S/L) = 0$ for all $i > 0$.*

Proof. This is by [18] Corollary 17.5 and Theorem 17.6. □

Lemma 6.5. *Let $I, J \subseteq S$ be ideals. Let $\ell \subseteq V$ be a regular linear space for the ideals I, J and $I + J$. Then ℓ is a regular linear space for $I \cap J$ and in the quotient ring S/L (where $L = (\ell)$) we have:*

$$\frac{(I \cap J) + L}{L} = \left(\frac{I + L}{L}\right) \cap \left(\frac{J + L}{L}\right). \tag{12}$$

Proof. We have a short exact sequence

$$0 \to S/(I \cap J) \to S/I \oplus S/J \to S/(I + J) \to 0.$$

By tensoring with S/L, using the long exact Tor-sequence, and Lemma 6.4, we get that ℓ is a regular linear space for $I \cap J$. We also get an exact sequence:

$$0 \to \frac{S/L}{((I \cap J) + L)/L} \to \frac{S/L}{(I + L)/L} \oplus \frac{S/L}{(J + L)/L} \to \frac{S/L}{(I + J + L)/L} \to 0,$$

which gives Equation (12). □

The following rather trivial result will also be used.

Lemma 6.6. *Let $K, K_u, u \in U$ be graded ideals in $S = k[V]$, and suppose $K = \cap_{u \in U} K_u$. Let a be a new variable. These ideals generate ideals $(K), (K_u)$ in $k[V \oplus (a)]$.*

a. *We have $(K) = \cap_{u \in U}(K_u)$ in $k[V \oplus (a)]$.*

b. *In the quotient ring $k[V] \cong k[V \oplus (a)]/(a)$ we have*

$$\frac{(\cap_u (K_u)) + (a)}{(a)} = \frac{(K) + (a)}{(a)} = \bigcap_{u \in U} \left(\frac{(K_u) + (a)}{(a)}\right). \tag{13}$$

Proof. Note that all ideals above are $\mathbb{Z}^2$-graded, with (d, d_a) denoting the degree of a homogeneous polynomial where d is the degree of terms in $k[V]$, and d_a is the power of a occurring.

a. It is clear that $(K) \subseteq \cap_u (K_u)$. Let $p = p_0 + ap_1 + a^2 p_2 \cdots$ be in the latter intersection. Due to the $\mathbb{Z}^2$-grading each p_i is in each (K_u) and so in K_u. Thus each p_i is in K and so p is in (K).

b. Let $p \in k[V]$. Write $\overline{p}$ when considered as element in $k[V \oplus \langle a \rangle]/(a)$. If $\overline{p}$ is in the intersection on the right of Equation (13), then for each $u \in U$, $p + aq_u$ is in (K_u) for some q_u. Due to $\mathbb{Z}^2$-grading, $p \in K_u$ for each u, and so p is in K and so $\overline{p}$ in the left side of Equation (13). □

6.3. Minimal irreducible decompositions. Let now $T \subseteq A_1 \times \cdots \times A_n$ be an acyclic subset. It gives the ideal $I(T)$ with Alexander dual ideal $J(T)$, a polarization of an Artin monomial ideal.

Writing $A_i = \{a_1^i, a_2^i, \ldots\}$, for each i the variable differences $a_k^i - a_l^i$ generate a linear subspace ℓ_i. Let $\ell = \ell_1 + \ell_2 + \cdots + \ell_n$. It generates an ideal $L = (\ell) \subseteq S$, and

$$k[A_1 \cup A_2 \cup \cdots \cup A_n]/L \cong k[x_1, x_2, \ldots, x_n].$$

Let $R_\bullet \subseteq^* A_\bullet$. It gives $U = R_1 \times \cdots \times R_n$. We write $I(R_\bullet), J(R_\bullet)$ for $I(U), J(U)$. Then

$$J(U) = J(R_\bullet) = (m(R_1), \ldots, m(R_n)).$$

Thus

$$\frac{J(U)+L}{L} \subseteq \frac{k[A_1 \cup \cdots \cup A_n]}{L} \tag{14}$$

identifies as

$$(x_1^{r_1}, x_2^{r_2}, \cdots, x_n^{r_n}) \subseteq k[x_1, \ldots, x_n], \quad r_i = |R_i|. \tag{15}$$

Theorem 6.7. *Let $T \subseteq A_1 \times \cdots \times A_n$ be an acyclic subset, which is equivalent to $J(T)$ being a polarization of an Artin monomial ideal $\mathfrak{j}$ (by Corollary 5.7 and Proposition 3.6). Let $\Pi(R_\bullet^k), k = 1, \ldots, m$ be the maximal cells in $C(T)$. Denote $\mathfrak{j}_k = (x_1^{r_1^k}, x_2^{r_2^k}, \ldots, x_n^{r_n^k})$, the irreducible monomial ideal where r_i^k is the cardinality of R_i^k.*

a.
$$J(T) = J(R_\bullet^1) \cap J(R_\bullet^2) \cap \cdots \cap J(R_\bullet^m).$$

b. *The minimal irreducible decomposition of $\mathfrak{j}$ is*

$$\mathfrak{j} = \mathfrak{j}_1 \cap \mathfrak{j}_2 \cap \cdots \cap \mathfrak{j}_m. \tag{16}$$

So the components of the minimal irreducible decomposition of $\mathfrak{j}$ are in bijection to the maximal cells of $C(T)$.

Proof. Part a. We have

$$I(T) = I(R_\bullet^1) + I(R_\bullet^2) + \cdots + I(R_\bullet^m).$$

The Alexander dual is then

$$J(T) = J(R_\bullet^1) \cap J(R_\bullet^2) \cap \cdots \cap J(R_\bullet^m).$$

Part b. We must show:

$$\mathfrak{j} = \frac{J(T)+L}{L} = \frac{J(R_\bullet^1)+L}{L} \cap \frac{J(R_\bullet^2)+L}{L} \cap \cdots \cap \frac{J(R_\bullet^m)+L}{L} \tag{17}$$

which shows (16). In addition we must show the minimality of the decomposition.

We have

$$I(T) = I(T^a) + I(T_{-a}), \quad I(T_{/a}) = I(T^a) \cap I(T_{-a}). \tag{18}$$

The first is clear. To see the second, since $I(T^a) = I(T_a) + I(T_{/a})$, the intersection

$$I(T^a) \cap I(T_{-a}) = I(T_a) \cap I(T_{-a}) + I(T_{/a}) \cap I(T_{-a}). \tag{19}$$

The last term is $I(T_{/a})$. As for the middle term, by the last arguments of Proposition 4.8 it has $(n+1)$-linear resolution. A monomial generator of this intersection must have the form $aa_1a_2\cdots a_n$ where $a_i \in A_i$. Then $a_1 \cdots a_n \in I(T_{-a})$ and $aa_2 \cdots a_n \in I(T_a)$. So $(a_1, a_2, \ldots, a_n) \in T_{/a}$. Thus the left side of (19) is $I(T_{/a})$.

On the Alexander dual side, (18) becomes:

$$J(T) = J(T^a) \cap J(T_{-a}), \quad J(T_{/a}) = J(T^a) + J(T_{-a}).$$

If $C(T)$ has only one maximal cell we are done by Equations (14) and (15). Suppose now $C(T)$ has at least two maximal cells. Let a be an element of some A_i, say $a \in A_1$ which is on at least one maximal cell but not all. Then $J(T^a)$ and $J(T_{-a})$ are ideals properly containing J, and both T^a, T_{-a} and $T_{/a}$, are acyclic subsets of $A_1 \times \cdots \times A_n$, by Proposition 5.12. Thus by Lemma 6.5

$$\frac{J(T)+L}{L} = \left(\frac{J(T^a)+L}{L}\right) \cap \left(\frac{J(T_{-a})+L}{L}\right).$$

Both T^a and T_{-a} are properly included in T. Furthermore a maximal cell $\Pi(R_\bullet^j)$ of $C(T)$ is either in $C(T^a)$ (if $a \in R_1^j$) or in $C(T_{-a})$ (if $a \notin R_1^j$). Let M^a denote the index set of j in the former, and M_{-a} the j in the latter. Then by induction and Lemma 6.6:

$$\frac{J(T)+L}{L} = \cap_{j \in M^a} \left(\frac{J(R^j_\bullet)+L}{L} \right) \bigcap \cap_{j \in M_{-a}} \left(\frac{J(R^j_\bullet)+L}{L} \right).$$

This show Equation (17).

To show minimality, suppose there was some $p \in [m]$ such that

$$\cap_{i \in [m]\setminus\{p\}} \mathfrak{j}_i \subseteq \mathfrak{j}_p. \tag{20}$$

Let $\mathrm{Mon}(n)$ be the monomials in the polynomial ring $k[x_1,\dots,x_n]$, and if $K \subseteq k[x_1,\dots,x_n]$ is a monomial ideal, write $\mathrm{Mon}(n) = N(K) \sqcup \mathrm{Mon}(K)$ as a disjoint decomposition, where $\mathrm{Mon}(K)$ are the monomials in K and $N(K)$ are the normal monomials, those not in K. Expression (20) corresponds to

$$\cup_{i \in [m]\setminus\{p\}} N(\mathfrak{j}_i) \supseteq N(\mathfrak{j}_p).$$

But writing $\mathfrak{j}_p = (x_1^{r_1^p},\dots,x_n^{r_n^p})$, the maximal element (for the division order in $\mathrm{Mon}(n)$) of $N(\mathfrak{j}_p)$ is the monomial $x_1^{r_1^p-1}x_2^{r_2^p-1}\cdots x_n^{r_n^p-1}$. If this is in $N(\mathfrak{j}_i)$, then each $r_k^i \geqslant r_k^p, k=1,\dots,n$. But this contradicts Proposition 6.2. Thus we cannot have the situation of (20). $\square$

For $S \subseteq A = A_1 \cup \cdots \cup A_n$, write $S_i = S \cap A_i$. We say S *covers* $T \subseteq A_1 \times \cdots \times A_n$ if for every maximal cell $\Pi(R_\bullet)$ in $C(T)$, there is some i such that $R_i \subseteq S_i$. Note in particular that each $A_i \subseteq A$ covers T.

Corollary 6.8. *The polarization $J(T)$ is generated by the monomials $m(S_\bullet) = \prod_{i=1}^n m(S_i)$ such that S covers T.*

Proof. It is clear that S covers T iff $m(S_\bullet) \in J(R_\bullet)$ for each maximal cell $\Pi(R_\bullet)$ of $C(T)$. $\square$

For an *irreducible* Artin monomial ideal $\mathfrak{j} = (x_1^{r_1}, x_2^{r_2},\dots,x_n^{r_n})$ define its *power size* to be

$$\mathrm{psize}\,(\mathfrak{j}) = \sum_{i=1}^n (r_i - 1).$$

(It is the number of variable powers $x_i^{s_i}$ in the quotient ring $k[x_1,\dots,x_n]/\mathfrak{j}$.) The following will also be useful.

Lemma 6.9. *For any maximal cells $\Pi(R^i_\bullet)$ and $\Pi(R^j_\bullet)$ in $C(T)$, the intersection $\Pi(R^i_\bullet) \cap \Pi(R^j_\bullet)$ has dimension $\leqslant$ psize $(\mathfrak{j}^i + \mathfrak{j}^j)$.*

Proof. Let $P_k = R^i_k \cap R^j_k$. Then the intersection $\Pi(R^i_\bullet) \cap \Pi(R^j_\bullet) = \Pi(P_\bullet)$. Thus $I(R^i_\bullet) \cap I(R^j_\bullet) \supseteq I(P_\bullet)$. Taking Alexander duals this gives

$$J(R^i_\bullet) + J(R^j_\bullet) \subseteq J(P_\bullet).$$

Then for the associated irreducible Artin ideals in S/L we have:

$$\mathfrak{j}^i + \mathfrak{j}^j \subseteq \mathfrak{j}', \quad \mathfrak{j}' = \frac{J(P_\bullet)+L}{L}.$$

Whence

$$\mathrm{psize}\,(\mathfrak{j}^i + \mathfrak{j}^j) \geqslant \mathrm{psize}\,(\mathfrak{j}') = \dim \Pi(P_\bullet). \qquad \square$$

6.4. Standard polarization. Fix a total order on each A_i. So $A_i = \{a^i_1 < a^i_2 < \cdots < a^i_{\alpha_i}\}$. For each $0 \leqslant r_i \leqslant \alpha_i$ define $m^{r_i}(A_i) = \prod_{j=1}^{r_i} a^i_j$. Each $\mathbf{r} = (r_1, r_2, \ldots, r_n) \in \mathbb{N}^n$ gives a monomial $m(\mathbf{r}) = x_1^{r_1} \cdots x_n^{r_n}$. The *standard polarization* of the monomial $m(\mathbf{r})$ is

$$m^{\mathbf{r}}(A_\bullet) = \prod_{i=1}^{n} m^{r_i}(A_i).$$

If $\mathfrak{j} = (x^{\mathbf{r}^1}, \ldots, x^{\mathbf{r}^N})$, its *standard polarization* is

$$J = (m^{\mathbf{r}^1}(A_\bullet), \ldots, m^{\mathbf{r}^N}(A_\bullet)).$$

Example 6.10. Consider $\mathfrak{j} = (x_1^3x_2, x_2^4x_3^2, x_1x_3^3)$. Rename $A := A_1, B := A_2$ and $C := A_3$, and write b_i instead of a^2_i for the elements of B, and similarly for C. The standard polarization of $\mathfrak{j}$ is then:

$$J^{st} = (a_1a_2a_3b_1, b_1b_2b_3b_4c_1c_2, a_1c_1c_2c_3).$$

It is well-known that the standard polarization of any monomial ideal is a polarization, [28, Prop.1.6.2]

Proposition 6.11. *Let $\mathfrak{j}$ be an Artin monomial ideal, with $\mathfrak{j} = \cap_{k=1}^m \mathfrak{j}_k$ its minimal irreducible decomposition, where $\mathfrak{j}_k = (x_1^{r^k_1}, x_2^{r^k_2}, \ldots, x_n^{r^k_n})$. Then the standard polarization of $\mathfrak{j}$ is*

$$J^{st} = \cap_{k=1}^m J^{st}_k.$$

where $J^{st}_k = (m^{r^k_1}(A_1), \ldots, m^{r^k_n}(A_n))$.

Proof. That $J^{st} \subseteq J^{st}_k$ is immediate to see. Let then m be a monomial in the intersection. Each J^{st}_k has a generator $m^{r^k_{i_k}}(A_{i_k})$ which divides m. Then m is divisible by the least common multiple of these monomials. This least common multiple is $m^{\mathbf{r}}(A_\bullet)$ for a suitable $\mathbf{r}$. Furthermore $x^{\mathbf{r}}$ is the least common multiple of the $x_{i_k}^{r_{i_k}}$. So this is in the intersection $\cap_{k=1}^m \mathfrak{j}_k$, and so $x^{\mathbf{r}}$ is in $\mathfrak{j}$. Write $x^{\mathbf{r}} = x^{\mathbf{r}'}x^{\mathbf{p}}$ where $x^{\mathbf{p}}$ is a minimal generator for $\mathfrak{j}$. Then $m^{\mathbf{p}}(A_\bullet) \in J^{st}$, and $m^{\mathbf{r}}(A_\bullet)$ is a multiple of $m^{\mathbf{p}}(A_\bullet)$ and so m is in J^{st}. ☐

Corollary 6.12. *Let I^{st} be the Alexander dual of the standard polarization J^{st}, so $I^{st} = I(T^{st})$ for a subset $T^{st} \subseteq A_1 \times \cdots \times A_n$. The maximal cells of $C(T^{st})$ are the $\Pi(R^k_\bullet)$ where*

$$R^k_\bullet = \{(a^1_{i_1}, \ldots, a^n_{i_n}) \,|\, 1 \leqslant i_j \leqslant r^k_j \text{ for } j = 1, \ldots, n\}.$$

So this cell is

$$\Delta(R^k_1) \times \cdots \times \Delta(R^k_m)$$

where R^k_i is the initial segment of A_i with r^k_i terms. (Note that $(a^1_1, a^2_1, \ldots, a^n_1)$ is a common vertex of all such cells.)

Remark 6.13. As polarizations preserve all graded Betti numbers, by Theorem 5.5 every cell complex $C(T)$ has the same f-vector as the cell complex $C(T^{st})$ from the standard polarization. This is useful in the examples of Section 7.

7. Simplicial complexes and hypercubes

We look at the case when $\Pi(A_\bullet)$ is a hypercube. The Artin monomial ideal then contains all squares of the variables. The nice feature is that such ideals are equivalent to give a square free monomial ideal and so a simplicial complex. In a number of examples, for various simplicial complexes, we describe the possible associated r-acycliccell complexes $C(T)$. This enables the description of all possible polarizations, up to isomorphism.

7.1. Hypercubes and ideals containing the squares of variables. A subset $S \subseteq V$, gives rise to a geometric simplex in $\mathbb{R}^V$:

$$\Delta(S) = \left\{ \sum_{s \in S} \delta_s \cdot s \,|\, 0 \leqslant \delta_s \leqslant 1,\ \sum_s \delta_s = 1 \right\}.$$

It also gives rise to a (hyper)cube in $\mathbb{R}^V$.

$$\square(S) = \left\{ \sum_{s \in S} \gamma_s \cdot s \,|\, 0 \leqslant \gamma_s \leqslant 1 \right\}.$$

From a combinatorial simplicial complex X on a set V, we get a geometric realization $\Delta(X) \subseteq \mathbb{R}^V$ consisting of all $\Delta(S)$ for $S \in X$. We also get a cubical cell complex $\square(X)$ consisting of all $\square(S)$ for $S \in X$. (Note that $\square(X)$ is contractible, as each $\square(S)$ contains the origin.)

Remark 7.1. The graph which is the one-skeleton of $\square(X)$ is called the *simplex graph* in the literature. The book [47] considers graphs which may be embedded on hypercubes.

Consider now those Artin monomial ideals $\mathfrak{j}$ which contain the squares of all variables. We may then write

$$\mathfrak{j} = \mathfrak{m}_\square + \mathfrak{j}_\Delta, \quad \mathfrak{m}_\square = (x_1^2, x_2^2, \ldots, x_n^2),$$

where $\mathfrak{j}_\Delta$ is a squarefree monomial ideal corresponding to a simplicial complex X. Each $A_i = \{a_i, b_i\}$ has then cardinality two. The hypercube

$$\Delta(A_1) \times \Delta(A_2) \times \cdots \times \Delta(A_n). \tag{21}$$

may be identified with $[0,1]^n$. For $S \subseteq [n]$, a vertex $e_S = \sum_{i \in S} e_i$ of $[0,1]^n$ corresponds to $(u_1, \ldots, u_n)$ in (21) with $u_i = b_i$ if $i \in S$ and $u_i = a_i$ otherwise. The hypercube (21) has a natural monomial labelling, where a vertex may be labelled with the monomial $u_1 u_2 \cdots u_n$, see Figure 7.

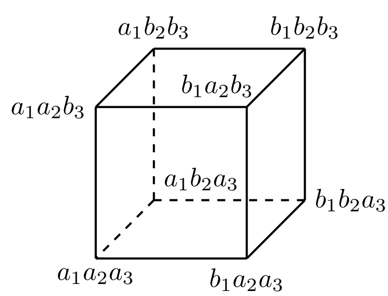

Figure 7. The cube with monomial labelling of vertices

In the following, for a subset S of $[n]$ write $x(S)$ for the ideal $(x_i)_{i \in S}$ and $x^2(S) = (x_i^2)_{i \in S}$. It is well known that the minimal primary decomposition of I_X is

$$I_X = \bigcap_{\text{facet} F \text{ of } X} x(F^c), \tag{22}$$

see [28, Lemma 1.5.4] or [39, Theorem 1.7].

Proposition 7.2. *The minimal irreducible decomposition of* $\mathfrak{j} = \mathfrak{m}_\square + \mathfrak{j}_\Delta$ *is*

$$\mathfrak{j} = \bigcap_{\text{facet} F \text{of} X} (x(F^c) + x^2(F)).$$

In particular in any polarization of $\mathfrak{j}$*, the maximal cells of* $C(T)$ *are in bijection with the facets* F *of* X*. The dimension of a maximal cell is the cardinality* $|F|$*, one more than the dimension of the face* F*. Each maximal cell is a sub-hypercube of the hypercube* (21)*.*

Proof. It is clear that we have

$$\mathfrak{j} = \mathfrak{m}_{\Box} + \mathfrak{j}_{\Delta} \subseteq \bigcap_{\text{facet } F \text{ of } X} (x(F^c) + x^2(F)).$$

It is clear that any monomial containing the square of a variable is contained in the left side. If m is a squarefree monomial contained in the right side, it is also clear by (22), that m is in the left side. Whence we have equality above. $\square$

Proposition 7.3. *Let* $J^{st} = J(T^{st})$ *be the standard polarization of* $\mathfrak{j}$*, where* $\mathfrak{j}_{\Delta}$ *corresponds to a simplicial complex* X*. The cell complex* $C(T^{st})$ *is the canonical cubical complex* $\Box(X)$*. With its natural monomial labelling, it gives a free cellular resolution of the Alexander dual rainbow ideal* I^{st} *of* J^{st}*.*

Proof. This follows by Theorem 5.5 and Corollary 6.12. $\square$

7.2. Examples. In the following X will be the simplicial complex whose Stanley-Reisner ideal is $\mathfrak{j}_{\Delta}$. In our examples the dimension of X will be $\leqslant 1$. In a polarization with associated cell complex $C(T)$, a vertex of X corresponds to an edge of $C(T)$ and an edge of X gives a square in $C(T)$. This correspondence is one-to-one between facets of X and maximal cells of $C(T)$ (but not in general between faces and cells).

Furthermore, if two squares in $C(T)$ have an edge in common, the corresponding edges in X have a corresponding vertex in common, by Lemma 6.9. But conversely, if two edges in X have a vertex in common, the corresponding squares in $C(T)$ may or may not have an edge in common, in fact the squares may even be disjoint (see Example 7.10). The essential thing concerning $C(T)$ is that it is r-acyclic.

Instead of writing the variables as $x_1, x_2, \ldots$, we write $x, y, \ldots$. We write $A_x = \{x_a, x_b\}$ for the variables mapping to x, and similarly A_y and so on.

Example 7.4. Let $\mathfrak{j} = (x^2, y^2) \subseteq k[x, y]$. It has a unique polarization J with Alexander dual I:

$$J = (x_a x_b, y_a y_b) \subseteq S = k[x_a, x_b, y_a, y_b], \quad I = (x_a y_a, x_a y_b, x_b y_a, x_b y_b).$$

The associated T is the whole $A_x \times A_y$. So $C(T)$ is the square with vertices labelled by monomials as in the left in Figure 8.

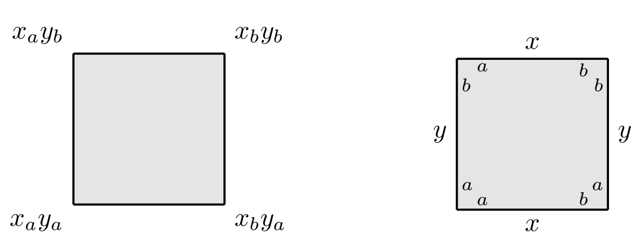

Figure 8. The cube with monomial labelling

We write it somewhat more stylistic as in the right of Figure 8. The free cellular resolution of I becomes:

$$I \leftarrow S(-2)^4 \leftarrow S(-3)^4 \leftarrow S(-4).$$

EXAMPLE 7.5.

$$\mathfrak{j} = (x^2, y^2, z^2) + (xy, xz, yz).$$

This is the same Artin monomial ideal we considered in Example 1.3 in the Introduction. The simplicial complex associated to $\mathfrak{j}_\Delta$ is three isolated points, labelled respectively x, y and z. In a polarization $J(T)$, by Theorem 6.7 the cell complex $C(T)$ will have three line segments, labelled respectivey x, y and z. The cell complex is r-acyclic and so must be a tree. This gives, up to isomorphism, the two possibilities given in Figure 9.

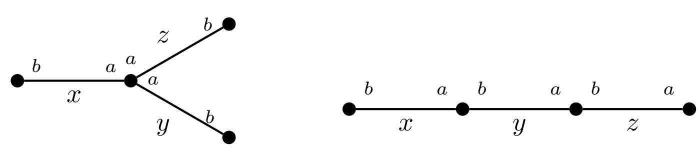

FIGURE 9. The $C(T)$ are trees with labellings

These are embedded on the cube as in Figure 10. Using Corollary 6.8, the corresponding polarizations are then

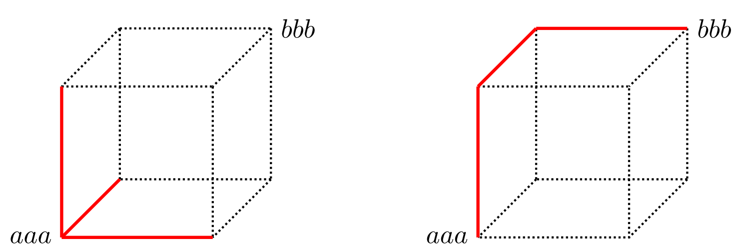

FIGURE 10. The $C(T)$-trees with their embedding on cubes

$$J_1 = (x_a x_b, y_a y_b, z_a z_b) + (x_a y_a, x_a z_a, y_a z_a)$$
$$J_2 = (x_a x_b, y_a y_b, z_a z_b) + (x_a y_b, x_a z_b, y_a z_b)$$

More simply, using Corollary 6.8 to obtain the polarization J_2, amounts to observe for the tree to the right in Figure 9: On the path from edge x to edge y the label on the vertex of x is a, and the vertex on y is b. Similarly for the paths form x to z and the path from y to z. See also how to obtain the ideal J_3 below from the tree in Figure 11.

The ideals J_1 and J_2 are the ideals given in Example 1.3 in the Introduction. In Figure 9 for the edges labelled x we are free to label its two vertices with a and b as we like, and similarly for the y, z edges. We could have labelled the line graph with a and b as in Figure 11, which would have given the embedding on the cube as shown there.

This gives a polarization isomorphic to J_2:

$$J_3 = (x_a x_b, y_a y_b, z_a z_b) + (x_b y_b, x_b z_a, y_a z_a)$$

In general for the powers $(x_1, \ldots, x_n)^2$, its polarizations are classified, up to isomorphism, by trees with n edges, [3, Section 7]. A comprehensive study of ideals obtained from such polarizations is done in [22], see also the present Subsection 12.2.

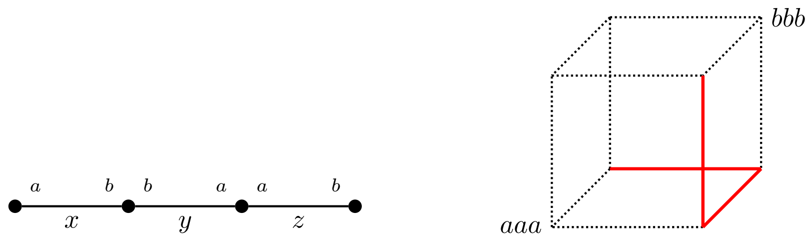

Figure 11. Alternative labelling of a and b, with corresponding embedding on cube

Example 7.6.

$$\mathfrak{j} = (x^2, y^2, z^2, w^2) + (xz, xw, yz, yw, zw).$$

The simplicial complex X associated to $\mathfrak{j}_\Delta$ is given in Figure 12.

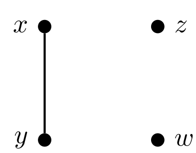

Figure 12. Simplicial complex: A line segment and two isolated points

In a polarization $J(T)$, by Theorem 6.7 the cell complex $C(T)$ will then have one square, corresponding to the line segment labelled xy, and two line segments corresponding to the points labelled z and w. The cell complex $C(T)$ must also be r-acyclic. This gives, up to isomorphism, the four possibilities for $C(T)$ given in Figure 13. We

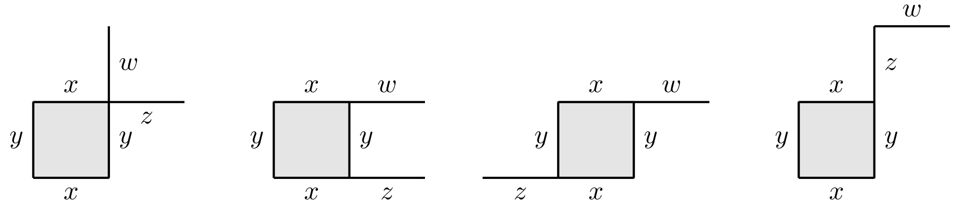

Figure 13. Restriction-acyclic $C(T)$ with one square and two line segments

thus get, up to isomorphism, four polarizations of $\mathfrak{j}$. Using Theorem 6.7 and Corollary 6.8 they are seen to be:

$$\begin{aligned}
J_1 &= (x_a x_b, y_a y_b, z_a z_b, w_a w_b) + (x_a z_a, x_a w_a, y_a z_a, y_a w_a, z_a w_a)\\
J_2 &= (x_a x_b, y_a y_b, z_a z_b, w_a w_b) + (x_a z_a, x_a w_a, y_b z_a, y_a w_a, z_a w_a)\\
J_3 &= (x_a x_b, y_a y_b, z_a z_b, w_a w_b) + (x_b z_a, x_a w_a, y_b z_a, y_a w_a, z_a w_a)\\
J_4 &= (x_a x_b, y_a y_b, z_a z_b, w_a w_b) + (x_a z_a, x_a w_a, y_a z_a, y_a w_a, z_b w_a)
\end{aligned}$$

(Again for each variable, the labels a and b may be switched (or not) and we will get isomorphic polarizations, which are embedded distinctly on the four-dimensional hypercube.) The standard polarization J_1 corresponds to the first cell complex in Figure 13. Taking the Alexander duals of the above, we get rainbow ideals:

$$I_1 = (x_ay_az_aw_a, x_by_az_aw_a, x_ay_bz_aw_a, x_by_bz_aw_a, x_ay_az_bw_a, x_ay_az_aw_b)$$
$$I_2 = (x_ay_az_aw_a, x_by_az_aw_a, x_ay_bz_aw_a, x_by_bz_aw_a, x_ay_bz_bw_a, x_ay_az_aw_b)$$
$$I_3 = (x_ay_az_aw_a, x_by_az_aw_a, x_ay_bz_aw_a, x_by_bz_aw_a, x_by_bz_bw_a, x_ay_az_aw_b)$$
$$I_4 = (x_ay_az_aw_a, x_by_az_aw_a, x_ay_bz_aw_a, x_by_bz_aw_a, x_ay_az_bw_a, x_ay_az_bw_b)$$

The minimal free resolution for each of these is the free cellular complex given by $C(T)$. It has in each case the form

$$S(-4)^6 \leftarrow S(-5)^6 \leftarrow S(-6).$$

The triangulated balls defined by the ideals J_i each have six facets which are complements of the variable sets in each of the generators of the rainbow ideals

$$F1: \begin{matrix}\{x_b, y_b, z_b, w_b\} \\ \{x_a, y_b, z_b, w_b\}\end{matrix}, \begin{matrix}\{x_b, y_a, z_b, w_b\} \\ \{x_a, y_a, z_b, w_b\}\end{matrix}, \begin{matrix}\{x_b, y_b, z_a, w_b\} \\ \{x_b, y_b, z_b, w_a\}\end{matrix}$$
$$F2: \begin{matrix}\{x_b, y_b, z_b, w_b\} \\ \{x_a, y_b, z_b, w_b\}\end{matrix}, \begin{matrix}\{x_b, y_a, z_b, w_b\} \\ \{x_a, y_a, z_b, w_b\}\end{matrix}, \begin{matrix}\{x_b, y_a, z_a, w_b\} \\ \{x_b, y_b, z_b, w_a\}\end{matrix}$$
$$F3: \begin{matrix}\{x_b, y_b, z_b, w_b\} \\ \{x_a, y_b, z_b, w_b\}\end{matrix}, \begin{matrix}\{x_b, y_a, z_b, w_b\} \\ \{x_a, y_a, z_b, w_b\}\end{matrix}, \begin{matrix}\{x_a, y_a, z_a, w_b\} \\ \{x_b, y_b, z_b, w_a\}\end{matrix}$$
$$F4: \begin{matrix}\{x_b, y_b, z_b, w_b\} \\ \{x_a, y_b, z_b, w_b\}\end{matrix}, \begin{matrix}\{x_b, y_a, z_b, w_b\} \\ \{x_a, y_a, z_b, w_b\}\end{matrix}, \begin{matrix}\{x_b, y_b, z_a, w_b\} \\ \{x_b, y_b, z_a, w_a\}\end{matrix}$$

REMARK 7.7. The third cell complex in Figure 13 can be embedded on the 3-dimensional cube. The w-label is changed to z. Note however that for this embedding the cell complex is *not* r-acyclic, as restricting to one of the horizontal levels, the cell complex becomes two separate points.

EXAMPLE 7.8.

$$\mathfrak{j} = (x^2, y^2, z^2, w^2, t^2, u^2) + (xz, xw, yz, yw, xt, xu, yt, yu, zt, zu, wt, wu).$$

The simplicial complex X associated to $\mathfrak{j}_\Delta$ is given in Figure 14. In a polarization

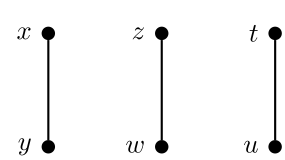

FIGURE 14. Three line segments

$J(T)$, the cell complex $C(T)$ will then have three squares, corresponding to the lines labelled xy, zw and tu. The cell complex $C(T)$ must also be r-acyclic. By Corollary 6.9, each pair of squares can intersect in at most a point. This gives, up to isomorphism, the three possibilities for $C(T)$ given in Figure 15. We thus get, up to isomorphism, three polarizations of $\mathfrak{j}$. Using Theorem 6.7 and Corollary 6.8 they are seen to be (write $M_\square = (x_ax_b, y_ay_b, z_az_b, w_aw_b, t_at_b, u_au_b)$)

$$J_1 = M_\square + (x_az_b, x_aw_b, y_az_b, y_aw_b, x_at_b, x_au_a, y_at_b, y_au_a, z_at_b, z_au_a, w_bt_b, w_bu_a)$$
$$J_2 = M_\square + (x_az_b, x_aw_b, y_az_b, y_aw_b, x_at_b, x_au_b, y_at_b, y_au_b, z_at_b, z_au_b, w_at_b, w_au_b)$$
$$J_3 = M_\square + (x_az_a, x_aw_a, y_az_a, y_aw_a, x_at_a, x_au_a, y_at_a, y_au_a, z_at_a, z_au_a, w_at_a, w_au_a)$$

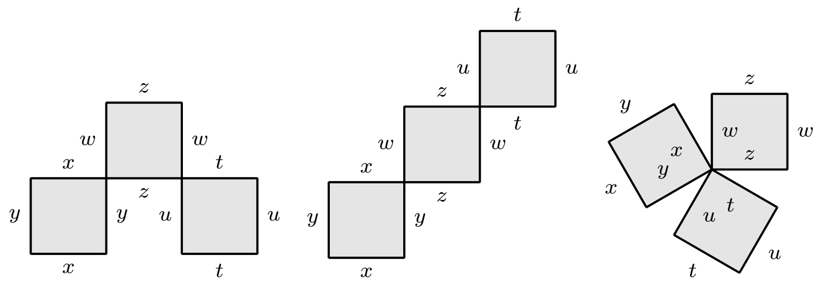

Figure 15. Restriction-acyclic $C(T)$ of three squares without edge intersections

Example 7.9.

$$\mathfrak{j} = (x^2, y^2, z^2, w^2) + (xz, xw, yw, zw).$$

The simplicial complex X associated to $\mathfrak{j}_\Delta$ is given in Figure 16.

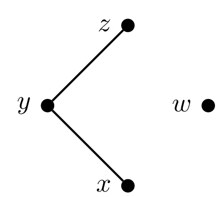

Figure 16. Two line segments and an isolated point

In a polarization $J(T)$, the cell complex $C(T)$ will then have two squares, corresponding to the lines labelled xy and yz, and one line segment, labelled w. The cell complex $C(T)$ must be r-acyclic. This gives, up to isomorphism, the two possibilities for $C(T)$ given in Figure 17. We thus get, up to isomorphism, two polarizations of $\mathfrak{j}$.

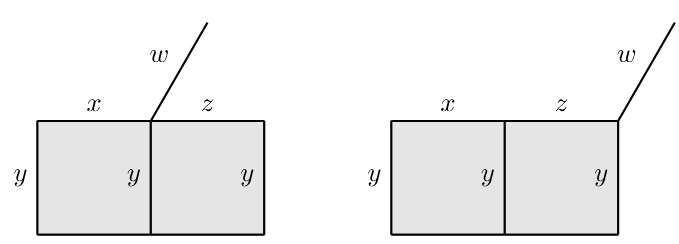

Figure 17. Restriction-acyclic $C(T)$ with two adjacent squares and one line segment

Using Theorem 6.7 and Corollary 6.8 they are seen to be:

$$J_1 = (x_a x_b, y_a y_b, z_a z_b, w_a w_b) + (x_a z_a, x_a w_a, y_a w_a, z_a w_a)$$
$$J_2 = (x_a x_b, y_a y_b, z_a z_b, w_a w_b) + (x_a z_a, x_a w_a, y_a w_a, z_b w_a)$$

Example 7.10.

$$\mathfrak{j} = (x^2, y^2, z^2, w^2) + (yz, yw, zw).$$

The simplicial complex X associated to $\mathfrak{j}_\Delta$ is given in Figure 18. In a polarization

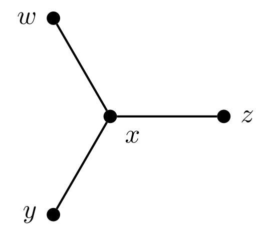

Figure 18. The star with three edges

$J(T)$, the cell complex $C(T)$ will then have three squares, corresponding to the lines labelled xy, xz and xw. The cell complex $C(T)$ must be r-acyclic. This gives, up to isomorphism, the two possibilities for $C(T)$ given in Figure 19. We thus get, up to

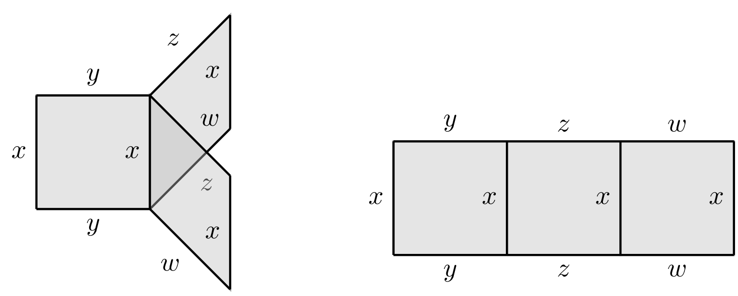

Figure 19. Restriction-ayclic $C(T)$ with three adjacent squares

isomorphism, two polarizations of $\mathfrak{j}$. They are seen to be:

$$J_1 = (x_a x_b, y_a y_b, z_a z_b, w_a w_b) + (y_a z_a, y_a w_a, z_a w_a)$$
$$J_2 = (x_a x_b, y_a y_b, z_a z_b, w_a w_b) + (y_a z_b, y_a w_b, z_a w_b)$$

Example 7.11.
$$\mathfrak{j} = (x^2, y^2, z^2, w^2, t^2) + (xz, yw, zt, wx, ty).$$

The simplicial complex X associated to $\mathfrak{j}_\Delta$ is given in Figure 20. In a polarization

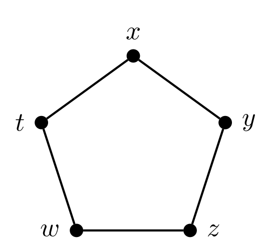

Figure 20. The five-cycle

$J(T)$, the cell complex $C(T)$ will then have five squares. Up to isomorphism there is only one possible $C(T)$ given in Figure 21. The corresponding polarization is the standard polarization:

$$J = (x_a x_b, y_a y_b, z_a z_b, w_a w_b, t_a t_b) + (x_a z_a, y_a w_a, z_a t_a, w_a x_a, t_a y_a).$$

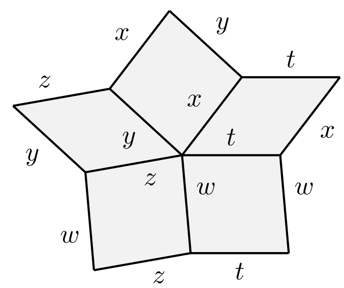

Figure 21. Restriction-acyclic $C(T)$ with five squares in a cycle

In general, if $\mathfrak{j}_\Delta$ corresponds to the simplicial complex which is a cycle graph, then $\mathfrak{j}$ has only the standard polarization. This is a consequence of the following, which is an analog of [22, Corollary 2.3].

Proposition 7.12. *Suppose $\mathfrak{j} = \mathfrak{m}_\square + \mathfrak{j}_\Delta$, where the simplicial complex X corresponding to $\mathfrak{j}_\Delta$ is flag, i.e. all generators of $\mathfrak{j}_\Delta$ are quadratic. Then $\mathfrak{j}$ has non-standard polarizations if and only if there is a vertex v on the simplicial complex such that removing v and all its neighbours, the induced simplicial complex on the remaining vertices is disconnected.*

Proof. Let the ring be $k[V]$. So the vertices of the simplicial complex is V. Let $v \in V$ be a vertex. Let $W \subseteq V \backslash \{v\}$ be the vertices which are not neighbours of v. Then the generators of $\mathfrak{j}_\Delta$ are the vw where $w \in W$ (as v varies). We have a non-standard polarization involving v iff there is a disjoint union $A \sqcup B = W$ with non-empty parts such that replacing the generators vw with $v_a w$ for $w \in A$ and $v_b w$ for $w \in B$, gives an ideal $\mathfrak{j}'$ which is a separation of $\mathfrak{j}$. Pick then $w^a \in A$ and $w^b \in B$. Then $(v_a - v_b)w^a w^b$ is zero in the quotient ring $S'/\mathfrak{j}'$. As $(v_a - v_b)$ is a non-zero divisor, this gives that $w^a w^b \in \mathfrak{j}'$, and so $w^a w^b \in \mathfrak{j}$. But as $w^a \in A$ and $w^b \in B$ were chosen arbitrarily, this means that there is no edge between the vertices of A and of B. Thus we have a separation of $\mathfrak{j}$ iff such a situation occurs. This separation may then further be standard polarized. The result will be a non-standard polarizaton, as a variable has been separated in the mixed term generators of $\mathfrak{j}$. $\square$

8. Liaison for squarefree monomial ideals

Liaison is a standard and well-developed technique in algebraic geometry, [36, 30, 48]. Here we apply it to squarefree monomial ideals where it takes a simple form. Our objective is to show that if $T \subseteq A_1 \times \cdots \times A_n$ is acyclic, then the complement subset T^c is also acyclic. Our argument for this is entirely algebraic. One could envision a more topological argument using Alexander duality, but it is a challenge to construct a suitable retract onto $C(T)$ or $C(T^c)$. In general it would be interesting to understand more about how the homology of the free cellular complexes $\tilde{\mathcal{C}}_\bullet(T)$ and $\tilde{\mathcal{C}}_\bullet(T^c)$ are related.

8.1. Liaison.

We now consider linkage of squarefree monomial ideals with respect to the complete intersection ideal

$$C = (m(A_1), m(A_2), \ldots, m(A_n)). \tag{23}$$

Two ideals J and K containing C are *linked* if their heights (codimensions) are all equal (so is n), and $K = (C : J)$ and $J = (C : K)$.

For simplicial complexes, we have the following concrete, simple, and general result. We thank an anonymous referee for the below statement.

Proposition 8.1. *Let X, Y be simplicial sub-complexes of a simplicial complex Z. Then the ideal $I_Y = (I_Z : I_X)$ if and only if Y is the simplicial complex generated by $Z \backslash X$.*

In particular, under this assumption, for F a facet of Z:

i. *If F is not in X, it is a facet of Y,*
ii. *If F is in X, it is not a facet of Y,*
iii. *The set of facets of Y are contained in the set of facets of Z.*

Proof. Let Y' be the simplicial complex generated by $Z \backslash X$, and $I_Y = (I_Z : I_X)$. We show firstly $Y \subseteq Y'$ and secondly $Y' \subseteq Y$. If F is a face of Y, the monomial $m(F) \notin I_Y$. Then there is $m(R) \in I_X$ (equivalently $R \notin X$), such that $m(F) \cdot m(R) \notin I_Z$. This monomial has $m(F \cup R)$ as a factor so $F \cup R$ is in Z and not in X. The upshot is that $Y \subseteq Y'$.

Conversely let G be a facet of Y'. Then $m(G) \in I_X$ and $m(G) \notin I_Z$. If $m(G) \in I_Y$, then $m(G) \cdot m(G) \in I_Z$. But since I_Z is squarefree, we have $m(G) \in I_Z$, against assumption. So $m(G) \notin I_Y$, giving $G \in Y$. Hence $Y' \subseteq Y$.

The three consequences are immediate. ☐

Corollary 8.2. *We have $I_Y = (I_Z : I_X)$ and $I_X = (I_Z : I_Y)$ iff we have a disjoint union of facets*

$$\mathit{facets}(Z) = \mathit{facets}(X) \sqcup \mathit{facets}(Y).$$

In our specific situation we have:

Proposition 8.3. *Let J and K be squarefree monomial ideals containing the complete intersection C of* (23)*, and let X and Y be the corresponding simplicial complexes via the SR-correspondence (which are subcomplexes of $\partial\Delta(A_\bullet)$). Then J and K are linked by C iff the facets of $\partial\Delta(A_\bullet)$ is the disjoint union of the facets of X and Y.*

The following is Claim 1 in Lemma 3.6 in [45].

Corollary 8.4. *Let $T \subseteq A_1 \times \cdots \times A_n$ and let J be the Alexander dual $I(T)^\vee$. Let K be a squarefree monomial ideal containing C. Then J and K are linked by C iff K is the Alexander dual $I(T^c)^\vee$.*

Proof. If J and K are linked by C, the facets of $\partial\Delta(A_\bullet)$ is the disjoint union of the facets of X and Y. The facets of X are then $[\mathbf{a}]^c$ for $\mathbf{a} \in T$, and so the facets of Y must then be the $[\mathbf{a}]^c$ for $\mathbf{a} \in T^c$. Hence K is the Alexander dual of $I(T^c)$.

Conversely if K is the Alexander dual of $I(T^c)$, the facets of Y and X are the $[\mathbf{a}]^c$ for respectively $\mathbf{a} \in T^c$ and $\mathbf{a} \in T$. So $\partial\Delta(A_\bullet)$ is the disjoint union of these facets, and so J and K are linked. ☐

We have a diagram:

$$\begin{tikzcd}
I(T) \arrow[r, leftrightarrow, "\text{Alexander dual}"] \arrow[d, leftrightarrow, "\text{complement rainbow ideals}"'] & J(T) \arrow[d, leftrightarrow, "\text{linked}"] \\
I(T^c) \arrow[r, leftrightarrow, "\text{Alexander dual}"] & K(T)
\end{tikzcd}$$

Corollary 8.5. *If the rainbow monomial ideal $I(T)$ has a linear resolution, the complement rainbow ideal $I(T^c)$ also has a linear resolution. Hence T is acyclic iff the complement T^c is acyclic.*

Proof. Let J be the Alexander dual of $I(T)$. It is Cohen-Macaulay since $I(T)$ has a linear resolution. The linked ideal K of J is also Cohen-Macaulay by [18, Thm. 21.23]. The Alexander dual of K is $I(T^c)$ and so this ideal also has linear resolution. $\square$

Corollary 8.6. *If X is a full-dimensional triangulated ball on $\partial\Delta(A_\bullet)$, its full-dimensional complement $Y = X(T^c)$ on $\partial\Delta(A_\bullet)$ is also a triangulated ball.*

As stated after Theorem 4.4, a full-dimensional CM subcomplex of $\partial\Delta(A_\bullet)$ may not be shellable. H.T. Hall [25] gives a full-dimensional subcomplex Y of a 12-dimensional hyperoctahedron H which is shellable, but for no facet F outside of Y is $Y \cup \overline{F}$ shellable, where $\overline{F}$ is the simplicial complex generated by F. (One says H is not extendably shellable.) The consequence of this fact is the following.

Proposition 8.7. *The complement sub-complex X of Y, generated by $H\backslash Y$, is a full-dimensional CM subcomplex of H which is not shellable. (But by Theorem 4.4, X is constructible.)*

Proof. By Proposition 8.3, X and Y are linked. Since Y is CM, X is also CM by [18, Thm. 21.23].

If $F_1, \dots, F_m$ is a shelling of X, the simplicial complex generated by $X\backslash\overline{F_m}$ is shellable by $F_1, \dots, F_{m-1}$ and so is CM. Its complement in turn is $Y \cup \overline{F_m}$, which is the linked complex and so would also be CM. By Lemma 2.2, $Y \cap \overline{F_m}$ would be CM of dimension one less than $\overline{F_m}$, and so a pure codimension one subcomplex of $\overline{F_m}$. This would imply shellability of $Y \cup \overline{F_m}$, contradicting the result of [25]. $\square$

9. The boundary

For an acyclic $T \subseteq A_1 \times \cdots \times A_n$, we have the partition into two acyclic sets $T \sqcup T^c = A_1 \times \cdots \times A_n$. Let $X(T)$ and $X(T^c)$ be the corresponding simplicial balls, whose union $X(T) \cup X(T^c)$ is $\partial\Delta(A_\bullet)$. Recall that J is the SR-ideal of $X(T)$ and K is the SR-ideal of $X(T^c)$ (formerly mostly denoted Y). We also denote $L(T) = I(T^c)$.

Proposition 9.1. *For acyclic T, the boundaries of $X(T)$ and $X(T^c)$ are the same:*

$$\partial X(T) = \partial X(T^c) = X(T) \cap X(T^c).$$

So this is a simplicial sphere with defining ideal $J + K$, which is a Gorenstein ideal. The Alexander dual of $J + K$ is $I(T) \cap L(T)$, an ideal generated in degree $n + 1$.

Proof. Every codimension one face in $\partial\Delta(A_\bullet)$ is on two facets. The boundary of $X(T)$ consists of those codimension one faces on only one facet of $X(T)$. But then the other facet in $\partial\Delta(A_\bullet)$ containing this codimension one face must be in $X(T^c)$. Conversely, let F be a face in $X(T) \cap X(T^c)$. Then there are facets $[\mathbf{a}]^c$ in $X(T)$ and $[\mathbf{b}]^c$ in $X(T^c)$ containing F. If $\mathbf{a}$ and $\mathbf{b}$ have r distinct coordinates, consider a path from $\mathbf{a}$ to $\mathbf{b}$ of length r changing one coordinate at a time. Then every $\mathbf{a}_i$ on this path will have $F \subseteq [\mathbf{a}_i]^c$. At some stage $\mathbf{a}_i \in T$ and $\mathbf{a}_{i+1} \in T^c$. Their intersection is a codimension one face i) on the boundaries of $X(T^c)$ and $X(T)$, and ii) containing F. This shows the statement about the boundaries.

That $J + K$ is the defining ideal is clear since this is the ideal of $X(T) \cap X(T^c)$. Since $X(T)$ and $X(T^c)$ are simplicial balls, their boundaries are simplicial spheres (which have dimension one less that $X(T)$ and $X(T^c)$). Thus its defining ideal is Gorenstein [13, Corollary 5.6.5].

The complements of the facets of $X(T) \cap X(T^c)$, give the generators of $I(T) \cap L(T)$ and they must then have degree $n + 1$ (one more than the generators of $I(T)$). $\square$

REMARK 9.2. For ideals J and K linked by a complete intersection C, the sum $J+K$ may not be a Gorenstein ideal. But this holds if $J \cap K = C$ [48, Thm. 2.8]. This is the situation here, since the Alexander dual of $J \cap K$ is $I(T) + L(T)$, which is the ideal B generated by all rainbow monomials. Since C is the Alexander dual of B this follows.

REMARK 9.3. In general for Z a d-dimensional simplicial sphere, and X a d-dimensional Cohen-Macaulay sub-complex, the complement simplicial complex Y generated by $Z \backslash X$ is also Cohen-Macaulay, and their boundaries $\partial X = \partial Y = X \cap Y$. The latter may be shown using that Cohen-Macaulay simplicial complexes are gallery connected (strongly connected), [28, Prop. 9.1.12].

REMARK 9.4. A concrete, effective description of the SR-ideal of the boundary is given in [14, Thm. 5.2] for simplicial spheres defined by letterplace ideals.

10. ARTIN MONOMIAL IDEALS FROM A BOX

Subsection 3.3 considered polarizations J of Artin monomial ideals $\mathfrak{j} \subseteq k[x_1, \ldots, x_n]$ containing the complete intersection

$$\mathfrak{c} = (x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) \tag{24}$$

Section 8 considered linkage of two squarefree monomial ideals J and K in $k[A]$ with respect to the complete intersection

$$C = (m(A_1), m(A_2), \ldots, m(A_n)).$$

We show this corresponds to linkage downstairs of Artin monomial ideals.

For an integer $m \geqslant 0$ let $[m]_0$ denote the chain $\{0 < 1 < \cdots < m\}$. For $i = 1, \ldots, n$ let A_i be a set with $\alpha_i \geqslant 1$ elements.

Recall that an up-set U in a poset P is a set such that $x \in U$ and $x \leqslant y$ implies $y \in U$. Similarly we have the notion of down-set. Consider an up-set U in the product poset

$$\alpha = [\alpha_1 - 1]_0 \times [\alpha_2 - 1]_0 \times \cdots \times [\alpha_n - 1]_0,$$

We get a monomial ideal $\mathfrak{i}(U) = (x^{\mathbf{b}} \,|\, \mathbf{b} \in U)$. This is an $(\alpha_1 - 1, \ldots, \alpha_n - 1)$-*determined* Artin monomial ideal (a notion introduced in [38], see also [39, Def. 5.20]). Let $\mathfrak{i} \subseteq k[x_1, \ldots, x_n]$ be the Artin monomial ideal $\mathfrak{c} + \mathfrak{i}(U)$.

Let D be the complement of U in α, so D is a down-set in α. Let α^{op} be the opposite poset of α. There is an isomorphism of posets $\alpha^{\mathrm{op}} \cong \alpha$ sending:

$$\mathbf{i}^{\mathrm{op}} = (i_1^{\mathrm{op}}, i_2^{\mathrm{op}}, \ldots, i_n^{\mathrm{op}}) \mapsto (\alpha_1 - 1 - i_1, \alpha_2 - 1 - i_2, \ldots, \alpha_n - 1 - i_n) =: \bar{\mathbf{i}}.$$

This sends the the up-set D^{op} in α^{op} to an up-set $\overline{D}$ in α. Let $\mathfrak{k} \subseteq k[x_1, \ldots, x_n]$ be the Artin ideal $\mathfrak{c} + \mathfrak{i}(\overline{D})$, the *dual* of $\mathfrak{i}$ with respect to $\mathfrak{c}$.

REMARK 10.1. The ideal $\mathfrak{i}(\overline{D})$ is the *Alexander dual of* $\mathfrak{i}(U)$ *with respect to* $(\alpha_1 - 1, \alpha_2 - 1, \ldots, \alpha_n - 1)$, see [39, Def. 5.20] or [37]. The definition in [39] is different from the one above (and more general in scope), but Lemma 10.3 below and [39, Cor. 5.25] show they coincide in our case.

EXAMPLE 10.2. Let $n = 3$ and $\alpha_1, \alpha_2, \alpha_3 = 2$, so $\alpha = \{0, 1\}^3$. Let the up-set of α be:

$$U : 111, 101, 011.$$

It gives an Artin monomial ideal:

$$\mathfrak{i} = \mathfrak{i}(U) + \mathfrak{c} = (x_1^2, x_1x_3, x_2x_3, x_2^2, x_3^2).$$

The corresponding down-set is:

$$D : 000, 100, 010, 001, 110,$$

and so

$$\overline{D} : 111, 011, 101, 110, 001.$$

The dual Artin monomial of $\mathfrak{i}$ with respect to (x_1^2, x_2^2, x_3^2) is:

$$\mathfrak{k} = \mathfrak{i}(\overline{D}) + \mathfrak{c} = (x_1^2, x_1x_2, x_2^2, x_3).$$

Lemma 10.3. *The ideals* $\mathfrak{i}$ *and* $\mathfrak{k}$ *are linked with respect to the complete intersection* $\mathfrak{c}$.

Proof. Note that $(b_1, \dots, b_n) \in \alpha$ is in $\overline{D}$ iff $(\alpha_1 - 1 - b_1, \dots, \alpha_n - 1 - b_n)$ is in D, i.e. is not in U. That is, for *every* $(a_1, \dots, a_n)$ in U there is some j with $\alpha_j - 1 - b_j < a_j$, or equivalently $a_j + b_j \geqslant \alpha_j$.

Now let us show that $\mathfrak{k} = \mathfrak{c} + \mathfrak{i}(\overline{D})$ equals $(\mathfrak{c} : \mathfrak{i}) = (\mathfrak{c} : \mathfrak{i}(U))$. That $x_1^{b_1} \cdots x_n^{b_n} \in \mathfrak{i}(\overline{D})$ is by the above equivalent to for *every* $x_1^{a_1} \cdots x_n^{a_n}$ in $\mathfrak{i}(U)$ there is some j with $a_j + b_j \geqslant \alpha_j$. But this is equivalent to $x_1^{b_1} \cdots x_n^{b_n} \cdot \mathfrak{i}(U)$ being in $\mathfrak{c}$. □

Proposition 10.4. *Let J be a polarization of $\mathfrak{j}$. Let K be the link of J by C, and $\mathfrak{k}$ the link of $\mathfrak{j}$ by $\mathfrak{c}$, so $\mathfrak{k}$ is the dual of $\mathfrak{j}$. Then K is a polarization of $\mathfrak{k}$.*

Proof. Set $A_l = \{a_1^l, \dots, a_{\alpha_l}^l\}$ for every $l = 1, \dots, n$. We divide the quotient rings $k[A]/J, k[A]/K$ and $k[A]/C$ by all the variable differences $a_j^l - a_{j-1}^l$ for $j = 2, \dots, \alpha_l$ and $l = 1, \dots, n$. We get quotient rings $k[x_1, \dots, x_n]/\mathfrak{j}$, $k[x_1, \dots, x_n]/\mathfrak{k}'$ and $k[x_1, \dots, x_n]/\mathfrak{c}$, where $\mathfrak{j}$, $\mathfrak{k}'$ and $\mathfrak{c}$ are Artin. By [30, Proposition 2.8] when we cut down by a regular sequence for $k[A]/C$ the ideals we get are still linked, and it is a regular sequence also for $k[A]/J$ and $k[A]/K$. Thus $\mathfrak{k}'$ is linked to $\mathfrak{j}$ by $\mathfrak{c}$, so so $\mathfrak{k}' = \mathfrak{k}$ and K is a polarization of this. □

To sum up we have a diagram:

$$\begin{tikzcd}
 & I(T) \arrow[ld, <->, "\text{complement rainbow ideals}"'] \arrow[rr, <->, "\text{Alexander dual}"] & & J(T) \arrow[d, "\text{polarization}"] \arrow[ld, <->, "\text{linked}"'] \\
L(T) \arrow[rr, <->, "\text{Alexander dual}"] & & K(T) \arrow[d, "\text{polarization}"'] & \mathfrak{j}. \\
 & & \mathfrak{k} \arrow[ru, <->, "\text{linked}"'] &
\end{tikzcd}$$

The left-right arrows are correspondences, and the vertical arrows are maps of ideals. But note that the process of polarization goes from the lower ideal to the upper ideal.

11. Squeezed balls and spheres

This section is somewhat independent of the rest of the article. We show that the squeezed balls constructed by G. Kalai [32] (their boundaries are squeezed spheres), have Stanley-Reisner ideals which are regular quotients by variable differences of a special class of letterplace ideals. Letterplace ideals are polarizations of Artin monomial ideals by Theorem 5.12 and Corollary 2.4 in [21].

In all, this shows that the two classical ways of constructing large numbers of triangulated balls and spheres, those of Bier balls and spheres [11], [14, Cor.5.3] and squeezed balls and spheres [32], both are part of the setting we give here. They are actually at the opposite ends, as Bier balls arise from letterplace ideals of discrete posets (antichains) [14, Cor.5.3.], while squeezed balls arise from letterplace ideals of totally ordered posets (chains), see Subsection 11.3.

11.1. Squeezed balls. Given an integer $e \geqslant 1$. Consider sequences of natural numbers $\mathbf{i} : 1 \leqslant i_1 < i_2 < \cdots < i_e$ such that each difference $i_p - i_{p-1} \geqslant 2$. Let F_e be the set of such sequences. There is a partial order on F_e by $\mathbf{i} \leqslant \mathbf{j}$ if each $i_p \leqslant j_p$. Let $F_e(m)$ be such sequences with $i_e < m$.

Each such sequence induces a subset of natural numbers of cardinality $2e$:

$$\hat{\mathbf{i}} : i_1, i_1 + 1, i_2, i_2 + 1, \ldots, i_e, i_e + 1.$$

Let I be a *finite* down-set (order ideal) in F_e for its partial order. Kalai shows [32, Sec.4] that the simplicial complex $B(I)$ with facets $\hat{\mathbf{i}}$ for $\mathbf{i} \in I$, is a simplicial ball, termed a *squeezed ball*, with boundary a *squeezed sphere*. By a counting argument, there are so many of these that most of these balls and spheres do not have convex realization.

11.2. Poset homomorphisms. Let P and Q be posets. We consider order-preserving maps $\phi : P \to Q$, and denote by $\mathrm{Hom}(P, Q)$ the set of such. This becomes itself a poset (this is an internal Hom), by $\phi \preceq \psi$ if $\phi(p) \leqslant \psi(p)$ for every $p \in P$. There are order-preserving maps:

$$P \times \mathrm{Hom}(P, Q) \to P \times Q \quad \text{inducing} \quad \mathrm{Hom}(P, Q) \xrightarrow{G} \mathrm{Hom}(P, P \times Q),$$

and for $\phi : P \to Q$ the image of the map $G(\phi)$ is the graph $\Gamma(\phi) \subseteq P \times Q$.

Denote by $[n]$ the chain $\{1 < 2 < \cdots < n\}$, and $[n]_0$ the chain $[n] \cup \{0\}$. Note that $[n]_0 \cong \mathrm{Hom}([n], \{0 < 1\})$. We have a composition:

$$\begin{aligned} [n] \times \mathrm{Hom}([n], [e]_0) \to [n] \times [e]_0 &\xrightarrow{\alpha} [n + 2e] \\ (i, j) &\mapsto i + 2j \end{aligned} \tag{25}$$

giving

$$\Phi : \mathrm{Hom}([n], [e]_0) \to \mathrm{Hom}([n], [n + 2e]).$$

We also have a composition:

$$\begin{aligned} [e] \times \mathrm{Hom}([e], [n]_0) \to [e] \times [n]_0 &\xrightarrow{\beta} [n + 2e - 1] \\ (j, i) &\mapsto (2j - 1) + i \end{aligned}$$

giving

$$\Psi : \mathrm{Hom}([e], [n]_0) \to \mathrm{Hom}([e], [n + 2e - 1]).$$

Note that the domains of Φ and Ψ identify:

$$\begin{aligned} \mathrm{Hom}([n], [e]_0) =& \mathrm{Hom}([n], \mathrm{Hom}([e], \{0 < 1\})) \cong \mathrm{Hom}([n] \times [e], \{0 < 1\}) \\ \cong& \mathrm{Hom}([e], \mathrm{Hom}([n], \{0 < 1\})) = \mathrm{Hom}([e], [n]_0). \end{aligned}$$

This is just mapping a partition to its dual partition. Given $\phi : [n] \to [e]_0$, it may be characterized by a sequence

$$0 = a_0 \leqslant a_1 \leqslant \cdots \leqslant a_e \leqslant a_{e+1} = n,$$

such that on the interval $[a_p + 1, a_{p+1}]$, the value of ϕ is p. These a's give the dual map $\phi' : [e] \to [n]_0$.

Lemma 11.1.

a. *The image of* Ψ *is* $F_e(n + 2e)$ *and this gives an isomorphism of posets between* $\mathrm{Hom}([e], [n]_0)$ *and* $F_e(n + 2e)$*.*
b. *Let* $\mathbf{i} = im\, \Psi(\phi')$*. Then* $\hat{\mathbf{i}} = (im\, \Phi(\phi))^c$*.*

Proof. a. Let $\phi \in \mathrm{Hom}([n],[e]_0)$ and $\phi' \in \mathrm{Hom}([e],[n]_0)$ correspond, and let $a_p = \phi'(p)$. The image of the map $\Psi(\phi')$ is $\mathbf{i}$ where $i_p = a_p + 2p - 1$ for $p = 1, \dots, e$. This gives a bijection between $\mathrm{Hom}([e],[n]_0)$ and $F_e(n+2e)$, and it is clear that the partial orders correspond.

b. The gaps in the image of $\Phi(\phi)$ in $[n+2e]$ are as follows: If

$$a_{p-1} < a_p = \cdots = a_q < a_{q+1},$$

there is a gap ranging from $(a_p + 2(p-1)) + 1$ to $(a_q + 1 + 2q) - 1$. This gap has even size $2q - 2p + 2$. The values for $\mathbf{i}$ are

$$\begin{aligned} i_p = a_p + 2p - 1, \quad i_{p+1} &= a_{p+1} + 2(p+1) - 1 = a_p + 2p + 1, \\ \dots \quad i_q &= a_q + 2q - 1 = a_p + 2q - 1, \end{aligned}$$

with each step in this range of size two. This shows the last claim. □

11.3. Letterplace ideals. For a finite set R let $k[x_R]$ be the polynomial ring with variables $x_r, r \in R$. If $S \subseteq R$ let x_S be the monomial $\prod_{s \in S} x_s$.

Let P be a finite poset, and $\mathcal{I}$ a down-set in (the internal Hom) $\mathrm{Hom}(P,[m])$. Each $\phi \in \mathcal{I}$ gives a graph $\Gamma(\phi) \subseteq P \times [m]$. The associated *co-letterplace ideal* $L(P,[m];\mathcal{I})$ is the ideal in $k[x_{P\times[m]}]$ generated by monomials $x_{\Gamma\phi}$ for $\phi \in \mathcal{I}$. The associated *letterplace ideal* $L(\mathcal{I};P,[m])$ is the Alexander dual of $L(P,[m];\mathcal{I})$. The letterplace ideal is a polarization of an Artin monomial ideal by Corollary 2.4 and Theorem 5.12 in [21]. Its associated triangulated ball is described in [14].

Now consider a down-set $\mathcal{I} \subseteq \mathrm{Hom}([n],[e]_0)$. It gives a co-letterplace ideal $L([n],[e]_0;\mathcal{I})$ and a letterplace ideal $L(\mathcal{I};[n],[e]_0)$. The co-letterplace ideal is a rainbow ideal with n color classes each of size $e+1$. The map $\alpha : [n]\times[e]_0 \to [n+2e]$ from (25) is given by $(i,j) \mapsto i+2j$. It gives a map of polynomial rings $\tilde{\alpha} : k[x_{[n]\times[e]_0}] \to k[x_{[n+2e]}]$. We may think of this as collapsing each fiber of α into a single variable. If a fiber has t elements, we are dividing by $t-1$ variable differences, and we run through all fibers. The variable differences are of type $x_{i,j+1} - x_{i+2,j}$, so $x_{i,j+1}$ and $x_{i+2,j}$ are in different color classes. Let $L^\alpha(\mathcal{I};[n],[e]_0)$ be the image ideal of $L(\mathcal{I};[n],[e]_0)$ by $\tilde{\alpha}$, an ideal in the polynomial ring $k[x_{[n+2e]}]$.

Theorem 11.2. *Let the poset ideal $\mathcal{I} \subseteq \mathrm{Hom}([n],[e]_0)$ correspond to the poset ideal $I \subseteq F_e(n+2e)$ by the isomorphism Ψ. The Stanley-Reisner ideal of the squeezed ball $B(I)$ is the ideal $L^\alpha(\mathcal{I};[n],[e]_0)$.*

Proof. Each fiber of α is seen to be a *bistrict* chain in $[n] \times [e]_0^{\mathrm{op}}$. That is, if (p,u) and (q,v) are in the same fiber of α, either $p > q$ and $u < v$, *or* $p < q$ and $u > v$. By Theorem 7.3 in [21], the images by $\tilde{\alpha}$

$$L^\alpha(\mathcal{I};[n],[e]_0) \text{ of } L(\mathcal{I};[n],[e]_0), \quad L^\alpha([n],[e]_0;\mathcal{I}) \text{ of } L([n],[e]_0;\mathcal{I})$$

are still square-free and they are Alexander dual ideals.

The facets of the simplicial complex with Stanley-Reisner ideal $L^\alpha(\mathcal{I};[n],[e]_0)$ are the complement in $[n+2e]$ of the indices of the minimal generators of $L^\alpha([n],[e]_0;\mathcal{I})$. These index sets are $\mathrm{im}\,\Phi(\phi)$ for $\phi \in \mathcal{I}$ and by Lemma 11.1 part b their complements give the facets of the squeezed ball $B(I)$. □

12. Further problems

We sketch three open research areas.

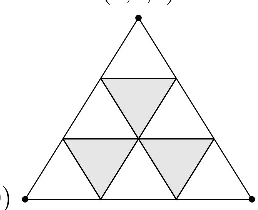

FIGURE 22. Vertices correspond to generators of $(x_1, x_2, x_3)^3$

12.1. ACYCLIC SUBSETS $T \subseteq \mathbf{A}$. Section 7 illustrates well how one may make acyclic subsets on a hypercube. When the cardinalities of the A_i become $\geqslant 3$, things get more complicated.

PROBLEM 1. *Give machines for constructing acyclic sets $T \subseteq \mathbf{A}$.*

One such machine is the following.

Co-letterplace ideals. This is from [21], which is again inspired by [19]. Recall from Subsection 11.3 that for a poset P an order-preserving map $\phi : P \to [n]$ gives a graph

$$\Gamma\phi = \{(p, \phi(p)) \,|\, p \in P\} \subseteq P \times [n].$$

The graph may be considered as an element of $[n]^P = \times_{p \in P}[n]$, the cartesian product of $[n]$'s, one factor for each $p \in P$. For $\mathcal{I}$ a down-set in $\mathrm{Hom}(P, [n])$, let

$$T = \{\Gamma\phi \,|\, \phi \in \mathcal{I}\} \subseteq [n]^P.$$

The ideal $I(T)$ is the co-letterplace ideal from [21], and has linear resolution, in fact it has linear quotients. In [14] the simplicial balls $X(T)$ are studied. Their boundaries, simplicial spheres, have a particularly nice and compact description, [14, Sec.5]. The classical cases are:

- Bier balls and spheres [9, 11]: P is an antichain and $n = 2$, and in [43] for any n.
- Squeezed balls and spheres [32]: P is a chain. Or rather they are derived from this case, see Section 11.

D. Lu and Z. Wang, [34], extend this. They replace the down-set $\mathcal{I}$ of $\mathrm{Hom}(P, [n])$ with larger classes of subsets of $\mathrm{Hom}(P, [n])$.

Polarizations of powers of maximal graded ideals $(x_1, \ldots, x_n)^m$ are characterized in [3]. When $n = 3$, there is a nice visual procedure for this:

Planar triangulations $n = 3$. The generators of $(x_1, x_2, x_3)^m$ corresponds to the set $\Delta_3(m)$ consisting of all triples $(a_1, a_2, a_3) \in \mathbb{N}_0^3$ with $a_1 + a_2 + a_3 = m$. Figure 22 displays in the case $m = 3$ these ten triples as vertices. The figure has three "down-triangles". Polarizations of $(x_1, x_2, x_3)^m$ are in bijection with removals of either zero or one edge from each down-triangle.

12.2. REGULAR QUOTIENTS OF RAINBOW IDEALS. If the simplicial ball $X(T)$ is a one-dimensional ball, then $\partial\Delta(A_\bullet)$ must be one-dimensional. This occurs only for $n = 2$ and $|A_1| = |A_2| = 2$, when $\partial\Delta(A_\bullet)$ is a square (perimeter), or $n = 1$ and $|A_1| = 3$, when it is a triangle (perimeter). Hence the only one-dimensional balls $X(T)$ are:

In particular the line graphs with $\geqslant 5$ vertices:

are not triangulated balls we can obtain by our constructions here.

Concerning two-dimensional $X(T)$, only triangulations of m-gons where $m \leqslant 6$ are obtained this way. From this perspective our setting seems somewhat limited. However this is not the case: By [22] *every* line graph or triangulation of a polygon may be obtained from $X(T)$'s by *joining* variables: Divide $k[A]/J(T)$ by a regular sequence of variable differences.

In our article here, the regular sequences we divide out by to get the Artin monomial ideals, consists of variable differences $a^i_j - a^{i'}_{j'}$ where $i = i'$ (the terms are in same color class A_i). However we may also divide out by variable differences where $i \neq i'$. Starting form $k[A]/J$ we then get a ring $k[A']/J'$, coming from a surjection $A \to A'$. This is how we obtain any line graph and any triangulation of a polygon in [22]. We thus have the processes:

$$\begin{tikzcd} & J(T) \arrow[dr, "\text{joining variables}"] & \\ & & J'. \\ \mathfrak{j} \arrow[uur, "\text{separating variables}"] & & \end{tikzcd}$$

Each of the arrows above preserves the graded Betti numbers. Hence every J' has the same graded Betti numbers as $\mathfrak{j}$. In Section 11 we obtained in this way the squeezed balls and spheres of G. Kalai [32].

Problem 2. *Given a subset $T \subseteq \mathbf{A}$. Which sets of variable differences $\{a^i_j - a^{i'}_{j'}\}$ are such that:*

i. *they are a regular sequence for $k[A]/J$, where $J = J(T)$ is the Alexander dual of the rainbow ideal $I(T)$?*
ii. *in addition the resulting ideal J' is square free?*
iii. *in addition the ideal J' defines a simplicial ball?*

[22] achieves a full understanding of this for acyclic T constructed from trees (see Example 7.5). Consequences in [20] are relations between partitions of vertices and facets of classes of simplicial complexes. General as the results of [20, 22] may seem, they consider only a tiny fraction of Artin monomial ideals: the second powers $(x_1, \ldots, x_n)^2$. In [40] they do the above for standard polarizations I^{st} of any monomial ideal I. They look at J' whose standard polarization J'^{st} is isomorphic to I^{st}. They describe how to give such J', and they give a poset structure on the family of J's.

12.3. Triangulations and convex polytopes. There are many combinatorial constructions of simplicial or simple polytopes. (The dual of a simple polytope is a simplicial polytope.) For instance the nestohedra of [1, 49] are simple polytopes.

Our results here construct simplicial balls whose boundaries are simplicial spheres. These make a much wider class as most of these simplicial spheres do not have convex realizations [11]. Furthermore there is an additional aspect: our constructions also triangulate the interior of the sphere.

Problem 3. *Which simplicial balls $X(T)$ have boundaries $\partial X(T)$ corresponding to the polytopes or duals of polytopes mentioned above* [1, 49]*, or to other simple or simplicial polytopes?*

Two classes of such are Bier spheres of threshold simplicial complexes and of simplicial complexes of “short sets” [31].

An $X(T)$ can only be a triangulated *polygon* of size n for $n \leqslant 6$. Triangulated polygons of sizes $n \geqslant 7$ are not of this type. (However their ideals may be separated to give SR-ideals of $X(T)$'s, [22].) For generalized permutahedra [49], one may also only construct n-gons for $n \leqslant 6$, but not for $n \geqslant 7$. Does this indicate some connection?

There is a vast literature on triangulations of polyhedral complexes, see [15, 33]. In particular we mention regular triangulations [12, 50].

Problem 4. *For classes of triangulations described in the literature, which can be described as $X(T)$'s or regular quotients of these?*

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