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Corrigendum to "A colourful path to matrix-tree theorems"

Adrien Kassel & Thierry Lévy

Abstract Some statements in the original publication relied on an implicit assumption. This corrigendum provides the necessary modifications when that assumption does not hold.

In [1, Eq. (1)], we defined a notion of τ-determinant for a general tracial map τ. However, in our combinatorial expansion results, we implicitly made the assumption that $\tau(1) = 1$. In this short note, we reformulate the results of [1], using the same notation, in the case where this equality is not assumed.

To be clear, the results of [1] are correct as we originally stated them under the assumption that $\tau(1) = 1$. Without this assumption, and using the same proofs taking into account the value of $\tau(1)$ in the computations (which adds a weight $\tau(1)$ per "black arrow" in the graphical expansions), one easily finds the following modified statements.

Theorem 1 (Corrects [1, Theorem 3.1]). *In the ring $S = K[a_{ij} \colon (i,j) \in \mathsf{E}]$,*

$$\det_\tau \Delta_{[m]} = \tau(1)^m \sum_{\mathsf{F} \in \vec{\mathscr{F}}_m} a_{\mathsf{F}} \prod_{c \in \mathscr{C}(\mathsf{F})} \Big(1 - \tau(1)^{-\ell(c)} \tau(h_c)\Big) .$$

Corollary 2 (Corrects [1, Corollary 3.2]). *In the quotient $S/\big(a_{ij} - a_{ji} \colon (i,j) \in \mathsf{E}\big)$,*

$$\det_\tau \Delta_{[m]} = \tau(1)^m \sum_{[\mathsf{F}] \in \mathscr{F}_m} a_{\mathsf{F}} \prod_{\substack{c \in \mathscr{C}(\mathsf{F}) \\ \ell(c)=2}} \Big(1 - \tau(1)^{-\ell(c)} \tau(h_c)\Big) \prod_{\substack{c \in \mathscr{C}(\mathsf{F}) \\ \ell(c)\geqslant 3}} \Big(2 - \tau(1)^{-\ell(c)} \big(\tau(h_c) + \tau(h_{c^{-1}})\big)\Big) .$$

Theorem 3 (Corrects [1, Theorem 5.1]). *In the ring $S = K[a_{ij} \colon (i,j) \in \mathsf{E}]$,*

$$\det_\tau \Delta_{[Nm]} = \tau(1)^{Nm} \sum_{\mathsf{F} \in \mathscr{H}\vec{\mathscr{F}}_{m,N}} a_{\mathsf{F}} \prod_{c \in \mathscr{H}(\mathsf{F})} \big(1 - \tau(1)^{-\ell(c)} \tau(h_c^{\rotatebox{90}{$\bowtie$}})\big) \prod_{c \in \mathscr{S}(\mathsf{F})} \big(- \tau(1)^{-\ell(c)} \tau(h_c^{\rotatebox{90}{$\bowtie$}})\big) .$$

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Corollary 4 (Corrects [1, Corollary 5.2]). *In the quotient* $S/(a_{ij} - a_{ji} \colon (i,j) \in \mathsf{E})$,

$$\begin{aligned}\det_\tau \Delta_{[Nm]} =& \tau(1)^{Nm} \sum_{[\mathsf{F}] \in \mathscr{HF}_{m,N}} a_{\mathsf{F}} \prod_{\substack{c \in \mathscr{H}(\mathsf{F}) \\ \ell(c)=2}} \big(1 - \tau(1)^{-\ell(c)} \tau(h_c)\big) \\ & \prod_{\substack{c \in \mathscr{H}(\mathsf{F}) \\ \ell(c)\geqslant 3}} \Big(2 - \tau(1)^{-\ell(c)} \big(\tau(h_c) + \tau(h_{c^{-1}})\big)\Big) \\ & \prod_{\substack{c \in \mathscr{S}(\mathsf{F}) \\ \ell(c)=2}} (-\tau(1))^{-\ell(c)} \tau(h_c) \prod_{\substack{c \in \mathscr{S}(\mathsf{F}) \\ \ell(c)\geqslant 3}} (-\tau(1))^{-\ell(c)} \big(\tau(h_c) + \tau(h_{c^{-1}})\big) .\end{aligned}$$

Theorem 5 (Corrects [1, Theorem 6.1]). *In the ring* $S = K[a_{ij} \colon (i,j) \in \mathsf{E}]$,

$$\det_\tau \Delta_{[m]} = \left[\frac{\tau(1)}{d}\right]^m \sum_{\mathsf{F} \in \mathscr{F}_m^{\rightarrow}} a_{\mathsf{F}} \prod_{c \in \mathscr{C}(\mathsf{F})} \left(1 - \left[\frac{d}{\tau(1)}\right]^{\ell(c)} \varepsilon_c \tau(h_c)\right) .$$

Theorem 6 (Corrects [1, Theorem 7.1]). *In the ring* $S = K[a_{ij} \colon (i,j) \in \mathsf{E}]$,

$$\det_\tau \Delta_{[Nm]} = \left[\frac{\tau(1)}{N}\right]^{mN} \sum_{\mathsf{F} \in \mathscr{F}_{m,N}^{\rightarrow}} a_{\mathsf{F}} \prod_{c \in \mathscr{C}(\mathsf{F})} \left(1 - \left[\frac{N}{\tau(1)}\right]^{\ell(c)} \tau(h_c^{\between})\right) .$$

In particular, when $\tau(1) = N$ (which is the case when taking $H = M_N(\mathbb{C})$, $K = \mathbb{C}$, and $\tau : H \to K$ given by the trace $\tau(\cdot) = \mathrm{Tr}(\cdot)$) the last equation simplifies nicely to give the following corollary which did not appear in [1].

Corollary 7 (Not stated in [1]). *Assume that* $\tau(1) = N$. *Then the following equality holds in the ring* $S = K[a_{ij} : (i,j) \in \mathsf{E}]$*:*

$$\det_\tau \Delta_{[Nm]} = \sum_{\mathsf{F} \in \mathscr{F}_{m,N}^{\rightarrow}} a_{\mathsf{F}} \prod_{c \in \mathscr{C}(\mathsf{F})} \Big(1 - \tau(h_c^{\between})\Big) .$$

References

[1] Adrien Kassel and Thierry Lévy, *A colourful path to matrix-tree theorems*, Algebr. Comb. **3** (2020), no. 2, 471–482.

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