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Mark S. MacLean & Štefko Miklavič

ABSTRACT Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$, and intersection numbers c_i, b_i ($0 \leq i \leq D$). By a *pseudo cosine sequence* of Γ we mean a sequence of complex scalars $\sigma_0, \sigma_1, \dots, \sigma_D$ such that $\sigma_0 = 1$ and $c_i\sigma_{i-1} + b_i\sigma_{i+1} = k\sigma_i$ for $1 \leq i \leq D-1$. By an *associated pseudo primitive idempotent* of Γ , we mean a nonzero scalar multiple of the matrix $\sum_{i=0}^D \sigma_i A_i$, where A_0, A_1, \dots, A_D are the distance matrices of Γ . Given pseudo primitive idempotents E, F of Γ , we define the pair E, F to be *taut* whenever the entry-wise product $E \circ F$ is not a scalar multiple of a pseudo primitive idempotent, but is a linear combination of two pseudo primitive idempotents of Γ . In this paper, we determine all the taut pairs of pseudo primitive idempotents of Γ .

1. INTRODUCTION

Let Γ denote a distance-regular graph with diameter $D \geq 4$ and Bose–Mesner algebra M . In [14], Terwilliger and Weng introduced the notion of a *pseudo primitive idempotent*. Given any complex number θ , a nonzero matrix $E \in M$ is said to be a *pseudo primitive idempotent* of Γ for θ whenever $(A - \theta I)E$ is a scalar multiple of A_D , where A is the adjacency matrix and A_D is the D th distance matrix of Γ . We call θ the associated *pseudo eigenvalue* of E . By the associated *pseudo cosine sequence*, we mean a sequence of scalars $\sigma_0, \sigma_1, \dots, \sigma_D$ such that $\sigma_0 = 1$ and

$$(1) \quad c_i\sigma_{i-1} + a_i\sigma_i + b_i\sigma_{i+1} = \theta\sigma_i \quad (0 \leq i \leq D-1),$$

where σ_{-1} is indeterminate.

There have been a number of papers concerning pseudo primitive idempotents and their cosine sequences [5, 6, 7, 12, 14]. Given a pair of pseudo primitive idempotents E, F , in [12], Pascasio and Terwilliger defined the pair to be *tight* whenever the entry-wise product $E \circ F$ is a scalar multiple of a single pseudo primitive idempotent. In their paper they determine all the tight pairs of pseudo primitive idempotents whose pseudo eigenvalues are real.

It is natural to consider the case where the entry-wise product $E \circ F$ is not a scalar multiple of a pseudo primitive idempotent, but instead is a linear combination of two pseudo primitive idempotents. We define such a pair E, F to be *taut*. To make this

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problem manageable, we restrict our attention to bipartite distance-regular graphs. In [11], MacLean and Terwilliger posed the following two problems.

PROBLEM 1.1. ([11, Problem 17.8]) *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$. Find all the taut pairs of pseudo primitive idempotents of Γ .*

PROBLEM 1.2. ([11, Conjecture 17.9]) *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$, and fix vertex x . Let T denote the Terwilliger algebra of Γ with respect to x . Assume that up to isomorphism there exist exactly two irreducible T -modules with endpoint 2, and they are both thin. Let E, F denote pseudo primitive idempotents associated with these modules. We conjecture the pair E, F is taut.*

For a bipartite distance-regular graph with valency k , a pseudo eigenvalue θ is said to be *trivial* whenever $\theta = \pm k$. A pseudo primitive idempotent or pseudo cosine sequence is said to be trivial whenever its associated pseudo eigenvalue is trivial. With this definition, we note Lang has already made some progress in attacking Problems 1.1, 1.2. Regarding Problem 1.1, see Theorem 5.4 for his main result. Regarding Problem 1.2, Lang proved in [7] that if Γ is Q -polynomial with $D \geq 5$ and exactly two non-isomorphic irreducible T -modules with endpoint 2, then the pair of pseudo primitive idempotents associated with the local eigenvalues of these modules do indeed form a taut pair. Lang also provides a feasible intersection array with diameter 4 for which the conjecture in Problem 1.2 is *not* satisfied. However, at this time no distance-regular graph is known to exist with this intersection array.

In this paper we solve Problem 1.1. Our results are organized as follows. Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$, and define a parameter Δ_2 in terms of the intersection numbers by

$$\Delta_2 = (k - 2)(c_3 - 1) - (c_2 - 1)p_{22}^2.$$

Let E, F denote nontrivial pseudo primitive idempotents with associated pseudo eigenvalues θ, θ' , respectively. We determine all taut pairs E, F in the case $D = 4$ in Theorem 4.9. Now assume $D \geq 5$. In Section 5, we determine all taut pairs E, F for which $\theta = \pm\theta'$. Now assume E, F is a taut pair for which $\theta \neq \pm\theta'$. If one of θ, θ' is zero, this taut pair is described in Lemma 7.3. Now assume θ, θ' are both nonzero. If $\Delta_2 = 0$, this taut pair is described in Section 8. If $\Delta_2 \neq 0$, this taut pair is described in Section 9.

We are very interested in seeing a solution to Problem 1.2 in the future, since we are interested in classifying the bipartite distance-regular graphs with exactly two irreducible T -modules of endpoint 2, both of which are thin.

2. PRELIMINARIES

In this section we review some definitions and basic results concerning distance-regular graphs. See the book of Brouwer, Cohen and Neumaier [2] for more background information.

Let \mathbb{C} denote the complex number field and let X denote a nonempty finite set. Let $\text{Mat}_X(\mathbb{C})$ denote the \mathbb{C} -algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in \mathbb{C} . Let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set \mathcal{R} . Let ∂ denote the path-length distance function for Γ , and set $D := \max\{\partial(x, y) \mid x, y \in X\}$. We call D the *diameter* of Γ . For a vertex $x \in X$ and an integer i let $\Gamma_i(x)$ denote the set of vertices at distance i from x . We abbreviate $\Gamma(x) := \Gamma_1(x)$. For an integer $k \geq 0$ we say Γ is *regular with valency k* whenever $|\Gamma(x)| = k$ for all $x \in X$. We say Γ

is distance-regular whenever for all integers h, i, j ($0 \leq h, i, j \leq D$) and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p_{ij}^h := |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of x and y . The p_{ij}^h are called the *intersection numbers* of Γ .

For the rest of this paper we assume Γ is distance-regular with diameter $D \geq 4$ and valency $k \geq 3$. Note that $p_{ij}^h = p_{ji}^h$ for $0 \leq h, i, j \leq D$. For convenience set $c_i := p_{1,i-1}^i$ ($1 \leq i \leq D$), $a_i := p_{1i}^i$ ($0 \leq i \leq D$), $b_i := p_{1,i+1}^i$ ($0 \leq i \leq D-1$), $k_i := p_{ii}^0$ ($0 \leq i \leq D$), and $c_0 = b_D = 0$. By the triangle inequality the following hold for $0 \leq h, i, j \leq D$:

- (i) $p_{ij}^h = 0$ if one of h, i, j is greater than the sum of the other two;
- (ii) $p_{ij}^h \neq 0$ if one of h, i, j equals the sum of the other two.

In particular $c_i \neq 0$ for $1 \leq i \leq D$ and $b_i \neq 0$ for $0 \leq i \leq D-1$. We observe that Γ is regular with valency $k = k_1 = b_0$ and that

$$(2) \quad c_i + a_i + b_i = k \quad (0 \leq i \leq D).$$

Note that $k_i = |\Gamma_i(x)|$ for $x \in X$ and $0 \leq i \leq D$. By [2, p. 127],

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \quad (0 \leq i \leq D).$$

We recall the Bose–Mesner algebra of Γ . For $0 \leq i \leq D$ let A_i denote the matrix in $\text{Mat}_X(\mathbb{C})$ with (x, y) -entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \quad (x, y \in X).$$

We call A_i the i th *distance matrix* of Γ . We abbreviate $A := A_1$ and call this the *adjacency matrix* of Γ . We observe

- (ai) $A_0 = I$;
- (aii) $\sum_{i=0}^D A_i = J$;
- (aiii) $\overline{A_i} = A_i$ ($0 \leq i \leq D$);
- (aiv) $A_i^t = A_i$ ($0 \leq i \leq D$);
- (av) $A_i A_j = \sum_{h=0}^D p_{ij}^h A_h$ ($0 \leq i, j \leq D$),

where I (resp. J) denotes the identity matrix (resp. all 1's matrix) in $\text{Mat}_X(\mathbb{C})$. Using these facts we find A_0, A_1, \dots, A_D is a basis for a commutative subalgebra M of $\text{Mat}_X(\mathbb{C})$. We call M the *Bose–Mesner algebra* of Γ . It turns out that A generates M [1, p. 190]. Since each entry in the distance matrices is either 0 or 1, we have

$$A_i \circ A_j = \delta_{ij} A_i \quad (0 \leq i, j \leq D),$$

where \circ denotes the entry-wise matrix product. It follows that M is closed under \circ .

By [2, p. 45], M has a second basis E_0, E_1, \dots, E_D such that

- (ei) $E_0 = |X|^{-1} J$;
- (eii) $\sum_{i=0}^D E_i = I$;
- (eiii) $\overline{E_i} = E_i$ ($0 \leq i \leq D$);
- (eiv) $E_i^t = E_i$ ($0 \leq i \leq D$);
- (ev) $E_i E_j = \delta_{ij} E_i$ ($0 \leq i, j \leq D$).

We call E_0, E_1, \dots, E_D the *primitive idempotents* of Γ . For $0 \leq i \leq D$ define scalars θ_i by

$$A = \sum_{i=0}^D \theta_i E_i.$$

Then $AE_i = E_iA = \theta_i E_i$ for $0 \leq i \leq D$, and $\theta_0 = k$. The scalars $\theta_0, \theta_1, \dots, \theta_D$ are real and distinct, since A generates M [1, p. 197]. The scalars $\theta_0, \theta_1, \dots, \theta_D$ are known as the *eigenvalues* of Γ . We say that primitive idempotent E_i is *associated* with the eigenvalue θ_i ($0 \leq i \leq D$). For each integer i ($0 \leq i \leq D$), let m_i denote the rank of E_i ; we refer to m_i as the *multiplicity* of E_i .

Let E denote a primitive idempotent of Γ , and let m denote the multiplicity of E . By [1, §II.3], there exist real scalars $\sigma_0, \sigma_1, \dots, \sigma_D$ such that $\sigma_0 = 1$ and

$$E = |X|^{-1}m \sum_{i=0}^D \sigma_i A_i.$$

The sequence $\sigma_0, \sigma_1, \dots, \sigma_D$ is called the *cosine sequence* of Γ associated with E . We abbreviate $\sigma := \sigma_1$. By [2, Proposition 4.1.1], the following are equivalent:

- (i) θ is an eigenvalue of Γ and $\sigma_0, \sigma_1, \dots, \sigma_D$ is the associated cosine sequence;
- (ii) $\sigma_0 = 1$ and $c_i \sigma_{i-1} + a_i \sigma_i + b_i \sigma_{i+1} = \theta \sigma_i$ ($0 \leq i \leq D$),

where σ_{-1} and σ_{D+1} are indeterminates.

Recall Γ is *bipartite* whenever $a_i = 0$ for $0 \leq i \leq D$. Setting $a_i = 0$ in (2) we find

$$(3) \quad b_i + c_i = k \quad (0 \leq i \leq D).$$

Assume Γ is bipartite with diameter $D \geq 3$. By [2, Lemma 4.1.7], we have

$$(4) \quad p_{2i}^i = \frac{b_i(c_{i+1} - 1) + c_i(b_{i-1} - 1)}{c_2} \quad (1 \leq i \leq D - 1).$$

DEFINITION 2.1. Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. We define scalars Δ_i by

$$(5) \quad \Delta_i = (b_{i-1} - 1)(c_{i+1} - 1) - (c_2 - 1)p_{2i}^i \quad (2 \leq i \leq D - 1).$$

By [3, Theorem 12], we have $\Delta_i \geq 0$ for $2 \leq i \leq D - 1$. We note that $\Delta_i = 0$ for $2 \leq i \leq D - 2$ precisely when Γ is almost 2-homogeneous in the sense of Curtin [4].

3. PSEUDO PRIMITIVE IDEMPOTENTS AND PSEUDO COSINE SEQUENCES

Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. In this section we define pseudo eigenvalues and pseudo primitive idempotents of Γ and review some of their properties.

DEFINITION 3.1. Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let θ denote any complex number; we call θ a pseudo eigenvalue. By the pseudo cosine sequence associated with θ , we mean the sequence of complex numbers $\sigma_0, \sigma_1, \dots, \sigma_D$ such that $\sigma_0 = 1$ and

$$(6) \quad c_i \sigma_{i-1} + b_i \sigma_{i+1} = \theta \sigma_i \quad (0 \leq i \leq D - 1),$$

where σ_{-1} is an indeterminate. We abbreviate $\sigma := \sigma_1$. By a pseudo primitive idempotent associated with θ , we mean a nonzero scalar multiple of the matrix

$$(7) \quad \sum_{i=0}^D \sigma_i A_i.$$

With reference to Definition 3.1, we remark that $\theta \in \mathbb{R}$ is an eigenvalue of Γ if and only if its associated pseudo cosine sequence satisfies $c_D \sigma_{D-1} = \theta \sigma_D$.

REMARK 3.2. We remark that in the literature about pseudo primitive idempotents also the case $\theta = \infty$ is often considered. We will not consider this case here, as this situation was studied in [7] (see [7, Lemma 4.1, Lemma 4.2]).

LEMMA 3.3 ([7, Lemma 3.3]). Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ denote a pseudo cosine sequence of Γ with associated pseudo eigenvalue θ . Then for $0 \leq i \leq 4$, $\sigma_i = f_i(\theta)$, where

$$f_0(\lambda) = 1, \quad f_1(\lambda) = \frac{\lambda}{k}, \quad f_2(\lambda) = \frac{\lambda^2 - k}{kb_1},$$

$$f_3(\lambda) = \frac{\lambda^3 - (k + c_2b_1)\lambda}{kb_1b_2}, \quad f_4(\lambda) = \frac{\lambda^4 - (k + c_2b_1 + c_3b_2)\lambda^2 + c_3kb_2}{kb_1b_2b_3}.$$

LEMMA 3.4 (Christoffel–Darboux formula [1, Theorem III.1.3]). Let Γ denote a distance-regular graph with diameter $D \geq 4$. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ denote pseudo cosine sequences of Γ . Then

$$(8) \quad k(\sigma - \rho) \sum_{h=0}^i k_h \sigma_h \rho_h = k_i b_i (\sigma_{i+1} \rho_i - \sigma_i \rho_{i+1}) \quad (0 \leq i \leq D - 1).$$

Proof. See [12, Lemma 3.2]. We note that the proof in [12] assumes each $\sigma_i, \rho_i \in \mathbb{R}$, but their proof also works when $\sigma_i, \rho_i \in \mathbb{C}$. \square

For bipartite distance-regular graphs, we now obtain some other formulas similar to the Christoffel–Darboux equations. We will need the following two corollaries.

COROLLARY 3.5 ([12, Lemmas 3.3, 3.4]). Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ denote complex numbers. Then the following (i)–(iv) are equivalent.

- (i) $\sigma_0, \sigma_1, \dots, \sigma_D$ is a pseudo cosine sequence.
- (ii) $\sigma_0 = 1$ and

$$(9) \quad c_i(\sigma_{i-1} - \sigma_{i+1}) = k(\sigma\sigma_i - \sigma_{i+1}) \quad (1 \leq i \leq D - 1).$$

- (iii) $\sigma_0 = 1$ and

$$(10) \quad b_i(\sigma_{i+1} - \sigma_{i-1}) = k(\sigma\sigma_i - \sigma_{i-1}) \quad (1 \leq i \leq D - 1).$$

- (iv) $\sigma_0 = 1$ and

$$(11) \quad k(\sigma - 1) \sum_{h=0}^i k_h \sigma_h = k_i b_i (\sigma_{i+1} - \sigma_i) \quad (0 \leq i \leq D - 1).$$

Proof. We note that the proof in [12] assumes each $\sigma_i \in \mathbb{R}$, but their proof also works for $\sigma_i \in \mathbb{C}$. \square

COROLLARY 3.6. Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$, and let $\sigma_0, \sigma_1, \dots, \sigma_D$ denote a pseudo cosine sequence of Γ . Then for any integer i ($1 \leq i \leq D - 1$), the following are equivalent.

- (i) $\sigma_{i+1} = \sigma_{i-1}$.
- (ii) $\sigma\sigma_i = \sigma_{i-1}$.
- (iii) $\sigma\sigma_i = \sigma_{i+1}$.

LEMMA 3.7. Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ denote pseudo cosine sequences of Γ . For $1 \leq i \leq D - 1$,

$$(12) \quad k^2(\sigma^2 - \rho^2) \sum_{\substack{0 \leq h \leq i-1 \\ i-h \text{ odd}}} k_h \sigma_h \rho_h = k_i c_i b_i (\sigma_{i+1} \rho_{i-1} - \sigma_{i-1} \rho_{i+1}).$$

Proof. Similar to the proof of [8, Lemma 3.3]. \square

COROLLARY 3.8. *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ denote a pseudo cosine sequence of Γ . For $1 \leq i \leq D - 1$,*

$$(13) \quad k^2(\sigma^2 - 1) \sum_{\substack{0 \leq h \leq i-1 \\ i-h \text{ odd}}} k_h \sigma_h = k_i c_i b_i (\sigma_{i+1} - \sigma_{i-1}).$$

Proof. Similar to the proof of [8, Corollary 3.4]. □

4. TIGHT PAIRS AND TAUT PAIRS OF PSEUDO PRIMITIVE IDEMPOTENTS

Let Γ denote a distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. In this section we recall tight pairs and taut pairs of pseudo primitive idempotents of Γ .

DEFINITION 4.1. *Let Γ denote a distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let E, F denote pseudo primitive idempotents for Γ . We say the pair E, F is tight whenever the entry-wise product $E \circ F$ is a scalar multiple of a pseudo primitive idempotent of Γ . We say a pair of pseudo cosine sequences (or pseudo eigenvalues) is tight whenever its pair of associated pseudo primitive idempotents is tight.*

In [12], Pascasio and Terwilliger determine all tight pairs of pseudo primitive idempotents whose pseudo eigenvalues are real. In the following definition we consider pairs of pseudo primitive idempotents whose entry-wise product is a linear combination of exactly two pseudo primitive idempotents.

DEFINITION 4.2. *Let Γ denote a distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let E, F denote pseudo primitive idempotents for Γ . We say the pair E, F is taut whenever the pair E, F is not tight and the entry-wise product $E \circ F$ is a linear combination of two pseudo primitive idempotents of Γ . We say a pair of pseudo cosine sequences (or pseudo eigenvalues) is taut whenever its pair of associated pseudo primitive idempotents is taut. We note that if E, F is taut, then any pair of nonzero scalar multiples of E, F is taut.*

We shall restrict our attention to bipartite distance-regular graphs. We first remind the reader of the possible tight pairs of pseudo primitive eigenvalues for such graphs.

THEOREM 4.3 ([7, Lemmas 4.1, 4.2]). *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Then (i)–(iii) hold below.*

- (i) *The pair θ, k is tight for all $\theta \in \mathbb{C}$.*
- (ii) *The pair $\theta, -k$ is tight for all $\theta \in \mathbb{C}$.*
- (iii) *Γ has no further tight pairs.*

DEFINITION 4.4. *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let E denote a pseudo primitive idempotent with associated pseudo eigenvalue θ and pseudo cosine sequence $\sigma_0, \sigma_1, \dots, \sigma_D$. We say θ is trivial whenever $\theta \in \{-k, k\}$. We note that θ is trivial if and only if $\sigma \in \{-1, 1\}$. We say a pseudo primitive idempotent or pseudo cosine sequence is trivial whenever its associated pseudo eigenvalue is trivial.*

Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let E, F denote pseudo primitive idempotents with associated pseudo eigenvalues θ, θ' . In the results that follow, we will typically assume E, F are nontrivial in order to establish that the pair is not tight. Now assume the pair E, F is taut. In this section we develop some equations involving θ, θ' . First, we make a definition.

DEFINITION 4.5. *Throughout Sections 4 and 5, let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let $\theta_0 > \theta_1 > \dots > \theta_D$ denote the distinct eigenvalues of Γ , and for $0 \leq i \leq D$, let E_i denote the primitive idempotent*

of Γ associated with θ_i . Furthermore, let E and F denote nontrivial pseudo primitive idempotents of Γ . We let θ, θ' denote the corresponding pseudo eigenvalues, and we let $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ denote the corresponding pseudo cosine sequences. We let G and H denote pseudo primitive idempotents of Γ with cosine sequences $\gamma_0, \gamma_1, \dots, \gamma_D$ and $\epsilon_0, \epsilon_1, \dots, \epsilon_D$, respectively, for which we assume $\gamma \neq \epsilon$.

LEMMA 4.6. *With reference to Definition 4.5, the following (i), (ii) are equivalent.*

- (i) $E \circ F \in \text{span}\{G, H\}$.
- (ii) *There exist complex scalars a, b such that for $0 \leq i \leq D$,*

$$(14) \quad \sigma_i \rho_i = a\gamma_i + b\epsilon_i.$$

Suppose (i), (ii) hold. Then a, b are nonzero and are given by

$$(15) \quad a = \frac{\sigma\rho - \epsilon}{\gamma - \epsilon}, \quad b = \frac{\gamma - \sigma\rho}{\gamma - \epsilon}.$$

Proof. Similar to the proof of [8, Lemma 4.2]. □

LEMMA 4.7. *With reference to Definition 4.5, assume conditions (i), (ii) hold in Lemma 4.6. Then γ, ϵ are both roots of the following polynomial in $\mathbb{C}[x]$:*

$$(16) \quad (k - 1)k^2b_2x^2 - (k - 1)kc_2\theta\theta'x + \theta^2\theta'^2(c_2 - 1) + (\theta^2 + \theta'^2 - k^2)b_2.$$

In particular,

$$(17) \quad \gamma + \epsilon = \frac{c_2\theta\theta'}{kb_2}, \quad \gamma\epsilon = \frac{\theta^2\theta'^2(c_2 - 1) + (\theta^2 + \theta'^2 - k^2)b_2}{(k - 1)k^2b_2}.$$

Proof. Without loss of generality, we show ϵ is a root of (16). We first observe $\sigma\rho \neq \epsilon$, since otherwise $a = 0$, contradicting Lemma 4.6. Setting $i = 2$ in (14) and eliminating $\sigma_2, \rho_2, \gamma_2, \epsilon_2$ using Lemma 3.3, we may solve the resulting equation for γ . Now setting $i = 3$ in (14), applying Lemma 3.3, and replacing γ in the result by the expression just obtained, we find ϵ is a root of (16), as desired. □

THEOREM 4.8. *With reference to Definition 4.5, assume E, F is a taut pair. Then*

$$(18) \quad b_3((k - 2)b_2 - \theta^2(c_2 - 1))((k - 2)b_2 - \theta'^2(c_2 - 1)) = -b_1\Delta_2(\theta^2 - b_2)(\theta'^2 - b_2).$$

Proof. Observe conditions (i), (ii) hold in Lemma 4.6 for some pseudo primitive idempotents G, H . Using (15) and Lemma 3.3, one can show the expression

$$(19) \quad (k - 1)^2k^2b_2^2b_3^2(\sigma_4\rho_4 - a\gamma_4 - b\epsilon_4)$$

is equal to the expression

$$(k - 1)kb_2b_3(A((\gamma + \epsilon)\theta\theta' - k^2\gamma\epsilon) - k^2\theta\theta'(\gamma + \epsilon)((\gamma + \epsilon)^2 - 2\gamma\epsilon) + k^4\gamma\epsilon((\gamma + \epsilon)^2 - \gamma\epsilon) - k^3b_2^2c_3(b_3 - 1) - kb_2c_3(A(\theta^2 + \theta'^2) - \theta^4 - \theta'^4) + \theta^2\theta'^2(A - \theta^2)(A - \theta'^2),$$

where $A = b_2 + kc_2 + c_3b_2$. Replacing $\gamma + \epsilon, \gamma\epsilon$ in this latter expression using (17), and then using (4), (5), one can show this latter expression equals $(k^2 - \theta^2)(k^2 - \theta'^2)c_2(k - 1)^{-1}b_2^{-1}(b_2 - 1)^{-1}$ times the lefthand side of (18) minus the righthand side of (18). Since the expression in (19) equals 0, the result follows. □

With reference to Definition 4.5, our goal is to determine all the taut pairs of pseudo primitive idempotents of Γ . It turns out that the case $D = 4$ is somewhat different than the case $D \geq 5$. Therefore, we treat this case separately.

THEOREM 4.9. *With reference to Definition 4.5, assume that $D = 4$. Then the following (i), (ii) are equivalent.*

- (i) E, F is a taut pair.
- (ii) θ, θ' satisfy (18) and the two roots of (16) are distinct.

Proof.

(i) \Rightarrow (ii). If E, F is a taut pair, then (18) holds by Theorem 4.8. Clearly, the roots of (16) are distinct in this case.

(ii) \Rightarrow (i). Let γ and ϵ be the roots of polynomial (16), and let a, b be as in (15). Using (17), it is straightforward to show that (14) holds for $0 \leq i \leq 3$. It follows from the proof of Theorem 4.8 that since (18) holds we have also $\sigma_4\rho_4 = a\gamma_4 + b\epsilon_4$. The result follows. \square

With reference to Definition 4.5, for the rest of the paper we assume $D \geq 5$.

THEOREM 4.10. *With reference to Definition 4.5, assume that $D \geq 5$ and E, F is taut. Furthermore, assume θ, θ' are nonzero and that $(c_3 - c_2)^2 + (c_4 - c_3)(c_3 - c_2^2) \neq 0$. Then*

$$\begin{aligned} \theta^2\theta'^2 &= \frac{b_2^2(-(c_4 - 1)(b_2 + kc_2) + c_3^2(b_4 - c_2) + c_3(c_2c_4 + 3c_4 + kc_2 - 2k - 2c_2))}{(c_3 - c_2)^2 + (c_4 - c_3)(c_3 - c_2^2)}, \\ \theta^2 + \theta'^2 &= \frac{(b_4 - c_2)(k - 2c_2 + c_2^2)c_3^2 - (c_4 - 1)(b_2^2 + kc_2^2 - c_2^3 + kc_2^3) + c_3c_2^3(k + c_4 - 2)}{(c_3 - c_2)^2 + (c_4 - c_3)(c_3 - c_2^2)} \\ &\quad + \frac{c_3(c_2^2(kc_4 - k^2 - k + 3) + c_2(2k^2 - k - 4c_4 - kc_4) - 2k(k - 2c_4))}{(c_3 - c_2)^2 + (c_4 - c_3)(c_3 - c_2^2)}. \end{aligned}$$

Proof. Without loss of generality, we assume G, H are the pseudo primitive idempotents satisfying the equivalent conditions (i), (ii) in Lemma 4.6. Setting $i = 5$ in (14) and using Lemma 3.3, (6) and (3), we find

$$\begin{aligned} 0 &= b_1b_2b_3b_4[-k^4\theta\theta'(((\gamma + \epsilon)^2 - 2\gamma\epsilon)^2 - 2(\gamma\epsilon)^2) + k^6(\gamma\epsilon)^3 + k^6\gamma\epsilon(\gamma + \epsilon)((\gamma + \epsilon)^2 - 3\gamma\epsilon) \\ &\quad - k^4\theta\theta'\gamma\epsilon((\gamma + \epsilon)^2 - 2\gamma\epsilon) - k^2\gamma\epsilon(k^4((\gamma\epsilon)^2 - \gamma\epsilon(\gamma + \epsilon)) + k^2\theta\theta'\gamma\epsilon) \\ &\quad + (c_4b_3 + c_3b_2 + c_2b_1 + k)(k^2\theta\theta'((\gamma + \epsilon)^2 - 2\gamma\epsilon) - k^2\gamma\epsilon(k^2(\gamma + \epsilon) - \theta\theta'))] + C, \end{aligned}$$

where C is the constant term with no factors of γ, ϵ and which is easily isolated using computer algebra software (but which is too lengthy to include here). Now using (17), once we divide by a factor of $\theta\theta'(k^2 - \theta^2)(k^2 - \theta'^2)$, we get a linear equation in the expressions $\theta^2\theta'^2$ and $\theta^2 + \theta'^2$. Now multiplying out (18), we obtain another linear equation in the expressions $\theta^2\theta'^2$ and $\theta^2 + \theta'^2$. Solving this linear system, we get the desired result. \square

LEMMA 4.11. *With reference to Definition 4.5, assume that $D \geq 5$ and that the pair θ, θ' is taut. Furthermore, assume θ, θ' are nonzero. Then the following (i), (ii) are equivalent.*

- (i) $\Delta_3 = \Delta_2 = 0$.
- (ii) $(c_3 - c_2)^2 + (c_4 - c_3)(c_3 - c_2^2) = 0$.

Proof.

(i) \Rightarrow (ii). First, we show a couple of potential denominators are nonzero. Using [2, Proposition 4.1.6], we observe $(b_2 - c_3) + c_2(c_3 - c_2) \geq 0$. Suppose $(b_2 - c_3) + c_2(c_3 - c_2) = 0$. Then $b_2 = c_3 = c_2$, and so $c_2 = k/2$. Since $\Delta_2 = 0$, using (5) we find $k = 2$, a contradiction. Hence $(b_2 - c_3) + c_2(c_3 - c_2) > 0$. It's routine to show $k - 3c_2 + c_2^2 > 0$. Now using (4) and (5), we may solve for c_3, c_4 in terms of k, c_2 . It is then routine to check that (ii) holds.

(ii) \Rightarrow (i). Let N denote the numerator of the expression for $\theta^2\theta'^2$ in Theorem 4.10. By the proof of that theorem, since $(c_3 - c_2)^2 + (c_4 - c_3)(c_3 - c_2^2) = 0$, we also have $N = 0$. It's routine to show that

$$(20) \quad b_2^2(c_3 - 1)c_2\Delta_3 = (k - 2)b_2^2((c_3 - c_2)^2 + (c_4 - c_3)(c_3 - c_2^2)) - N(c_2 - 1).$$

It follows that either $\Delta_3 = 0$ or $c_3 = 1$. If $\Delta_3 = 0$, we are done by [9, Lemma 3.7]. Suppose $c_3 = 1$, so thus $c_2 = 1$. Using this information and the fact that $N = 0$, we find $c_4 = 1$. Hence $\Delta_3 = 0$ by (5). \square

5. THE CASE WHERE $\theta = \pm\theta'$

We say pseudo primitive idempotents E, F are *equivalent* whenever $\theta = \theta'$, where θ, θ' are the pseudo eigenvalues associated with E, F , respectively. In this case, E, F are scalar multiples of each other. We say E, F are *opposites* whenever $\theta = -\theta'$. With reference to Definition 4.5, suppose for the moment that E, F, G, H satisfy conditions (i), (ii) in Lemma 4.6. In this section, we consider the case when E, F are equivalent or opposites. We will need the following lemmas.

LEMMA 5.1. *With reference to Definition 4.5, suppose E, F are opposites. Then $\rho_i = (-1)^i\sigma_i$ ($0 \leq i \leq D$).*

Proof. Routine using (6) and induction. \square

LEMMA 5.2. *With reference to Definition 4.5, for any pseudo primitive idempotent K , let \hat{K} denote an opposite of K . Suppose $E \circ F \in \text{span}\{G, H\}$. Then $\hat{E} \circ \hat{F} \in \text{span}\{G, H\}$ and $E \circ \hat{F} \in \text{span}\{\hat{G}, \hat{H}\}$.*

Proof. Routine using Lemma 4.6 and Lemma 5.1. \square

In the following lemma, we recall this situation in the case when E, F, G, H are actual primitive idempotents, not just pseudo primitive idempotents.

LEMMA 5.3 ([8, Lemma 4.3]). *With reference to Definition 4.5, suppose conditions (i), (ii) hold in Lemma 4.6, and suppose E, F, G, H are primitive idempotents of Γ (not just pseudo idempotents). Then*

- (i) E, F are equivalent if and only if one of G, H is equal to E_0 .
- (ii) E, F are opposites if and only if one of G, H is equal to E_D .

Furthermore, suppose $E = F$. Then Γ is 2-homogeneous, and $E \in \{E_1, E_{D-1}\}$. Now suppose E, F are opposites. Then Γ is 2-homogeneous, and E, F is a permutation of E_1, E_{D-1} .

In [6], Lang has worked out part of this case when E, F are pseudo primitive idempotents.

THEOREM 5.4 ([6, Theorem 1.1]). *With reference to Definition 4.5, the following (i), (ii) are equivalent.*

- (i) *The entrywise product of E with itself is a linear combination of E_0 and a pseudo primitive idempotent of Γ .*
- (ii) *There exists a scalar λ such that $\sigma_{i-1} + \lambda\sigma_i + \sigma_{i+1} = 0$ for $1 \leq i \leq D - 1$.*

Moreover, Γ has such a pseudo cosine sequence and pseudo primitive idempotent if and only if Γ is almost 2-homogeneous with $c_2 \geq 2$.

The following result will be useful.

LEMMA 5.5. *With reference to Definition 4.5, assume that $D \geq 5$. Furthermore, assume that conditions (i), (ii) hold in Lemma 4.6, and that E, F are either equivalent or opposites. Then $\Delta_2 = 0$.*

Proof. First, observe θ, θ' are nonzero, since if $\theta = \theta' = 0$, we quickly reach a contradiction using (18). Now observe that if $(c_3 - c_2)^2 + (c_4 - c_3)(c_3 - c_2^2) = 0$, then $\Delta_2 = 0$ by Lemma 4.11 and we are done. Therefore, we assume $(c_3 - c_2)^2 + (c_4 - c_3)(c_3 - c_2^2) \neq 0$. Let $\theta = x + iy$ for some real numbers x, y and let $\theta' = \pm\theta$. Then we have that

$$(21) \quad \theta^2 + \theta'^2 = 2(x^2 - y^2) + 4ixy.$$

Note that by Theorem 4.10 we have that (21) is real. It follows that either $x = 0$ or $y = 0$. Consequently, θ^2 is real.

Setting $\theta' = \pm\theta$ in (18), we find

$$b_3((k - 2)b_2 - \theta^2(c_2 - 1))^2 = -b_1\Delta_2(\theta^2 - b_2)^2.$$

As θ^2 is real, the expression on the left above is nonnegative, and the expression on the right is nonpositive. Hence both sides must equal 0, and so we find $\Delta_2 = 0$. \square

THEOREM 5.6. *With reference to Definition 4.5, assume that $D \geq 5$ and that conditions (i), (ii) hold in Lemma 4.6. Furthermore, assume E, F are either equivalent or opposites. Then $c_2 > 1$, and $\theta^2 = \theta'^2 = (k - 2)b_2(c_2 - 1)^{-1}$. Furthermore, (i), (ii) hold below.*

- (i) E, F are equivalent if and only if one of G, H is a scalar multiple of E_0 , and the other has associated pseudo eigenvalue $(k - 2c_2)(c_2 - 1)^{-1}$.
- (ii) E, F are opposites if and only if one of G, H is a scalar multiple of E_D , and the other has associated pseudo eigenvalue $-(k - 2c_2)(c_2 - 1)^{-1}$.

Proof. Setting $\theta' = \pm\theta$ in (18), we find

$$b_3((k - 2)b_2 - \theta^2(c_2 - 1))^2 = -b_1\Delta_2(\theta^2 - b_2)^2.$$

As $\Delta_2 = 0$ by Lemma 5.5, we have that $(k - 2)b_2 = \theta^2(c_2 - 1)$. Note that $c_2 = 1$ forces $k = 2$, a contradiction. This shows that $\theta^2 = \theta'^2 = (k - 2)b_2(c_2 - 1)^{-1}$.

(i) Assume first that E, F are equivalent. Setting $\theta' = \theta = \pm\sqrt{(k - 2)b_2(c_2 - 1)^{-1}}$ in (16) and factoring, we find $\{\gamma, \epsilon\} = \{1, \frac{k - 2c_2}{k(c_2 - 1)}\}$. The result follows. Without loss of generality, assume now that G is a scalar multiple of E_0 and that H has associated pseudo eigenvalue $(k - 2c_2)(c_2 - 1)^{-1}$. Note that $\epsilon = (k - 2c_2)(k(c_2 - 1))^{-1}$, and so it follows from the equation on the left in (17) that $\theta\theta' = (k - 2)b_2/(c_2 - 1)^{-1}$. Plugging this and $x = \gamma = 1$ into (16) we find

$$\theta^2 + \theta'^2 = \frac{2(k - 2)b_2}{c_2 - 1} = 2\theta\theta'.$$

It follows that $\theta = \theta'$, and so E, F are equivalent.

- (ii) Similar to the proof of (i) above. \square

COROLLARY 5.7. *With reference to Definition 4.5, assume that $D \geq 5$. Then Γ has a taut pair of pseudo primitive idempotents that are equivalent or opposites if and only if Γ is almost 2-homogeneous with $c_2 \geq 2$.*

Proof. Follows from Theorems 5.4, 5.6, 5.5 and Lemma 5.2. \square

6. EQUATIONS INVOLVING COSINE SEQUENCES

With reference to Definition 4.5, in the previous section we determined all taut pairs E, F in which E, F are equivalent or opposites. In the sections that follow we will determine all other taut pairs of primitive idempotents. To rule out this special case, we make the following definition.

DEFINITION 6.1. Let Γ denote a bipartite distance-regular graph with diameter $D \geq 5$ and valency $k \geq 3$. Let E and F denote nontrivial pseudo primitive idempotents of Γ such that E, F are neither equivalent nor opposites. We let θ, θ' denote the corresponding pseudo eigenvalues, and we let $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ denote the corresponding pseudo cosine sequences.

LEMMA 6.2. With reference to Definition 6.1, let G, H denote nontrivial pseudo primitive idempotents with corresponding pseudo cosine sequences $\gamma_0, \gamma_1, \dots, \gamma_D$ and $\epsilon_0, \epsilon_1, \dots, \epsilon_D$, respectively. Fix an integer i ($0 \leq i \leq D - 1$) and suppose (14) holds for all nonnegative integers less than or equal to i , where a, b are from (15). Then

$$(22) \quad \frac{\sigma_{i+1}\rho_i - \sigma_i\rho_{i+1}}{\sigma - \rho} = a \frac{\gamma_{i+1} - \gamma_i}{\gamma - 1} + b \frac{\epsilon_{i+1} - \epsilon_i}{\epsilon - 1}.$$

Proof. Similar to the proof of [8, Lemma 4.4], using (8), (11). □

LEMMA 6.3. With reference to Definition 6.1, let G, H denote nontrivial pseudo primitive idempotents with corresponding pseudo cosine sequences $\gamma_0, \gamma_1, \dots, \gamma_D$ and $\epsilon_0, \epsilon_1, \dots, \epsilon_D$, respectively. Fix an integer i ($1 \leq i \leq D - 1$) and suppose (14) holds for all nonnegative integers less than i , where a, b are from (15). Then

$$(23) \quad \frac{\sigma_{i+1}\rho_{i-1} - \sigma_{i-1}\rho_{i+1}}{\sigma^2 - \rho^2} = a \frac{\gamma_{i+1} - \gamma_{i-1}}{\gamma^2 - 1} + b \frac{\epsilon_{i+1} - \epsilon_{i-1}}{\epsilon^2 - 1},$$

where a, b are from (15).

Proof. Similar to the proof of [8, Lemma 4.5], using (12), (13). □

With reference to Definition 6.1, in the rest of this section, we consider the equation

$$(24) \quad \sigma_{i+1}\rho_{i+1} - \sigma_{i-1}\rho_{i-1} = \alpha\sigma_i(\rho_{i+1} - \rho_{i-1}) + \beta\rho_i(\sigma_{i+1} - \sigma_{i-1}),$$

where

$$(25) \quad \alpha = \frac{\theta}{k} + \frac{\theta(k^2 - \theta^2)(b_2(k - 2) - \theta'^2(c_2 - 1))}{k(\theta^2 - \theta'^2)b_1b_2},$$

$$(26) \quad \beta = \frac{\theta'}{k} + \frac{\theta'(k^2 - \theta'^2)(b_2(k - 2) - \theta^2(c_2 - 1))}{k(\theta'^2 - \theta^2)b_1b_2}.$$

We shall see that this equation holds for $1 \leq i \leq D - 1$ whenever $E \circ F$ is a linear combination of pseudo primitive idempotents G, H . We have a comment. Observe that if equation (24) holds for $1 \leq i \leq D - 1$ for pair E, F , then it holds also for $1 \leq i \leq D - 1$ for pair E, \hat{F} . To see this, change β with $-\beta$ and use Lemma 5.1. We now need two lemmas addressing the situation when one of G, H is trivial.

LEMMA 6.4. With reference to Definition 6.1, assume (24) holds for $1 \leq i \leq D - 1$. If 1 is a root of the polynomial in (16), then for $1 \leq i \leq D$,

$$(27) \quad 0 = A(\sigma_i\rho_i + \sigma_{i-1}\rho_{i-1}) + B(\sigma_i\rho_{i-1} + \rho_i\sigma_{i-1}) + C,$$

where

$$\begin{aligned} A &= (k - 1)k^2(\theta + \theta')b_2, \\ B &= -\theta\theta'k^2(k - 1)c_2, \\ C &= -(k - 1)(\theta + \theta')(k^2b_2 + \theta\theta'(b_2 - kc_2)). \end{aligned}$$

Proof. By our assumptions, we first note $Q = 0$, where

$$(28) \quad Q = (k - 1)k^2b_2 - (k - 1)kc_2\theta\theta' + \theta^2\theta'^2(c_2 - 1) + (\theta^2 + \theta'^2 - k^2)b_2.$$

We proceed by induction on i . It is routine to show that equality holds in (27) for $i = 1$. Now suppose $i \geq 2$ and that

$$(29) \quad 0 = A(\sigma_{i-1}\rho_{i-1} + \sigma_{i-2}\rho_{i-2}) + B(\sigma_{i-1}\rho_{i-2} + \rho_{i-1}\sigma_{i-2}) + C.$$

We will show equality holds in (27). Define scalars α, β as in (25), (26). Using (29), (24), and the fact that $Q = 0$, we observe

$$\begin{aligned} & A(\sigma_i\rho_i + \sigma_{i-1}\rho_{i-1}) + B(\sigma_i\rho_{i-1} + \rho_i\sigma_{i-1}) + C \\ &= A(\sigma_i\rho_i + \sigma_{i-1}\rho_{i-1}) + B(\sigma_i\rho_{i-1} + \rho_i\sigma_{i-1}) + C \\ &\quad - (A(\sigma_{i-1}\rho_{i-1} + \sigma_{i-2}\rho_{i-2}) + B(\sigma_{i-1}\rho_{i-2} + \rho_{i-1}\sigma_{i-2}) + C) \\ &= A(\sigma_i\rho_i - \sigma_{i-2}\rho_{i-2}) + B(\sigma_{i-1}(\rho_i - \rho_{i-2}) + \rho_{i-1}(\sigma_i - \sigma_{i-2})) \\ &= A(\alpha\sigma_{i-1}(\rho_i - \rho_{i-2}) + \beta\rho_{i-1}(\sigma_i - \sigma_{i-2})) \\ &\quad + B(\sigma_{i-1}(\rho_i - \rho_{i-2}) + \rho_{i-1}(\sigma_i - \sigma_{i-2})) \\ &= (A\alpha + B)\sigma_{i-1}(\rho_i - \rho_{i-2}) + (A\beta + B)\rho_{i-1}(\sigma_i - \sigma_{i-2}) \\ &= \left(\frac{-k\theta Q}{\theta' - \theta}\right)\sigma_{i-1}(\rho_i - \rho_{i-2}) + \left(\frac{k\theta' Q}{\theta' - \theta}\right)\rho_{i-1}(\sigma_i - \sigma_{i-2}) \\ &= 0. \end{aligned} \quad \square$$

LEMMA 6.5. *With reference to Definition 6.1, assume 1 or -1 is a root of (16). Then the following (i), (ii) are equivalent.*

- (i) E, F is a taut pair.
- (ii) The polynomial in (16) has distinct roots, and there exist complex scalars α, β such that equality holds in (24) for all integers i ($1 \leq i \leq D - 1$).

Proof.

(i) \Rightarrow (ii). Let G, H denote pseudo primitive idempotents such that $E \circ F \in \text{span}\{G, H\}$, and let $\gamma_0, \gamma_1, \dots, \gamma_D$ and $\epsilon_0, \epsilon_1, \dots, \epsilon_D$ denote the pseudo cosine sequences associated with G and H , respectively. Note that $\gamma \neq \epsilon$ (as E, F is a taut pair) and that γ, ϵ are roots of (16).

First, assume 1 is a root of (16). Without loss of generality, we may assume $\epsilon = 1$ and $\gamma = c_2\theta\theta'(kb_2)^{-1} - 1$. Thus $\epsilon_i = 1$ ($0 \leq i \leq D$). Observe $Q = 0$, where Q is the expression in (28). We now proceed by induction. Let α, β be defined as in (25), (26). It's routine to check that (24) holds for $i = 1$. Now suppose $i \geq 2$ and that (24) holds for all positive integers less than i . Following the proof and notation of Lemma 6.4, we find

$$(30) \quad A(\sigma_i\rho_i + \sigma_{i-1}\rho_{i-1}) + B(\sigma_i\rho_{i-1} + \rho_i\sigma_{i-1}) + C = 0.$$

By this fact and since $Q = 0$, we find the expression

$$(31) \quad A(\theta' - \theta)(\sigma_i\rho_i + \sigma_{i-1}\rho_{i-1}) + \rho_i\sigma_{i-1}(B(\theta' - \theta) - k\theta'Q) + \sigma_i\rho_{i-1}(B(\theta' - \theta) + k\theta Q) + (\theta' - \theta)C$$

is zero. By (9) and Lemma 4.6,

$$(32) \quad 0 = a(c_i(\gamma_{i-1} - \gamma_{i+1}) - k(\gamma\gamma_i - \gamma_{i+1}))$$

$$(33) \quad = c_i(\sigma_{i-1}\rho_{i-1} - \sigma_{i+1}\rho_{i+1}) - k(\gamma\sigma_i\rho_i - \sigma_{i+1}\rho_{i+1} - b(\gamma - 1)),$$

where a, b are from (15). Now using (6) to eliminate σ_{i+1}, ρ_{i+1} throughout, we find $k(k - 1)b_2(\theta'^2 - \theta^2)$ times the expression in (33), minus the expression in (31), equals

$$(34) \quad k(k - 1)b_2b_i(\theta'^2 - \theta^2)(\sigma_{i+1}\rho_{i+1} - \sigma_{i-1}\rho_{i-1} - \alpha\sigma_i(\rho_{i+1} - \rho_{i-1}) - \beta\rho_i(\sigma_{i+1} - \sigma_{i-1})).$$

Since the expressions in (33) and (31) are both zero, we find (24) holds at i , as desired.

Next assume that -1 is a root of (16). By Lemma 5.2 the pair E, \hat{F} is also taut, and corresponding $\hat{\epsilon}$ equals 1. By the proof above, (24) holds for the pair E, \hat{F} . By the comment below (26), equation (24) holds also for the pair E, F .

(ii) \Rightarrow (i). First, assume 1 is a root of (16). We define γ, ϵ to be the roots of the quadratic equation in (16), where $\epsilon = 1$. Observe $\gamma = c_2\theta\theta'(kb_2)^{-1} - 1$ by (17). We define $\epsilon_i = 1$ ($0 \leq i \leq D$) so that $\epsilon_0, \epsilon_1, \dots, \epsilon_D$ is a trivial pseudo cosine sequence. Let a, b be as defined in (15). We first show $a \neq 0$. Suppose to the contrary that $a = 0$, so that $\sigma\rho = 1$ and $\theta\theta' = k^2$. Using (17), we routinely find $\theta^2 + \theta'^2 = 2k^2$. Thus $(\theta - \theta')^2 = \theta^2 + \theta'^2 - 2\theta\theta' = 0$, so $\theta = \theta'$, a contradiction. Hence $a \neq 0$. We now define $\gamma_i = \frac{\sigma_i\rho_i - b}{a}$ ($0 \leq i \leq D$). By Lemma 4.6 it suffices to show that $\gamma_0, \gamma_1, \dots, \gamma_D$ is a pseudo cosine sequence, which we will do in view of Corollary 3.5 by showing $b_i\gamma_{i+1} - k\gamma\gamma_i + c_i\gamma_{i-1} = 0$ for $1 \leq i \leq D - 1$. We observe

$$(35) \quad a(b_i\gamma_{i+1} - k\gamma\gamma_i + c_i\gamma_{i-1}) = b_i(\sigma_{i+1}\rho_{i+1} - b) - k\gamma(\sigma_i\rho_i - b) + c_i(\sigma_{i-1}\rho_{i-1} - b).$$

We evaluate the right-hand side of the equation in (35) by first eliminating the product $\sigma_{i+1}\rho_{i+1}$ using (24), and then eliminating σ_{i+1}, ρ_{i+1} in the result using (6). Recalling $\gamma = c_2\theta\theta'(kb_2)^{-1} - 1$ and using (25), (26), we find the expression $k(k - 1)b_2(\theta'^2 - \theta^2)a(b_i\gamma_{i+1} - k\gamma\gamma_i + c_i\gamma_{i-1})$ equals

$$(36) \quad A(\theta' - \theta)(\sigma_i\rho_i + \sigma_{i-1}\rho_{i-1}) + \rho_i\sigma_{i-1}(B(\theta' - \theta) - k\theta'Q) + \sigma_i\rho_{i-1}(B(\theta' - \theta) + k\theta Q) + (\theta' - \theta)C,$$

where Q is from (28). Since $Q = 0$, the expression in (36) equals 0 by Lemma 6.4, and the result follows.

Assume now that -1 is a root of (16). Then 1 is a root of the equation obtained from (16) by changing θ' with $-\theta'$. By the proof above, the pair E, \hat{F} is taut, and so by Lemma 5.2 also the pair E, F is taut. \square

THEOREM 6.6. *With reference to Definition 6.1, the following (i), (ii) are equivalent.*

- (i) E, F is a taut pair.
- (ii) *The polynomial in (16) has distinct roots, and there exist complex scalars α, β such that equality holds in (24) for all integers i ($1 \leq i \leq D - 1$).*

Furthermore, suppose conditions (i), (ii) hold above. Then (25), (26) hold.

Proof.

(i) \Rightarrow (ii). Let G, H denote pseudo primitive idempotents such that $E \circ F \in \text{span}\{G, H\}$. Let γ, ϵ denote the first pseudo cosines of G, H , respectively. Note that $\gamma \neq \epsilon$, and by Lemma 4.7, γ and ϵ are roots of (16). If $\gamma = \pm 1$ or $\epsilon = \pm 1$, then the result follows from Lemma 6.5. Suppose $\gamma \neq \pm 1$ and $\epsilon \neq \pm 1$. We first note that [8, Lemma 2.4] holds, since this proof works for pseudo primitive idempotents just as it does for primitive idempotents. The rest of our proof is similar to the proof of [8, Theorem 5.1], using [8, Lemma 2.4], (14), (22), (23).

(ii) \Rightarrow (i). By assumption, for all i ($1 \leq i \leq D - 1$), we have

$$(37) \quad \sigma_{i+1}\rho_{i+1} - \sigma_{i-1}\rho_{i-1} - \alpha\sigma_i(\rho_{i+1} - \rho_{i-1}) - \beta\rho_i(\sigma_{i+1} - \sigma_{i-1}) = 0.$$

Define real numbers γ, ϵ to be the roots of the quadratic equation in (16). Now let $\gamma_0, \gamma_1, \dots, \gamma_D$ and $\epsilon_0, \epsilon_1, \dots, \epsilon_D$ denote the scalars satisfying $\gamma_0 = \epsilon_0 = 1$ and

$$(38) \quad c_i\gamma_{i-1} + b_i\gamma_{i+1} = k\gamma\gamma_i, \quad c_i\epsilon_{i-1} + b_i\epsilon_{i+1} = k\epsilon\epsilon_i \quad (1 \leq i \leq D - 1).$$

Thus these sequences generated by γ, ϵ are pseudo cosine sequences by construction. If $\gamma = \pm 1$ or $\epsilon = \pm 1$, then the result follows from Lemma 6.5. So suppose $\gamma \neq \pm 1$, $\epsilon \neq \pm 1$. Define scalars a, b as in (15). We will now show by induction on j that

$$(39) \quad \sigma_j\rho_j = a\gamma_j + b\epsilon_j \quad (0 \leq j \leq D).$$

It's routine to show that (39) holds for $i = 0, 1$. Now fix an integer $1 \leq i \leq D - 1$ and assume (39) holds for all j , $0 \leq j \leq i$. We shall show (39) holds for $j = i + 1$. Adding (22) at i to (22) at $i - 1$, we obtain

$$(40) \quad \frac{\sigma_i(\rho_{i-1} - \rho_{i+1})}{\sigma - \rho} - \frac{\rho_i(\sigma_{i-1} - \sigma_{i+1})}{\sigma - \rho} = \frac{a(\gamma_{i+1} - \gamma_{i-1})}{\gamma - 1} + \frac{b(\epsilon_{i+1} - \epsilon_{i-1})}{\epsilon - 1}.$$

Evaluating the left-hand side of (23) using [8, Lemma 2.4], we find

$$(41) \quad \frac{\sigma\sigma_i(\rho_{i-1} - \rho_{i+1})}{\sigma^2 - \rho^2} - \frac{\rho\rho_i(\sigma_{i-1} - \sigma_{i+1})}{\sigma^2 - \rho^2} = \frac{a(\gamma_{i+1} - \gamma_{i-1})}{\gamma^2 - 1} + \frac{b(\epsilon_{i+1} - \epsilon_{i-1})}{\epsilon^2 - 1}.$$

Now consider the equation (*) which is (37), plus $(\gamma + \epsilon)$ times the equation in (40), minus the product of $(1 + \gamma + \epsilon + \gamma\epsilon)$ with (41). In the equation (*), the right-hand side is routinely shown to be equal to $a(\gamma_{i+1} - \gamma_{i-1}) + b(\epsilon_{i+1} - \epsilon_{i-1})$. Now using (25), (26), and (17), we find the left-hand side of the equation (*) equals $\sigma_{i+1}\rho_{i+1} - \sigma_{i-1}\rho_{i-1}$. By the inductive hypothesis, $\sigma_{i-1}\rho_{i-1} = a\gamma_{i-1} + b\epsilon_{i-1}$, and the result follows by Lemma 4.6.

Suppose now conditions (i), (ii) of the theorem hold. Solving (24) with $i = 1, 2$ for α and β we routinely find that α, β are equal to the scalars (25), (26). \square

7. WHEN CAN $\sigma_{i+1} = \sigma_{i-1}$?

With reference to Definition 6.1, assume conditions (i), (ii) hold in Theorem 6.6. In Section 9, we will obtain equations in which the expressions $\sigma\sigma_i - \sigma_{i-1}$, $\rho\rho_i - \rho_{i-1}$ appear as denominators. In this section we explore the circumstances under which these expressions can be zero. By Corollary 3.5, we note these expressions are zero precisely when $\sigma_{i+1} = \sigma_{i-1}$ and $\rho_{i+1} = \rho_{i-1}$. We will need the following two lemmas.

LEMMA 7.1. *With reference to Definition 6.1, at least one of $\sigma_{i-1} - \sigma_{i+1}$, $\sigma_i - \sigma_{i+2}$ is nonzero for $1 \leq i \leq D - 2$.*

Proof. Suppose, to the contrary, there exists an integer i ($1 \leq i \leq D - 2$) such that $\sigma_{i-1} - \sigma_{i+1} = 0$ and $\sigma_i - \sigma_{i+2} = 0$. Applying (9) at i and $i + 1$, we find

$$(42) \quad \sigma\sigma_i = \sigma_{i+1} \quad \text{and} \quad \sigma\sigma_{i+1} = \sigma_{i+2}.$$

Combining these two equations, we find $\sigma^2\sigma_i = \sigma_{i+2} = \sigma_i$. Suppose $\sigma_i = 0$. Then by (42), $\sigma_{i+1} = 0$. Using (6) and recursion, we find $\sigma_0 = 0$, a contradiction. Thus $\sigma_i \neq 0$ and hence $\sigma^2 = 1$. Thus θ is trivial, a contradiction. \square

LEMMA 7.2. *With reference to Definition 6.1, assume conditions (i), (ii) hold in Theorem 6.6. Then the following (i), (ii) are equivalent.*

- (i) $\alpha = \sigma$.
- (ii) $b_2(k - 2) - \theta^2(c_2 - 1) = 0$ or $\theta = 0$.

Moreover, suppose (i), (ii) hold above. Then

$$(43) \quad (\sigma_{i+1} - \sigma_{i-1})(\rho_{i+1} - (\beta + \rho)\rho_i + \rho_{i-1}) = 0 \quad (1 \leq i \leq D - 1).$$

Proof. The fact that (i), (ii) are equivalent follows from (25) and the fact that $\theta = k\sigma$. Now suppose (i), (ii) hold. Setting $\alpha = \sigma$ in (24), we find

$$(44) \quad (\sigma_{i+1} - \sigma\sigma_i)\rho_{i+1} - (\sigma_{i-1} - \sigma\sigma_i)\rho_{i-1} = \beta\rho_i(\sigma_{i+1} - \sigma_{i-1}) \quad (1 \leq i \leq D - 1).$$

Now using (9), (10), we find that for $1 \leq i \leq D - 1$,

$$(45) \quad \sigma_{i+1} - \sigma\sigma_i = \frac{c_i}{k}(\sigma_{i+1} - \sigma_{i-1}), \quad \sigma_{i-1} - \sigma\sigma_i = \frac{b_i}{k}(\sigma_{i-1} - \sigma_{i+1}).$$

Evaluating (44) using (45), we find

$$(46) \quad (\sigma_{i+1} - \sigma_{i-1})(c_i\rho_{i+1} - \beta k\rho_i + b_i\rho_{i-1}) = 0.$$

Using (6), it's routine to show $c_i\rho_{i+1} - \beta k\rho_i + b_i\rho_{i-1} = k(\rho_{i+1} - (\beta + \rho)\rho_i + \rho_{i-1})$, and the result follows. \square

With reference to Definition 6.1, in the following lemma we describe the taut pairs θ, θ' for which $\theta = 0$. We note this case is one instance in which $\sigma_{i+1} = \sigma_{i-1}$. In particular, $\sigma_{i+1} = \sigma_{i-1} = 0$ for even integers i ($1 \leq i \leq D - 1$).

LEMMA 7.3. *With reference to Definition 6.1, assume $\theta = 0$. Then the following (i), (ii) are equivalent.*

- (i) θ, θ' is a taut pair.
- (ii) $\exists \beta \in \mathbb{C}$ such that $\rho_{i+1} - (\beta + \rho)\rho_i + \rho_{i-1} = 0$ for i odd, $1 \leq i \leq D - 1$.

Assume (i), (i) hold above, and let G, H denote pseudo primitive idempotents such that $E \circ F \in \text{span}\{G, H\}$. Then $c_3 > 1$, $\theta'^2 = \frac{c_2(b_3-1)(k-1)}{c_3-1} + k$, and G, H are opposites. Furthermore, β is given by (26).

Proof. Using (6), we routinely find

$$(47) \quad \sigma_i = \begin{cases} 0 & \text{if } i \text{ odd} \\ (-1)^{\frac{i}{2}} \frac{c_1 c_3 \dots c_{i-1}}{b_1 b_3 \dots b_{i-1}} & \text{if } i \text{ even} \end{cases} \quad (0 \leq i \leq D).$$

(i) \Rightarrow (ii). By Lemma 7.2, (43) holds. The result now follows from (47).

(ii) \Rightarrow (i). Define $\alpha := 0$. Using (47) and the assumption, it's routine to show (24) holds for $1 \leq i \leq D - 1$. Since $\theta = 0$ and θ' is nontrivial, it is immediate that the polynomial in (16) has distinct roots. The result now follows from Theorem 6.6.

Now assume (i), (ii) hold, and let G, H denote pseudo primitive idempotents such that $E \circ F \in \text{span}\{G, H\}$. First, it follows from Theorem 6.6 that β is given by (26). Now suppose $c_3 = 1$. Then $c_2 = 1$, and hence $\Delta_2 = 0$ by (5). Thus $k = 2$ by (18), a contradiction. Hence $c_3 > 1$. The remaining results are routinely found using (18), (5), and (17). \square

With reference to Definition 6.1 assume conditions (i), (ii) hold in Theorem 6.6, and that $\alpha = \sigma$. If $\theta \neq 0$, then we have $b_2(k - 2) - \theta'^2(c_2 - 1) = 0$ by Lemma 7.2. We will show that this implies $\Delta_2 = 0$, and this case will be further studied in Section 8.

LEMMA 7.4. *With reference to Definition 6.1, assume conditions (i), (ii) hold in Theorem 6.6. Furthermore, assume $\alpha \neq \sigma, \beta \neq \rho$. Suppose $\sigma_{i+1} = \sigma_{i-1}$ or $\rho_{i+1} = \rho_{i-1}$ for some i , $1 \leq i \leq D - 1$. Then $b_{i-1} = 1$ and $c_2 = 1$. In particular, $\Delta_i = 0$, and $i > \frac{D}{2} + 1$.*

Proof. Let i denote the minimal integer ($1 \leq i \leq D - 1$) such that $\sigma_{i+1} = \sigma_{i-1}$ or $\rho_{i+1} = \rho_{i-1}$. Without loss of generality, assume $\sigma_{i+1} = \sigma_{i-1}$. It's routine to show $\sigma_2 \neq \sigma_0$, so $i \geq 2$. Thus $\sigma_i \neq \sigma_{i-2}$. By Corollary 3.6, $\sigma\sigma_i = \sigma_{i-1} = \sigma_{i+1}$. Setting $\sigma_{i+1} = \sigma\sigma_i$ and $\sigma_{i-1} = \sigma\sigma_i$ in (24) and simplifying, we find

$$(48) \quad \sigma_i(\sigma - \alpha)(\rho_{i+1} - \rho_{i-1}) = 0.$$

If $\sigma_i = 0$, then using (6) we find $\sigma_i = \sigma_{i-2} = 0$, a contradiction. Thus $\rho_{i+1} = \rho_{i-1}$, and $\rho_i \neq \rho_{i-2}$. Replacing i by $i - 1$ in (9) and recalling $\sigma_{i-1} = \sigma\sigma_i$, $\rho_{i-1} = \rho\rho_i$, we find

$$(49) \quad c_{i-1}(\sigma_{i-2} - \sigma_i) = k\sigma_i(\sigma^2 - 1),$$

$$(50) \quad c_{i-1}(\rho_{i-2} - \rho_i) = k\rho_i(\rho^2 - 1).$$

Combining these two equations, we find

$$(51) \quad (\sigma_{i-2} - \sigma_i)(\rho^2 - 1)\rho_i - (\rho_{i-2} - \rho_i)(\sigma^2 - 1)\sigma_i = 0.$$

Replacing i by $i - 1$ in (24) and using the facts that $\sigma_{i-1} = \sigma\sigma_i$, $\rho_{i-1} = \rho\rho_i$, we find

$$(52) \quad \sigma_i\rho_i - \sigma_{i-2}\rho_{i-2} - \alpha\sigma\sigma_i(\rho_i - \rho_{i-2}) - \beta\rho\rho_i(\sigma_i - \sigma_{i-2}) = 0.$$

Adding $\alpha\sigma\sigma_i - \sigma_{i-2}$ times (51) and $(\sigma^2 - 1)\sigma_i$ times (52) and simplifying using (25), (26), we find $\sigma_{i-2} = \sigma_i(k\sigma^2 - 1)(k - 1)^{-1}$. Plugging this into (49), we find $c_{i-1} = k - 1$. Thus $b_{i-1} = 1$. Now since $\Delta_i \geq 0$, using (5) we find $\Delta_i = 0$ and $c_2 = 1$. The final remark follows from [2, Proposition 4.1.6(ii)]. \square

COROLLARY 7.5. *With reference to Definition 6.1, assume conditions (i), (ii) hold in Theorem 6.6. Furthermore, assume $\Delta_2 \neq 0$, and θ, θ' are both nonzero. Then $\alpha \neq \sigma$, $\beta \neq \rho$, $\sigma_{i+1} \neq \sigma_{i-1}$, $\rho_{i+1} \neq \rho_{i-1}$ ($1 \leq i \leq D - 2$). If in addition $\Delta_{D-1} \neq 0$, then also $\sigma_D \neq \sigma_{D-2}$, $\rho_D \neq \rho_{D-2}$.*

Proof. Suppose $\alpha = \sigma$. Then by Lemma 7.2, $b_2(k - 2) = (c_2 - 1)\theta'^2$. Now by (18), we find $\theta^2 = b_2$. Setting $i = 4$ in (24) and using (5), we routinely find $\Delta_3 = 0$. But then also $\Delta_2 = 0$ by [9, Lemma 3.7], a contradiction. Hence $\alpha \neq \sigma$, and similarly, $\beta \neq \rho$. Assume now that $\sigma_{i+1} = \sigma_{i-1}$ or $\rho_{i+1} = \rho_{i-1}$ for some i ($1 \leq i \leq D - 2$). By Lemma 7.4 we have $b_{i-1} = 1$ and $c_2 = 1$. As $3 + (i - 1) \leq 3 + D - 3 = D$, by [2, Proposition 4.1.6(ii)] we also have that $c_3 \leq b_{i-1}$, implying $c_3 = 1$. But this forces $\Delta_2 = 0$, a contradiction. If $\Delta_{D-1} \neq 0$, then we have $\sigma_D \neq \sigma_{D-2}$, $\rho_D \neq \rho_{D-2}$ by Lemma 7.4. \square

8. THE CASE $\Delta_2 = 0$

With reference to Definition 6.1, in this section we consider the case when $\Delta_2 = 0$. We will need the following lemma.

LEMMA 8.1. *With reference to Definition 6.1, assume $\Delta_2 = 0$. Then $D \leq 5$ or $c_2 \leq 2$. Moreover, if Γ has a taut pair of pseudo eigenvalues, then $c_2 > 1$.*

Proof. By [10, Theorem 4.4], $D \leq 5$ or $c_2 \leq 2$. If Γ has a taut pair of pseudo eigenvalues, then we observe $c_2 \neq 1$ by Theorem 4.8. \square

We now consider the case when $\Delta_i = 0$ ($2 \leq i \leq D - 2$). We note that this is precisely the case when Γ is almost 2-homogeneous.

LEMMA 8.2. *With reference to Definition 6.1, assume $c_2 > 1$ and $\Delta_i = 0$ ($2 \leq i \leq D - 2$). Assume θ' satisfies $(k - 2)b_2 = (c_2 - 1)\theta'^2$. Then the following (i)–(iii) hold.*

- (i) *If $c_2 = 2$, then the pair θ, θ' is taut for all $\theta \in \mathbb{C}$.*
- (ii) *If $c_2 > 2$, then $D = 5$ and the pair θ, θ' is taut for all $\theta \in \mathbb{C} \setminus \{\pm\sqrt{-4b_2/(c_2 - 2)}\}$.*
- (iii) *Γ has no other taut pairs.*

Proof. (i) Using (5) and induction, we routinely find $c_i = i$ ($1 \leq i \leq D - 1$). Observe $\theta'^2 = (k - 2)^2$. We prove the result for $\theta' = k - 2$; the result for $\theta' = -(k - 2)$ then follows from Lemma 5.2. Let $\theta \in \mathbb{C}$ and let $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ denote the pseudo cosines sequences for θ and θ' , respectively. Using (6) and induction, we routinely find $\rho_i = (k - 2i)/k$ ($0 \leq i \leq D$). Setting $\alpha = \sigma$ and $\beta = 2 - \rho$ and using (6), we find (24) holds for $1 \leq i \leq D - 1$. Moreover, it's routine to show the polynomial in (16) has distinct roots in this case. Hence θ, θ' is a taut pair by Theorem 6.6.

(ii) First, observe $D = 5$ by Lemma 8.1. Using (5), we may solve for c_3, c_4 in terms of k, c_2 . Now using Lemma 3.3, (6), (25), and (26), it's routine to check that (24) holds for $1 \leq i \leq D - 1$. Using the fact that $\theta^2 \neq -4b_2/(c_2 - 2)$, it's routine to show the polynomial in (16) has distinct roots. Hence θ, θ' is a taut pair by Theorem 6.6.

(iii) Let $\theta, \Psi \in \mathbb{C}$, and assume this is a taut pair of pseudo eigenvalues. Setting $\Delta_2 = 0$ in (18), we find at least one of θ^2, Ψ^2 is equal to $(k - 2)b_2/(c_2 - 1)$, and the result follows. \square

In the rest of this section, we consider the case when $\Delta_2 = 0$ and there exists an integer i ($3 \leq i \leq D - 2$) for which $\Delta_i \neq 0$. With reference to Lemma 8.1, it is natural to treat cases $D = 5$ and $c_2 = 2$ separately. We shall classify the taut pairs of pseudo primitive idempotents in each case.

LEMMA 8.3. *With reference to Definition 6.1, assume $D = 5$. Furthermore, assume $\Delta_2 = 0$ and $\Delta_3 \neq 0$. Then the following (i), (ii) are equivalent.*

- (i) θ, θ' is a taut pair.
- (ii) $\{\theta^2, \theta'^2\} = \{0, \frac{b_2(k-2)}{c_2-1}\}$ or $\{\theta^2, \theta'^2\} = \{b_2 + c_2b_3, \frac{b_2(k-2)}{c_2-1}\}$.

Proof. First, observe that since $\Delta_2 = 0$, we find using (5) that

$$(53) \quad c_3 = \frac{b_2 - kc_2 - c_2^2 + kc_2^2}{k - 3c_2 + c_2^2}.$$

(i) \Rightarrow (ii). Since $\Delta_2 = 0$ and regarding (18), without loss of generality, we may assume $b_2(k - 2) - \theta'^2(c_2 - 1) = 0$. Thus by (26) we find

$$(54) \quad \beta = \frac{\theta'(kc_2 - b_2)}{kb_2}.$$

Observe (43) holds by Lemma 7.2. Using the above information and Lemma 3.3, we may verify that $\rho_{i+1} - (\beta + \rho)\rho_i + \rho_{i-1} = 0$ for $i = 1, 2, 3$, and cannot be zero for $i = 4$, or else Γ is almost 2-homogeneous by Theorem 5.4. Thus by (43), $\sigma_5 = \sigma_3$. Using (6) and (53), we routinely find that $\sigma_5 = \sigma_3$ implies either $\theta = 0$ or $\theta^2 = b_2 + c_2b_3$. The result follows.

(ii) \Rightarrow (i). Without loss of generality, assume $b_2(k - 2) - \theta'^2(c_2 - 1) = 0$. Let

$$(55) \quad \alpha = \sigma, \quad \beta = \frac{\theta'(kc_2 - b_2)}{kb_2}.$$

Using Lemma 3.3 and (6), it's now routine to show (24) holds for each i , $1 \leq i \leq 4$. It's also routine to show the polynomial in (16) has distinct roots. The result now follows from Theorem 6.6. \square

LEMMA 8.4. *With reference to Definition 6.1, assume $b_2(k - 2) = (c_2 - 1)\theta'^2$, and that there exists an integer i ($2 \leq i \leq D - 2$) such that*

$$(56) \quad \rho_i - (\beta + \rho)\rho_{i-1} + \rho_{i-2} = 0, \quad \rho_{i+2} - (\beta + \rho)\rho_{i+1} + \rho_i = 0,$$

where β is from (26). Then $\Delta_i = 0$.

Proof. Applying (9) at $i - 1$ and $i + 1$, we find

$$(57) \quad c_{i-1}(\rho_{i-2} - \rho_i) = k(\rho\rho_{i-1} - \rho_i), \quad c_{i+1}(\rho_i - \rho_{i+2}) = k(\rho\rho_{i+1} - \rho_{i+2}).$$

Now solving (56) for ρ_{i-2}, ρ_{i+2} and substituting into (57), we find

$$\begin{aligned} ((\beta + \rho)c_{i-1} - k\rho)\rho_{i-1} + (k - 2c_{i-1})\rho_i &= 0, \\ (-k + 2c_{i+1})\rho_i + (b_{i+1}(\beta + \rho) - k\rho)\rho_{i+1} &= 0. \end{aligned}$$

Furthermore, by (6),

$$(58) \quad c_i\rho_{i-1} - k\rho\rho_i + b_i\rho_{i+1} = 0.$$

We observe that we have three linear equations in the variables $\rho_{i-1}, \rho_i, \rho_{i+1}$. This system has nontrivial solutions, so the associated determinant must be zero. Setting this determinant equal to zero and using (5), (26), and the fact that $\theta'^2 = b_2(k - 2)(c_2 - 1)^{-1}$, we find $\Delta_i = 0$, as desired. \square

Recall that we are assuming that there exists an integer i ($3 \leq i \leq D-2$) for which $\Delta_i \neq 0$. Define a positive number f by

$$f = \min\{i \geq 3 \mid \Delta_i \neq 0\}$$

and note that $f \leq D-2$.

THEOREM 8.5. *With reference to Definition 6.1, assume that θ, θ' is a taut pair, and that $\Delta_2 = 0, c_2 = 2$. Then the following (i)–(iv) hold.*

- (i) *One of θ, θ' is equal to $\pm(k-2)$.*
- (ii) *$f = D-2$, where f is as defined above.*
- (iii) *$c_i = i$ for $1 \leq i \leq D-2$.*
- (iv) *Assume θ' (θ , respectively) is equal to $\pm(k-2)$. Then $\sigma_D = \sigma_{D-2}$ ($\rho_D = \rho_{D-2}$, respectively).*

Proof. Observe that by (18), we have that either θ^2 or θ'^2 is equal to $(k-2)b_2/(c_2-1)$. As $c_2 = 2$ and $b_2 = k-2$, this shows part (i) of the theorem.

For the rest of the proof, assume without loss of generality that $\theta'^2 = (k-2)^2$. By [13, Lemma 3.6(ii)], we have $c_i = i$ for $1 \leq i \leq f$. Using (6), it is now straightforward to show that for $0 \leq i \leq f+1$ we have that either $\rho_i = (k-2i)/k$ (if $\theta' = k-2$), or $\rho_i = (-1)^i(k-2i)/k$ (if $\theta' = -(k-2)$). Consider now equation (43) from Lemma 7.2. Clearly $\sigma_2 - \sigma_0 \neq 0$, and so $\rho_2 - (\beta + \rho)\rho_1 + \rho_0 = 0$ implies that $\beta + \rho$ is either 2 (if $\theta' = k-2$), or -2 (if $\theta' = -(k-2)$). It follows that $\rho_{i+1} - (\beta + \rho)\rho_i + \rho_{i-1} = 0$ for $1 \leq i \leq f$. Consider now $\rho_{f+2} - (\beta + \rho)\rho_{f+1} + \rho_f$. If this expression is equal to 0, then $\Delta_f = 0$ by Lemma 8.4, a contradiction. Therefore, $\sigma_{f+2} - \sigma_f = 0$ by (43).

Note that if $D = 5$, we have $f = D-2$ by our assumptions, so consider now the case $D \geq 6$. Assume that $f \leq D-3$. Then $\sigma_{f+3} - \sigma_{f+1} \neq 0$ by Lemma 7.1, and so $\rho_{f+3} - (\beta + \rho)\rho_{f+2} + \rho_{f+1} = 0$ by (43). By Lemma 8.4 we have that $\Delta_{f+1} = 0$, contradicting [13, Lemma 4.13(iv)]. This shows that $f = D-2$. Claims (iii) and (iv) of the theorem now follow from the comments above. \square

THEOREM 8.6. *With reference to Definition 6.1, assume $\Delta_2 = 0, c_2 = 2$, and $\Delta_{D-2} \neq 0$. Then the following (i), (ii) are equivalent.*

- (i) *The pair θ, θ' is taut.*
- (ii) *$c_i = i$ ($1 \leq i \leq D-2$), and either $\theta' = \pm(k-2)$ and $\sigma_D = \sigma_{D-2}$, or $\theta = \pm(k-2)$ and $\rho_D = \rho_{D-2}$.*

Proof.

(i) \Rightarrow (ii). This follows from Theorem 8.5.

(ii) \Rightarrow (i). Assume $\theta' = k-2$; the proof when $\theta' = -(k-2)$ (or $\theta = \pm(k-2)$) is similar. Furthermore, assume $c_i = i$ ($1 \leq i \leq D-2$) and $\sigma_D = \sigma_{D-2}$. Using (6) and induction, we find $\rho_i = (k-2i)/k$ ($0 \leq i \leq D-1$). Now define scalars $\alpha = \sigma$ and $\beta = 2 - \rho$. Using this information and (6), we find (24) holds for $1 \leq i \leq D-2$. Using the fact that $\sigma_D = \sigma_{D-2}$ and (6), we find (24) holds for $i = D-1$. The result now follows from Theorem 6.6. \square

REMARK 8.7. Observe that the equation $\sigma_D = \sigma_{D-2}$ gives us a polynomial of degree D in θ , and that $\theta = \pm k$ is a root of this polynomial.

9. THE CASE $\Delta_2 \neq 0$

In this section we will classify the taut pairs of pseudo eigenvalues in the case where $\Delta_2 \neq 0$. We will need the following definition.

DEFINITION 9.1. *With reference to Definition 6.1, by the weights of the pair of pseudo eigenvalues θ, θ' , we mean the complex scalars α, β given by the formulae in (25), (26).*

THEOREM 9.2. *With reference to Definition 6.1, assume that $\Delta_2 \neq 0$. Furthermore, assume that θ, θ' is a taut pair, with θ, θ' nonzero. Let α, β denote the corresponding weights. Then*

$$(59) \quad \frac{\sigma_{i+1} - \alpha\sigma_i}{\sigma\sigma_i - \sigma_{i-1}} = \frac{\beta\rho_i - \rho_{i-1}}{\rho\rho_i - \rho_{i-1}} \quad (1 \leq i \leq D - 2),$$

$$(60) \quad \frac{\rho_{i+1} - \beta\rho_i}{\rho\rho_i - \rho_{i-1}} = \frac{\alpha\sigma_i - \sigma_{i-1}}{\sigma\sigma_i - \sigma_{i-1}} \quad (1 \leq i \leq D - 2).$$

Observe the denominators in (59), (60) are nonzero by Corollaries 3.6, 7.5. Furthermore, if $\sigma_D \neq \sigma_{D-2}, \rho_D \neq \rho_{D-2}$, then (59), (60) also hold for $i = D - 1$.

Proof. First, observe that for $1 \leq i \leq D - 1$,

$$(61) \quad (\sigma\sigma_i - \sigma_{i+1})(\rho_{i-1} - \rho_{i+1}) = (\rho\rho_i - \rho_{i+1})(\sigma_{i-1} - \sigma_{i+1})$$

since both sides equal $c_i(\rho_{i-1} - \rho_{i+1})(\sigma_{i-1} - \sigma_{i+1})/k$ in view of (9). Multiplying out (61) and cancelling terms, we obtain

$$(62) \quad \sigma_{i+1}\rho_{i-1} - \sigma_{i-1}\rho_{i+1} = \sigma\sigma_i(\rho_{i-1} - \rho_{i+1}) - \rho\rho_i(\sigma_{i-1} - \sigma_{i+1}).$$

Multiplying (24) by $\sigma\sigma_i - \sigma_{i-1}$, multiplying (62) by $\sigma_{i+1} - \alpha\sigma_i$, and taking the difference between the two products, we find $\sigma_{i+1} - \sigma_{i-1}$ times

$$(63) \quad (\sigma_{i+1} - \alpha\sigma_i)(\rho\rho_i - \rho_{i-1}) - (\beta\rho_i - \rho_{i-1})(\sigma\sigma_i - \sigma_{i-1})$$

is zero. Observe $\sigma_{i+1} - \sigma_{i-1}$ is nonzero for $1 \leq i \leq D - 2$ by Corollary 7.5, so (63) is zero. Line (59) follows. We obtain (60) by interchanging the roles of σ_j, ρ_j ($0 \leq j \leq D - 1$) and the roles of α, β in the above argument. \square

LEMMA 9.3. *With reference to Definition 6.1, assume that $\Delta_2 \neq 0$. Furthermore, assume that θ, θ' is a taut pair, with θ, θ' nonzero. Let α, β denote the corresponding weights. Then*

$$(64) \quad \frac{k}{b_i} = \frac{\alpha\sigma_i - \sigma_{i-1}}{\sigma\sigma_i - \sigma_{i-1}} + \frac{\beta\rho_i - \rho_{i-1}}{\rho\rho_i - \rho_{i-1}} \quad (1 \leq i \leq D - 2).$$

Observe the denominators in (64) are nonzero by Corollaries 3.6, 7.5. Furthermore, if $\sigma_D \neq \sigma_{D-2}, \rho_D \neq \rho_{D-2}$, then (64) also holds for $i = D - 1$.

Proof. Using (10), we find that for $1 \leq i \leq D - 1$,

$$(65) \quad \frac{k}{b_i} = \frac{\sigma_{i+1} - \sigma_{i-1}}{\sigma\sigma_i - \sigma_{i-1}}$$

$$(66) \quad = \frac{\alpha\sigma_i - \sigma_{i-1}}{\sigma\sigma_i - \sigma_{i-1}} + \frac{\sigma_{i+1} - \alpha\sigma_i}{\sigma\sigma_i - \sigma_{i-1}}.$$

Evaluating the fraction on the right in (66) using (59), we obtain (64). \square

With reference to Definition 6.1, assume $\Delta_2 \neq 0$. To classify the taut pairs of nonzero pseudo eigenvalues θ, θ' , we will separate our results into two cases, depending on whether $\sigma_D = \sigma_{D-2}, \rho_D = \rho_{D-2}$. To see that there really are only two cases to consider, we need the following lemma.

LEMMA 9.4. *With reference to Definition 6.1, assume $\Delta_2 \neq 0$ and that the pair θ, θ' is taut, where θ, θ' are nonzero. If $\sigma_D = \sigma_{D-2}$, then $\rho_D = \rho_{D-2}$.*

Proof. Suppose to the contrary that $\sigma_D = \sigma_{D-2}$ but $\rho_D \neq \rho_{D-2}$. Observe $\alpha \neq \sigma$ by Corollary 7.5. Setting $i = D - 1$ in (6), (24), we find $\sigma_D = \sigma\sigma_{D-1}$ and $\sigma_D = \alpha\sigma_{D-1}$. Thus $\sigma\sigma_{D-1} = \alpha\sigma_{D-1}$. Since $\alpha \neq \sigma$, we find $0 = \sigma_{D-1} = \sigma_{D-2} = \sigma_D$. Using (6), we find $\sigma_{D-3} = 0$, contradicting Corollary 7.5. \square

THEOREM 9.5. *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 5$ and valency $k \geq 3$. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ denote two sequences of complex numbers satisfying $\sigma_D \neq \sigma_{D-2}, \rho_D \neq \rho_{D-2}$. Abbreviate $\sigma = \sigma_1, \rho = \rho_1$. Let α, β denote complex scalars. Assume $\Delta_2 \neq 0$ and $\sigma \neq \pm\rho$. Then the following (i), (ii) are equivalent.*

- (i) $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ form a taut pair of pseudo cosine sequences where $\sigma, \rho \neq 0$, and where α, β are the weights of the corresponding pseudo eigenvalues.
- (ii) $\sigma_0 = 1, \rho_0 = 1, \sigma \neq \alpha, \rho \neq \beta$, the polynomial in (16) has distinct roots (where $\theta = k\sigma, \theta' = k\rho$), and

$$(67) \quad \frac{\sigma_{i+1} - \alpha\sigma_i}{\sigma\sigma_i - \sigma_{i-1}} = \frac{\beta\rho_i - \rho_{i-1}}{\rho\rho_i - \rho_{i-1}} \quad (1 \leq i \leq D - 1),$$

$$(68) \quad \frac{\rho_{i+1} - \beta\rho_i}{\rho\rho_i - \rho_{i-1}} = \frac{\alpha\sigma_i - \sigma_{i-1}}{\sigma\sigma_i - \sigma_{i-1}} \quad (1 \leq i \leq D - 1),$$

$$(69) \quad \frac{k}{b_i} = \frac{\alpha\sigma_i - \sigma_{i-1}}{\sigma\sigma_i - \sigma_{i-1}} + \frac{\beta\rho_i - \rho_{i-1}}{\rho\rho_i - \rho_{i-1}} \quad (1 \leq i \leq D - 1),$$

and the denominators in (67)–(69) are nonzero.

Proof.

(i) \Rightarrow (ii). We assume $\sigma_0, \sigma_1, \dots, \sigma_D$ is a pseudo cosine sequence, so $\sigma_0 = 1$ by Corollary 3.5(ii). Similarly, $\rho_0 = 1$. Observe the polynomial in (16) has distinct roots by Theorem 6.6, and $\sigma \neq \alpha, \rho \neq \beta$ by Corollary 7.5. Equations (67), (68), (69) follow from Theorem 9.2 and Lemma 9.3.

(ii) \Rightarrow (i). Replacing the rightmost fraction in (69) using (67) and simplifying, we obtain

$$(70) \quad b_i(\sigma_{i+1} - \sigma_{i-1}) = k(\sigma\sigma_i - \sigma_{i-1}) \quad (1 \leq i \leq D - 1).$$

We assume $\sigma_0 = 1$, so $\sigma_0, \sigma_1, \dots, \sigma_D$ is a pseudo cosine sequence of Γ by Corollary 3.5. Similarly, $\rho_0, \rho_1, \dots, \rho_D$ is a pseudo cosine sequence of Γ . Observe the pseudo eigenvalues θ, θ' are nontrivial, since otherwise we would have a zero denominator in (67) at $i = 1$. We show θ, θ' is a taut pair. To do this, we show (24) holds for $1 \leq i \leq D - 1$. Fixing an integer i ($1 \leq i \leq D - 1$), we multiply the equations (67), (68) together and simplify to obtain

$$(71) \quad (\sigma_{i+1} - \alpha\sigma_i)(\rho_{i+1} - \beta\rho_i) = (\beta\rho_i - \rho_{i-1})(\alpha\sigma_i - \sigma_{i-1}).$$

Expanding (71), we routinely obtain (24). We have now shown (24) holds for $1 \leq i \leq D - 1$. Combining this with Theorem 6.6, we find θ, θ' is a taut pair with weights α, β . Finally, we note σ, ρ are nonzero by Lemma 7.2. \square

THEOREM 9.6. *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 5$ and valency $k \geq 3$. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ denote two sequences of complex numbers satisfying $\sigma_D = \sigma_{D-2}, \rho_D = \rho_{D-2}$. Abbreviate $\sigma = \sigma_1, \rho = \rho_1$. Let α, β denote complex scalars. Assume $\Delta_2 \neq 0$ and $\sigma \neq \pm\rho$. Then the following (i), (ii) are equivalent.*

- (i) $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ form a taut pair of pseudo cosine sequences where $\sigma, \rho \neq 0$, and where α, β are the weights of the corresponding pseudo eigenvalues.

- (ii) $\sigma_0 = 1, \rho_0 = 1, \sigma\sigma_{D-1} = \sigma_D, \rho\rho_{D-1} = \rho_D, \sigma \neq \alpha, \rho \neq \beta$, the polynomial in (16) has distinct roots (where $\theta = k\sigma, \theta' = k\rho$), and

$$(72) \quad \frac{\sigma_{i+1} - \alpha\sigma_i}{\sigma\sigma_i - \sigma_{i-1}} = \frac{\beta\rho_i - \rho_{i-1}}{\rho\rho_i - \rho_{i-1}} \quad (1 \leq i \leq D-2),$$

$$(73) \quad \frac{\rho_{i+1} - \beta\rho_i}{\rho\rho_i - \rho_{i-1}} = \frac{\alpha\sigma_i - \sigma_{i-1}}{\sigma\sigma_i - \sigma_{i-1}} \quad (1 \leq i \leq D-2),$$

$$(74) \quad \frac{k}{b_i} = \frac{\alpha\sigma_i - \sigma_{i-1}}{\sigma\sigma_i - \sigma_{i-1}} + \frac{\beta\rho_i - \rho_{i-1}}{\rho\rho_i - \rho_{i-1}} \quad (1 \leq i \leq D-2),$$

and the denominators in (72)–(74) are nonzero.

Furthermore, suppose (i), (ii) hold above. Then $b_{D-2} = b_{D-1} = c_2 = 1$.

Proof. Similar to the proof of Theorem 9.5. To finish the proof of (i) \Rightarrow (ii), we note that the equation $\sigma\sigma_{D-1} = \sigma_D$ follows from (9) and our assumption that $\sigma_D = \sigma_{D-2}$. Similarly, $\rho\rho_{D-1} = \rho_D$.

To finish the proof of (ii) \Rightarrow (i), we note that (24) trivially holds for $i = D-1$, since we assume $\sigma_D = \sigma_{D-2}, \rho_D = \rho_{D-2}$. The final statement follows from Lemma 7.4. \square

With reference to Definition 6.1, assume $\Delta_2 \neq 0$. In the following theorem, we show that Γ has essentially at most one taut pair of nonzero pseudo eigenvalues θ, θ' , and that any other taut pairs are obtained by taking plus or minus of these pseudo eigenvalues.

THEOREM 9.7. *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 5$ and valency $k \geq 3$. Assume that $\Delta_2 \neq 0$. Let $\theta, \theta', \psi, \psi'$ denote nonzero nontrivial pseudo eigenvalues. Suppose the pairs θ, θ' and ψ, ψ' are both taut. Then $\{\theta^2, \theta'^2\} = \{\psi^2, \psi'^2\}$.*

Proof. Immediate from Theorem 4.10 and Lemma 4.11. \square

REFERENCES

- [1] Eiichi Bannai and Tatsuro Ito, *Algebraic combinatorics I, association schemes*, Mathematics lecture note series, vol. 58, Benjamin/Cummings Publishing Company, Inc, 1984.
- [2] Andries E. Brouwer, Arjeh M. Cohen, and Arnold Neumaier, *Distance-regular graphs*, A Series of Modern Surveys in Mathematics, Springer-Verlag, 1989.
- [3] Brian Curtin, *2-homogeneous bipartite distance-regular graphs*, Discrete Mathematics **187** (1998), no. 1-3, 39–70.
- [4] ———, *Almost 2-homogeneous bipartite distance-regular graphs*, European Journal of Combinatorics **21** (2000), no. 7, 865–876.
- [5] Aleksandar Jurišić and Paul Terwilliger, *Pseudo 1-homogeneous distance-regular graphs*, Journal of Algebraic Combinatorics **28** (2008), no. 4, 509–529.
- [6] Michael S Lang, *Pseudo primitive idempotents and almost 2-homogeneous bipartite distance-regular graphs*, European Journal of Combinatorics **29** (2008), no. 1, 35–44.
- [7] ———, *Bipartite distance-regular graphs: the Q -polynomial property and pseudo primitive idempotents*, Discrete Mathematics **331** (2014), 27–35.
- [8] Mark S MacLean, *Taut distance-regular graphs of odd diameter*, Journal of Algebraic Combinatorics **17** (2003), no. 2, 125–147.
- [9] ———, *The local eigenvalues of a bipartite distance-regular graph*, European Journal of Combinatorics **45** (2015), 115–123.
- [10] Mark S MacLean, Štefko Miklavič, and Safet Penjić, *On the Terwilliger algebra of bipartite distance-regular graphs with $\Delta_2 = 0$ and $c_2 = 1$* , Linear Algebra and its Applications **496** (2016), 307–330.
- [11] Mark S MacLean and Paul Terwilliger, *Taut distance-regular graphs and the subconstituent algebra*, Discrete mathematics **306** (2006), no. 15, 1694–1721.
- [12] Arlene A Pascasio and Paul Terwilliger, *The pseudo-cosine sequences of a distance-regular graph*, Linear algebra and its applications **419** (2006), no. 2-3, 532–555.

- [13] Safet Penjić, *On the Terwilliger algebra of bipartite distance-regular graphs with $\Delta_2 = 0$ and $c_2 = 2$* , Discrete Mathematics **340** (2017), no. 3, 452–466.
- [14] Paul Terwilliger and Chih-wen Weng, *Distance-regular graphs, pseudo primitive idempotents, and the Terwilliger algebra*, European Journal of Combinatorics **25** (2004), no. 2, 287–298.

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