ALGEBRAIC COMBINATORICS

Carolina Benedetti & Nantel Bergeron

The antipode of linearized Hopf monoids

Volume 2, issue 5 (2019), p. 903-935.

<http://alco.centre-mersenne.org/item/ALCO_2019__2_5_903_0>

Algebraic Combinatorics
Volume 2, issue 5 (2019), p. 903–935
https://doi.org/10.5802/alco.53

The antipode of linearized Hopf monoids

Carolina Benedetti & Nantel Bergeron

Abstract In this paper, a Hopf monoid is an algebraic structure built on objects in the category of Joyal's vector species. There are two Fock functors, $\mathcal{K}$ and $\overline{\mathcal{K}}$, that map a Hopf monoid $\mathbf{H}$ to graded Hopf algebras $\mathcal{K}(\mathbf{H})$ and $\overline{\mathcal{K}}(\mathbf{H})$, respectively. There is a natural Hopf monoid structure on linear orders $\mathbf{L}$, and the two Fock functors are related by $\mathcal{K}(\mathbf{H}) = \overline{\mathcal{K}}(\mathbf{H} \times \mathbf{L})$. Unlike the functor $\overline{\mathcal{K}}$, the functor $\mathcal{K}$ applied to $\mathbf{H}$ may not preserve the antipode of $\mathbf{H}$. In view of the relation between $\mathcal{K}$ and $\overline{\mathcal{K}}$, one may consider instead of $\mathbf{H}$ the larger Hopf monoid $\mathbf{L} \times \mathbf{H}$ and study the antipode of $\mathbf{L} \times \mathbf{H}$. One of the main results in this paper provides a cancellation free and multiplicity free formula for the antipode of $\mathbf{L} \times \mathbf{H}$. As a consequence, we obtain a new antipode formula for the Hopf algebra $H = \mathcal{K}(\mathbf{H})$. We explore the case when $\mathbf{H}$ is commutative and cocommutative, and obtain new antipode formulas that, although not cancellation free, they can be used to obtain an antipode formula for $\overline{\mathcal{K}}(\mathbf{H})$ in some cases. We also recover many well-known identities in the literature involving antipodes of certain Hopf algebras. In our study of commutative and cocommutative Hopf monoids, hypergraphs and acyclic orientations play a central role. We obtain polynomials analogous to the chromatic polynomial of a graph, and also identities parallel to Stanley's (-1)-color theorem. An important consequence of our notion of acyclic orientation of hypergraphs is a geometric interpretation for the antipode formula for hypergraphs. This interpretation, which differs from the recent work of Aguiar and Ardila as the Hopf structures involved are different, appears in subsequent work by the authors.

Introduction

Computing antipode formulas in any graded Hopf algebra is a classical yet difficult problem. Recently, numerous results in this direction have been provided for various families of Hopf algebras [1, 4, 7, 10, 12, 15, 16]. Some motivations to find such formulas lie in their geometric interpretation [1], their use in quantum field theories [14], or their role in deriving combinatorial invariants of the discrete objects in play. A key example of this is the Hopf algebra of graphs $\mathcal{G}$ as given in [16], where the authors derive the antipode formula and use it to obtain the celebrated Stanley's (-1)-color theorem: the chromatic polynomial of a graph evaluated at -1 is, up to a sign, the number of acyclic orientations of the graph. A remarkable result in [1] shows that the antipode formula of a graph as given in [16] is encoded in the f-vector of the graphical zonotope corresponding to the underlying graph.

Manuscript received 23rd February 2018, revised 23rd November 2018, accepted 7th December 2018.

Keywords. Antipode, Hopf monoid, Hopf algebra, combinatorial identities, colorings, hypergraphs, orientations.

Acknowledgements. Carolina Benedetti thanks the faculty of Science of the University of Los Andes for its support.
With partial support of Bergeron's York University Research Chair and NSERC.

A general principle is that antipode formulas provide interesting identities for the combinatorial invariants of combinatorial objects. One of the key results in the theory of *Combinatorial Hopf algebras (CHAs)* gives us a canonical way of constructing combinatorial invariants with values in the space $QSym$ of quasisymmetric functions (see [2]). That is, letting $H = \bigoplus_{n\geqslant 0} H_n$ be a CHA over a field $\Bbbk$ and letting $\zeta\colon H \to \Bbbk$ be a *character of* H, there is a unique Hopf morphism $\Psi : H \to QSym$ such that $\zeta = \phi_1 \circ \Psi$ where $\phi_1\big(f(x_1, x_2, \ldots)\big) = f(1, 0, 0, \ldots)$. Moreover, there is a Hopf morphism $\phi_t\colon QSym \to \Bbbk[t]$ given by $\phi_t(M_a) = \binom{t}{\ell}$, where M_a is the monomial quasisymmetric function indexed by an integer composition $a = (a_1, a_2, \ldots, a_\ell)$. This Hopf morphism has the property that

$$\phi_t\big(f(x_1, x_2, \ldots)\big)\Big|_{t=1} = \phi_1(f)\,.$$

In particular,

$$\phi_t \circ \Psi\Big|_{t=1} = \big(\phi_t\big|_{t=1}\big) \circ \Psi = \phi_1 \circ \Psi = \zeta.$$

In the case when $H = \mathcal{G}$ and ζ is the character

$$\zeta(G) = \begin{cases} 1 & \text{if } G \text{ has no edges,} \\ 0 & \text{otherwise,} \end{cases}$$

we obtain that $\phi_t \circ \Psi(G) = \chi_G(t)$ is the chromatic polynomial of G as shown in [2, Example 4.5]. Stanley's (-1)-color theorem can be deduced in this Hopf setting using the fact that the antipode of $\Bbbk[t]$ is given by $S\big(p(t)\big) = p(-t)$. Hence

$$\chi_G(-1) = S \circ \phi_t \circ \Psi(G)\Big|_{t=1} = \phi_t \circ \Psi \circ S(G)\Big|_{t=1} = \zeta \circ S(G).$$

Moreover, if G is a graph on n vertices and $a(G)$ is the number of acyclic orientations of G, one has that the coefficient in $S(G)$ of the edgeless graph with n vertices is given by $(-1)^n a(G)$ (see [1, 10, 16]). Therefore, $\chi_G(-1) = \zeta \circ S(G) = (-1)^n a(G)$.

Here, we present a general framework that allows us to derive new formulas for the antipode of many of the graded Hopf algebras in the literature. Combinatorial objects which compose and decompose often give rise to Hopf monoids in Joyal's symmetric monoidal category of vector species. A Hopf monoid $\mathbf{H}$ is linearized if it can be described from a set species $\mathbf{h}$ as follows. For a finite set I, the vector space $\mathbf{H}[I]$ is the linear span of the set $\mathbf{h}[I]$, and the structure of $\mathbf{H}$ is obtained by linearization of the functions that define the structure on $\mathbf{h}$. We provide several examples of linearized Hopf monoids throughout.

There are two natural functors, the Fock functors $\mathcal{K}$ and $\overline{\mathcal{K}}$, that map Hopf monoids to graded Hopf algebras. Via these functors, it is sometimes possible to *lift* a Hopf algebra structure to the monoid level. All the objects and notions above are found in [4]. See also [19] where Hopf monoids are referred to as twisted Hopf algebras. The few basic notions and examples needed for our purposes are reviewed in Section 1, including the Hopf monoid of linear orders $\mathbf{L}$, the notion of linearized Hopf monoid, the Hadamard product and the Fock functors $\mathcal{K}$ and $\overline{\mathcal{K}}$.

The first goal of this paper is to construct a cancellation free and multiplicity free formula for the antipode of the Hadamard product $\mathbf{L} \times \mathbf{H}$ where $\mathbf{H}$ is a linearized Hopf monoid. This result will be developed in Section 2. One interesting fact is that even if the antipode formula of a Hopf monoid $\mathbf{H}$ is cancellation free, the Hopf algebra $\overline{\mathcal{K}}(\mathbf{H})$ may potentially have lots of cancellations in its antipode formula. However, $\overline{\mathcal{K}}$ gives us new ways of formulating antipodes and potentially new identities. We discuss this in Section 4.2.

In Section 3 we consider antipode formulas for commutative and cocommutative linearized Hopf monoids $\mathbf{H}$. This case is especially interesting as many of the Hopf monoids in combinatorics fall into this class. One consequence of our analysis is that the most relevant case to consider is the Hopf monoid of hypergraphs **HG** as defined in Section 1.5. The Hopf monoid **HG** contains all the information to compute antipodes for any other commutative and cocommutative linearized Hopf monoid **H**. This is a remarkable fact which we will unveil along the way. We give two antipode formulas for such **H**. One is derived in Section 3.3 from our work in Section 2 and one is obtained in Section 3.4 using orientations of hypergraphs. The second antipode formula is obtained using a sign reversing involution as in [10, 12]. Applications of our computations are presented in Section 4.1. In Section 4.3 we derive combinatorial identities using our antipode formulas. In particular we introduce a chromatic polynomial for total orders (permutations) and show an analogue of Stanley's (-1)-theorem.

We finish with a remark relating the results presented here to subsequent work obtained jointly with J. Machacek [8] in which we provide a geometric interpretation for the antipode of **HG** as encoded by a hypergraphic polytope. We point out that this interpretation differs from the one in [1]. Part of the difference relies on the fact that **HG** is cocommutative whereas the Hopf algebra of hypergraphs in [1] is not. We encourage the reader to see [8] for more details.

1. Hopf monoids

We review basic notions on Hopf monoids and illustrate definitions with three classical examples. We encourage the reader to see [4] for a deeper study on this topic. Throughout the paper $\Bbbk$ denotes an arbitrary field and all vector spaces are assumed to be over $\Bbbk$. In general, a Hopf monoid is defined in a symmetric monoidal category, but here we will use the term Hopf monoid in a much more restrictive manner. From now on, *Hopf monoid* stands for a connected Hopf monoid in the symmetric monoidal category of vector species using Cauchy product. Rather than defining these concepts in their full generality, we give here a brief description of the data that is needed for our purposes. However we highly encourage the reader to see [4] for more details on Hopf monoids and [11] for more details on species.

1.1. Species and Hopf monoids.

A *vector species* is a functor from the category of finite sets and bijections to the category of vector spaces and linear maps. Informally, a vector species $\mathbf{H}$ is a collection of vector spaces $\mathbf{H}[I]$, one for each finite set I, equivariant with respect to bijections $I \cong J$. A morphism of species $f : \mathbf{H} \to \mathbf{Q}$ is a collection of linear maps $f_I : \mathbf{H}[I] \to \mathbf{Q}[I]$ which commute with bijections.

A *set composition* of a finite set I is a finite sequence $(A_1, \dots, A_k)$ of disjoint non-empty subsets of I whose union is I. In this situation, we write $(A_1, \dots, A_k) \models I$.

A *Hopf monoid* consists of a vector species $\mathbf{H}$ equipped with two collections μ and Δ of equivariant linear maps

$$\mathbf{H}[A_1] \otimes \mathbf{H}[A_2] \xrightarrow{\mu_{A_1,A_2}} \mathbf{H}[I] \quad \text{and} \quad \mathbf{H}[I] \xrightarrow{\Delta_{A_1,A_2}} \mathbf{H}[A_1] \otimes \mathbf{H}[A_2]$$

subject to a number of axioms, of which the main ones follow.

Associativity. For each set composition $(A_1, A_2, A_3) \models I$, the diagrams

$$\begin{tikzcd} \mathbf{H}[A_1] \otimes \mathbf{H}[A_2] \otimes \mathbf{H}[A_3] \arrow[r, "\mathrm{id} \otimes \mu_{A_1,A_2}"] \arrow[d, "\mu_{A_1,A_2} \otimes \mathrm{id}"'] & \mathbf{H}[A_1] \otimes \mathbf{H}[A_2 \cup A_3] \arrow[d, "\mu_{A_1, A_2 \cup A_3}"] \\ \mathbf{H}[A_1 \cup A_2] \otimes \mathbf{H}[A_3] \arrow[r, "\mu_{A_1 \cup A_2, A_3}"'] & \mathbf{H}[I] \end{tikzcd} \tag{1}$$

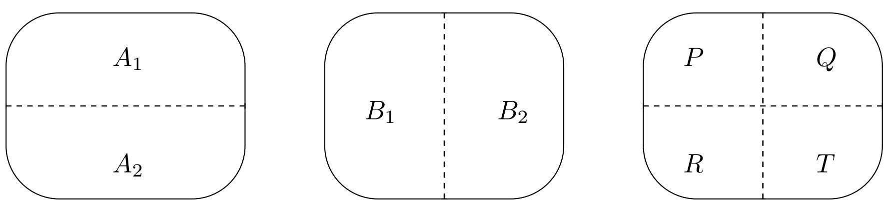
$$\begin{tikzcd} \mathbf{H}[I] \arrow[r, "\Delta_{A_1\cup A_2,A_3}"] \arrow[d, "\Delta_{A_1,A_2\cup A_3}"'] & \mathbf{H}[A_1\cup A_2]\otimes\mathbf{H}[A_3] \arrow[d, "\Delta_{A_1,A_2}\otimes\mathrm{id}"] \\ \mathbf{H}[A_1]\otimes\mathbf{H}[A_2\cup A_3] \arrow[r, "\mathrm{id}\otimes\Delta_{A_1,A_2}"'] & \mathbf{H}[A_1]\otimes\mathbf{H}[A_2]\otimes\mathbf{H}[A_3] \end{tikzcd} \tag{2}$$

commute.

Compatibility. Fix two set compositions (A_1, A_2) and (B_1, B_2) of I, and consider the resulting pairwise intersections:

$$P := A_1 \cap B_1,\ Q := A_1 \cap B_2,\ R := A_2 \cap B_1,\ T := A_2 \cap B_2,$$

as illustrated below

(3) 

.

For any such pair of set compositions, the diagram

$$\begin{tikzcd} \mathbf{H}[P]\otimes\mathbf{H}[Q]\otimes\mathbf{H}[R]\otimes\mathbf{H}[T] \arrow[rr, "\cong"] & & \mathbf{H}[P]\otimes\mathbf{H}[R]\otimes\mathbf{H}[Q]\otimes\mathbf{H}[T] \arrow[d, "\mu_{P,R}\otimes\mu_{Q,T}"] \\ \mathbf{H}[A_1]\otimes\mathbf{H}[A_2] \arrow[u, "\Delta_{P,Q}\otimes\Delta_{R,T}"] \arrow[r, "\mu_{A_1,A_2}"'] & \mathbf{H}[I] \arrow[r, "\Delta_{B_1,B_2}"'] & \mathbf{H}[B_1]\otimes\mathbf{H}[B_2] \end{tikzcd} \tag{4}$$

must commute. The top arrow stands for the map that interchanges the middle factors.

In addition, we require that the Hopf monoid $\mathbf{H}$ is *connected*, that is, $\mathbf{H}[\varnothing] \cong \Bbbk$ and the maps

$$\mathbf{H}[I]\otimes\mathbf{H}[\varnothing] \underset{\Delta_{I,\varnothing}}{\overset{\mu_{I,\varnothing}}{\rightleftarrows}} \mathbf{H}[I] \qquad \text{and} \qquad \mathbf{H}[\varnothing]\otimes\mathbf{H}[I] \underset{\Delta_{\varnothing,I}}{\overset{\mu_{\varnothing,I}}{\rightleftarrows}} \mathbf{H}[I]$$

are the canonical identifications.

The collection μ is the *product* and the collection Δ is the *coproduct* of the Hopf monoid $\mathbf{H}$. For any Hopf monoid $\mathbf{H}$ the antipode map $S\colon \mathbf{H}\to\mathbf{H}$ is computed using Takeuchi's formula (see Section 8.4.2 of [4]). More precisely, for any finite set I and a set composition $A = (A_1, \dots, A_k) \models I$, if $k = 1$ we let $\mu_{A_1} = \Delta_{A_1} = \mathbf{1}_I$ the identity map on $\mathbf{H}[I]$, and if $k > 1$ we let

$$\mu_{A_1,\dots,A_k} = \mu_{A_1,I\smallsetminus A_1}(\mathbf{1}_{A_1}\otimes\mu_{A_2,\dots,A_k}) \quad \text{and} \quad \Delta_{A_1,\dots,A_k} = (\mathbf{1}_{A_1}\otimes\Delta_{A_2,\dots,A_k})\Delta_{A_1,I\smallsetminus A_1}.$$

We then have

$$S_I = \sum_{k=1}^{|I|} \sum_{(A_1,\dots,A_k)\models I} (-1)^k \mu_{A_1,\dots,A_k}\Delta_{A_1,\dots,A_k} = \sum_{A\models I} (-1)^{\ell(A)}\mu_A\Delta_A\,. \tag{5}$$

A Hopf monoid is (co)commutative if the left (right) diagram below commutes for all set compositions $(A_1, A_2) \models I$.

$$
\begin{tikzcd}
\mathbf{H}[A_1]\otimes\mathbf{H}[A_2] \arrow[rr, "\tau_{A_1,A_2}"] \arrow[dr, "\mu_{A_1,A_2}"'] & & \mathbf{H}[A_2]\otimes\mathbf{H}[A_1] \arrow[dl, "\mu_{A_2,A_1}"] \\
& \mathbf{H}[I] &
\end{tikzcd}
\tag{6}
$$

$$
\begin{tikzcd}
\mathbf{H}[A_1]\otimes\mathbf{H}[A_2] \arrow[rr, "\tau_{A_1,A_2}"] & & \mathbf{H}[A_2]\otimes\mathbf{H}[A_1] \\
& \mathbf{H}[I] \arrow[ul, "\Delta_{A_1,A_2}"] \arrow[ur, "\Delta_{A_2,A_1}"'] &
\end{tikzcd}
$$

The arrow τ_{A_1,A_2} stands for the map that interchanges the factors.

A morphism of Hopf monoids $f : \mathbf{H} \to \mathbf{Q}$ is a morphism of species that commutes with μ and Δ.

1.2. The Hopf monoid of linear orders $\mathbf{L}$ ([4]). For any finite set I let $\mathbf{l}[I]$ be the set of all linear orders on I. For instance, if $I = \{a, b, c\}$,

$$\mathbf{l}[I] = \{abc,\ bac,\ acb,\ bca,\ cab,\ cba\}.$$

The vector species $\mathbf{L}$ is such that $\mathbf{L}[I] := \Bbbk\mathbf{l}[I]$ is the vector space with basis $\mathbf{l}[I]$.

Given $(A_1, A_2) \models I$ and linear orders α_1, α_2 on A_1, A_2, respectively, their concatenation $\alpha_1 \cdot \alpha_2$ is the linear order on I given by α_1 followed by α_2. Given a linear order α on I and $P \subseteq I$, the restriction $\alpha|_P$ is the ordering in P given by the order α. These operations give rise to maps

$$
\begin{aligned}
\mathbf{l}[A_1] \times \mathbf{l}[A_2] &\to \mathbf{l}[I] & \qquad \mathbf{l}[I] &\to \mathbf{l}[A_1] \times \mathbf{l}[A_2] \\
(\alpha_1, \alpha_2) &\to \alpha_1 \cdot \alpha_2 & \alpha &\to (\alpha|_{A_1}, \alpha|_{A_2}).
\end{aligned}
\tag{7}
$$

Extending by linearity, we obtain linear maps

$$\mu_{A_1,A_2} : \mathbf{L}[A_1] \otimes \mathbf{L}[A_2] \to \mathbf{L}[I] \quad \text{and} \quad \Delta_{A_1,A_2} : \mathbf{L}[I] \to \mathbf{L}[A_1] \otimes \mathbf{L}[A_2]$$

which turn $\mathbf{L}$ into a cocommutative but not commutative Hopf monoid. The reader should check that all the required axioms for a Hopf monoid are indeed satisfied for this and the upcoming examples.

1.3. The Hopf monoid of set partitions $\boldsymbol{\pi}$ ([4]). A *partition* of a finite set I is a collection X of disjoint nonempty subsets whose union is I. The subsets are the *blocks* of X.

Given a partition X of I and $P \subseteq I$, the restriction $X|_P$ is the partition of P whose blocks are the nonempty intersections of the blocks of X with P. Given $(A_1, A_2) \models I$ and partitions X_i of A_i, $i = 1, 2$, the union $X_1 \cup X_2$ is the partition of I whose blocks are the blocks of X_1 and the blocks of X_2.

Let $\boldsymbol{\pi}[I]$ denote the set of partitions of I and $\boldsymbol{\pi}[I] = \Bbbk\boldsymbol{\pi}[I]$ the vector space with basis $\boldsymbol{\pi}[I]$. A Hopf monoid structure on $\boldsymbol{\pi}$ is defined and studied in [3, 4, 7, 13]. Among its various linear bases, we are interested in the *power-sum* basis on which the operations are as follows. The product

$$\mu_{A_1,A_2} : \boldsymbol{\pi}[A_1] \otimes \boldsymbol{\pi}[A_2] \to \boldsymbol{\pi}[I]$$

is given by

$$\mu_{A_1,A_2}(X_1 \otimes X_2) = X_1 \cup X_2. \tag{8}$$

for $X_i \in \boldsymbol{\pi}[A_i]$ and extended linearly. The coproduct

$$\Delta_{A_1,A_2} : \boldsymbol{\pi}[I] \to \boldsymbol{\pi}[A_1] \otimes \boldsymbol{\pi}[A_2]$$

is given by

$$\Delta_{A_1,A_2}(X) = \begin{cases} X|_{A_1} \otimes X|_{A_2} & \text{if } A_1 \text{ is the union of some blocks of } X, \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

for $X \in \boldsymbol{\pi}[I]$ and extended linearly. These operations turn the species $\boldsymbol{\pi}$ into a Hopf monoid that is both commutative and cocommutative.

1.4. The Hopf monoid of simple graphs $\mathbf{G}$. A *(simple) graph* g on a finite set I is a collection E of subsets of I of size 2. The elements of I are the *vertices* of g. There is an *edge* between two vertices i, j if $e = \{i, j\} \in E$. In this case we say that e is *incident* to i and j.

Given a graph g on I and $P \subseteq I$, the restriction $g|_P$ is the graph on the vertex set P whose edges are the edges of g incident to elements of P only. Let $(A_1, A_2) \models I$ and consider the graphs g_i of A_i, for $i = 1, 2$. The union $g_1 \cup g_2$ is the graph on I whose edges are those of g_1 and those of g_2.

Let $\mathbf{g}[I]$ denote the set of graphs on I and $\mathbf{G}[I] = \Bbbk\mathbf{g}[I]$ the vector space with basis $\mathbf{g}[I]$. A Hopf monoid structure on $\mathbf{G}$ is defined using the maps

$$\begin{array}{cc} \mathbf{g}[A_1] \times \mathbf{g}[A_2] \to \mathbf{g}[I] & \mathbf{g}[I] \to \mathbf{g}[A_1] \times \mathbf{g}[A_2] \\ (g_1, g_2) \to g_1 \cup g_2 & g \to (g|_{A_1}, g|_{A_2}). \end{array} \tag{10}$$

Extending linearly, we obtain linear maps

$$\mu_{A_1,A_2} : \mathbf{G}[A_1] \otimes \mathbf{G}[A_2] \to \mathbf{G}[I] \quad \text{and} \quad \Delta_{A_1,A_2} : \mathbf{G}[I] \to \mathbf{G}[A_1] \otimes \mathbf{G}[A_2].$$

These operations turn the species $\mathbf{G}$ into a Hopf monoid that is both commutative and cocommutative.

1.5. The Hopf monoid of simple hypergraphs $\mathbf{HG}$. Let 2^I denote the collection of subsets of I. Let $\mathbf{HG}[I] = \Bbbk\mathbf{hg}[I]$ be the space spanned by the basis $\mathbf{hg}[I]$ where

$$\mathbf{hg}[I] = \big\{h \subseteq 2^I \,: U \in h \text{ implies } |U| \geqslant 2\big\}.$$

An element $h \in \mathbf{hg}[I]$ is a *hypergraph on* I. For $(P, T) \models I$ and $h, k \in \mathbf{hg}[I]$, the multiplication is given by $\mu_{P,T}(h, k) = h \cup k$ and the comultiplication is given by $\Delta_{P,T}(h) = h|_P \otimes h|_T$ where $h|_P = \{U \in h : U \cap P = U\}$. Extending these definitions linearly we have that $\mathbf{HG}$ is commutative and cocommutative Hopf monoid.

1.6. The Hadamard product. Given two species $\mathbf{H}$ and $\mathbf{Q}$, their *Hadamard product* is the species $\mathbf{H} \times \mathbf{Q}$ defined by

$$(\mathbf{H} \times \mathbf{Q})[I] = \mathbf{H}[I] \otimes \mathbf{Q}[I],$$

where $\otimes$ is the usual tensor product of vector spaces over $\Bbbk$. If $\mathbf{H}$ and $\mathbf{Q}$ are Hopf monoids, then so is $\mathbf{H} \times \mathbf{Q}$, with the following operations. For $(A_1, A_2) \models I$, the product on $\mathbf{H} \times \mathbf{Q}$ is depicted in the diagram:

$$\begin{tikzcd}
(\mathbf{H}\times\mathbf{Q})[A_1] \otimes (\mathbf{H}\times\mathbf{Q})[A_2] \arrow[r, dashed] \arrow[d, equal] & (\mathbf{H}\times\mathbf{Q})[I] \arrow[dd, equal] \\
\mathbf{H}[A_1] \otimes \mathbf{Q}[A_1] \otimes \mathbf{H}[A_2] \otimes \mathbf{Q}[A_2] \arrow[d, "\cong"'] & \\
\mathbf{H}[A_1] \otimes \mathbf{H}[A_2] \otimes \mathbf{Q}[A_1] \otimes \mathbf{Q}[A_2] \arrow[r, "\mu^{\mathbf{H}}_{A_1,A_2} \otimes \mu^{\mathbf{Q}}_{A_1,A_2}"'] & \mathbf{H}[I] \otimes \mathbf{Q}[I]
\end{tikzcd}$$

The coproduct is defined similarly. If $\mathbf{H}$ and $\mathbf{Q}$ are (co)commutative, then so is $\mathbf{H}\times\mathbf{Q}$.

1.7. Linearized Hopf monoids. A *set species* $\mathbf{h}$ is a collection of sets $\mathbf{h}[I]$, one for each finite set I, equivariant with respect to bijections $I \cong J$. We say that $\mathbf{h}$ is a basis for a Hopf monoid $\mathbf{H}$ if for every finite set I we have that $\mathbf{H}[I] = \Bbbk\mathbf{h}[I]$, the vector space with basis $\mathbf{h}[I]$. We say that the monoid $\mathbf{H}$ is *linearized* in the basis $\mathbf{h}$ if the product and coproduct maps have the following properties. The product

$$\mu_{A_1,A_2} : \mathbf{H}[A_1] \otimes \mathbf{H}[A_2] \to \mathbf{H}[I]$$

is the linearization of a map

$$\mu_{A_1,A_2} : \mathbf{h}[A_1] \times \mathbf{h}[A_2] \to \mathbf{h}[I] \tag{11}$$

and the coproduct

$$\Delta_{A_1,A_2} : \mathbf{H}[I] \to \mathbf{H}[A_1] \otimes \mathbf{H}[A_2]$$

is the linearization of a map

$$\Delta_{A_1,A_2} : \mathbf{h}[I] \to (\mathbf{h}[A_1] \times \mathbf{h}[A_2]) \cup \{0\}\,. \tag{12}$$

It is understood that for any bijection $\sigma\colon I \to J$, the linear isomorphism $\mathbf{H}[\sigma]\colon \mathbf{H}[I] \to \mathbf{H}[j]$ maps the basis $\mathbf{h}[I]$ to the basis $\mathbf{h}[J]$ via the bijection $\mathbf{h}[\sigma]$. From now on, we will use capital letters for vector species and lower case for set species.

The Hopf monoids $\mathbf{L}$, $\boldsymbol{\pi}$, $\mathbf{G}$ and $\mathbf{HG}$ are linearized in the bases $\mathbf{l}$, $\boldsymbol{\pi}$, $\mathbf{g}$ and $\mathbf{hg}$ respectively. As remarked in [18], many of the Hopf monoids in the literature are linearized in some basis.

1.8. Functors $\mathcal{K}$ and $\overline{\mathcal{K}}$. As describe in [4, Section III], there are some interesting functors from the category of species to the category of graded vector spaces. Let $[n] := \{1, 2, \ldots, n\}$ and assume throughout that $\mathrm{char}(\Bbbk) = 0$. Given a species $\mathbf{H}$, we write $\mathbf{H}[n]$ instead of $\mathbf{H}[[n]]$. The symmetric group S_n acts on $\mathbf{H}[n]$ by relabelling. Define the functors $\mathcal{K}$ and $\overline{\mathcal{K}}$ as

$$\mathcal{K}(\mathbf{H}) = \bigoplus_{n\geqslant 0} \mathbf{H}[n] \qquad\qquad \overline{\mathcal{K}}(\mathbf{H}) = \bigoplus_{n\geqslant 0} \mathbf{H}[n]_{S_n}$$

where

$$\mathbf{H}[n]_{S_n} = \mathbf{H}[n]\Big/\langle x - \mathbf{H}[\sigma](x) \mid \ \sigma \in S_n;\ x \in \mathbf{H}[n]\rangle$$

denotes the quotient space of equivalence classes under the S_n-action. When $\mathbf{H}$ is a Hopf monoid, we can build a product and coproduct on $\mathcal{K}(\mathbf{H})$ and $\overline{\mathcal{K}}(\mathbf{H})$ from those of $\mathbf{H}$ together with certain canonical transformations. For example, one has that

$$\overline{\mathcal{K}}(\mathbf{L}) \cong \Bbbk[t]$$

is the polynomial algebra on one generator, while $\mathcal{K}(\mathbf{L})$ is the Hopf algebra introduced by Patras and Reutenauer in [19]. The antipode map $S : \mathbf{L} \to \mathbf{L}$ is such that for $\alpha = a_1 \cdots a_n \in \mathbf{l}[n]$

$$S_{[n]}(\alpha) = (-1)^n a_n \cdots a_1.$$

However, the antipode of the graded Hopf algebra $\mathcal{K}[\mathbf{L}]$ is not given by the formula above (see Section 4.2). On the other hand, in the Hopf algebra $\overline{\mathcal{K}}[\mathbf{L}] \cong \Bbbk[t]$, the antipode is given by $S(t^n) = (-1)^n t^n$ and it is the functorial image of the map above. This is not an accident: the functor $\mathcal{K}$ may not preserve the antipode but the functor $\overline{\mathcal{K}}$ always does.

A very interesting relation between the functors $\mathcal{K}$ and $\overline{\mathcal{K}}$ is given in [4, Theorem 15.13] as follows

$$\overline{\mathcal{K}}(\mathbf{L} \times \mathbf{H}) \cong \mathcal{K}(\mathbf{H}), \tag{13}$$

where $\mathbf{H}$ is an arbitrary Hopf monoid. In this paper we aim to make use of this relation to study the antipode problem for some Hopf algebras.

2. Antipode for linearized Hopf Monoid $\mathbf{L} \times \mathbf{H}$

In this section we show a multiplicity free and cancellation free formula for the antipode of Hopf monoids of the form $\mathbf{L} \times \mathbf{H}$ where $\mathbf{H}$ is linearized in some basis. Thus, by (13), we obtain an antipode formula for $\mathcal{K}(\mathbf{H})$ as well. However, in $\mathcal{K}(\mathbf{H})$ this antipode formula may not be cancellation free.

2.1. Antipode Formula for $\mathbf{L}\times\mathbf{H}$. Let $\mathbf{H}$ be a Hopf monoid linearized in the basis $\mathbf{h}$. We intend to resolve the cancellations in the Takeuchi formula for the antipode of $\mathbf{L} \times \mathbf{H}$. For a fixed finite set I let $(\alpha, x) \in (\mathbf{l} \times \mathbf{h})[I]$, that is, α is a linear ordering on I and x is an element of $\mathbf{h}[I]$. From (5) we have

$$S_I(\alpha, x) = \sum_{A \models I} (-1)^{\ell(A)} \mu_A \Delta_A(\alpha, x) = \sum_{\substack{A \models I \\ \Delta_A(x) \neq 0}} (-1)^{\ell(A)} (\alpha_A, x_A), \tag{14}$$

summing over all $A = (A_1, \dots, A_k) \models I$, where α_A denotes the element in $\mathbf{l}[I]$ given by

$$\alpha_A = \alpha|_{A_1} \cdot \alpha|_{A_2} \cdots \alpha|_{A_k} .$$

Also, provided $\Delta_A(x) \neq 0$ we set

$$x_A = \mu_A \Delta_A(x) \in \mathbf{h}[I] .$$

Each composition A gives rise to single elements α_A and x_A since $\mathbf{L}$ and $\mathbf{H}$ are linearized in the basis $\mathbf{l}$ and $\mathbf{h}$, respectively. We can thus rewrite equation (14) as

$$S_I(\alpha, x) = \sum_{(\beta, y) \in (\mathbf{l} \times \mathbf{h})[I]} \left(\sum_{\substack{A \models I \\ (\alpha_A, x_A) = (\beta, y)}} (-1)^{\ell(A)} \right) (\beta, y) . \tag{15}$$

Let

$$\mathcal{C}_{\alpha,x}^{\beta,y} = \left\{ A \models I : (\alpha_A, x_A) = (\beta, y) \right\} . \tag{16}$$

Using the notation above we have the following theorem which provides us a multiplicity-free and cancellation-free formula for the antipode of $\mathbf{L} \times \mathbf{H}$.

Theorem 2.1. *Let $\mathbf{H}$ be a linearized Hopf monoid in the basis $\mathbf{h}$. For $(\alpha, x) \in (\mathbf{l}\times\mathbf{h})[I]$ we have*

$$S_I(\alpha, x) = \sum_{(\beta, y) \in (\mathbf{l} \times \mathbf{h})[I]} c_{\alpha,x}^{\beta,y} \, (\beta, y), \qquad \text{where} \qquad c_{\alpha,x}^{\beta,y} = \sum_{A \in \mathcal{C}_{\alpha,x}^{\beta,y}} (-1)^{\ell(A)}.$$

In Section 2.3 we define a non nested graph $G_{\alpha,x}^{\beta,y}$ and see that $c_{\alpha,x}^{\beta,y} = c(G_{\alpha,x}^{\beta,y})$ where $c(G)$, defined in Section 2.4, is an invariant associated to a non-nesting graph G with values ± 1 or 0. We then have the cancellation free formula

$$S_I(\alpha, x) = \sum_{\beta, y} c(G_{\alpha,x}^{\beta,y})(\beta, y). \tag{17}$$

The proof of this theorem will be given in Section 2.4. We make use of the refinement order on set compositions to show that the set $\mathcal{C}_{\alpha,x}^{\beta,y}$ has a unique minimum. We will use this fact along with other properties to construct sign reversing involutions on $\mathcal{C}_{\alpha,x}^{\beta,y}$ and the result will follow once we understand the fixed points of such involutions.

2.2. Minimal element of $\mathcal{C}^{\beta,y}_{\alpha,x}$. Given set compositions $A = (A_1, \dots, A_k)$ and $B = (B_1, \dots, B_\ell)$ on a set I, we say that A *refines* B, and we write $A \leqslant B$, if the parts of B are unions of consecutive parts of A. For example

$$A = \big(\{1,4\}, \{2\}, \{5,7\}, \{3\}, \{9\}, \{6,8\}\big) \leqslant \big(\{1,4\}, \{2,3,5,7\}, \{6,8,9\}\big) = B$$

but A does not refine $\big(\{1,4,5,7\}, \{2\}, \{3\}, \{9\}, \{6,8\}\big)$. Denote by $(\mathcal{P}_I, \leqslant)$ the poset of set compositions of I, ordered by refinement. In what follows we will write $(14, 2, 57, 3, 9, 68)$ instead of $\big(\{1,4\}, \{2\}, \{5,7\}, \{3\}, \{9\}, \{6,8\}\big)$. Consider the order $\leqslant$ restricted to the set $\mathcal{C}^{\beta,y}_{\alpha,x}$.

Lemma 2.2. *If $\mathcal{C}^{\beta,y}_{\alpha,x} \neq \varnothing$, then there is a unique minimal element in $(\mathcal{C}^{\beta,y}_{\alpha,x}, \leqslant)$.*

Proof. Suppose that $A = (A_1, \dots, A_k)$ and $B = (B_1, \dots, B_\ell)$ are minimal in $\in \mathcal{C}^{\beta,y}_{\alpha,x}$ and $A \neq B$. We have that $\alpha_A = \beta = \alpha_B$ and $x_A = y = x_B$. Since $\alpha_A = \beta$, the parts of A appear consecutively in β and the same is true for the parts of B. For example if $\alpha = abcdef$ and $\beta = bcfade$, then for $A = (bc, f, ad, e)$ and $B = (bc, f, a, de)$ we have $\alpha_A = \alpha_B = \beta$.

Let $1 \leqslant i \leqslant k$ be the smallest index such that $A_i \neq B_i$, and assume without loss of generality that $|A_i| > |B_i|$. If $i = k$ then B refines A and this is a contradiction. Hence we assume that $i < k$ and we now build a composition C that refines A such that $C \in \mathcal{C}^{\beta,y}_{\alpha,x}$, which will contradict again the minimality of A. Since $\alpha_A = \alpha_B$ our choice of i implies that $B_i \subset A_i$. Let $U = A_i \smallsetminus B_i$. We claim that for $C = (A_1, \dots, A_{i-1}, B_i, U, A_{i+1}, ..., A_k)$

(a) $C < A$
(b) $\alpha_C = \alpha_A = \beta$
(c) $x_C = x_A = y$

Items (a) and (b) are straightforward. For (c) we show that for some P and T we get

$$x_A = \mu_{P,T}\Delta_{P,T}\mu_A\Delta_A(x) = \mu_C\Delta_C(x). \tag{18}$$

Let $P = B_1 \cup \dots \cup B_i$ and $T = B_{i+1} \cup \dots \cup B_\ell$. We claim that

$$x_B = \mu_{P,T}\Delta_{P,T}(x_B). \tag{19}$$

To see this, we use the associativity of μ to write $\mu_B = \mu_{P,T}(\mu_{B_1,\dots,B_i} \otimes \mu_{B_{i+1},\dots,B_\ell})$. Also, let $Q = R = \varnothing$ in the compatibility relation (4), we get $A_1 = P$, $A_2 = T$ and

$$\Delta_{P,T}\mu_{P,T} = \mathbf{1}_P \otimes \mathbf{1}_T,$$

where $\mathbf{1}_A$ denotes the identity map on A. Hence equation (19) follows from

$$\begin{aligned}\mu_{P,T}\Delta_{P,T}(x_B) &= \mu_{P,T}\Delta_{P,T}\mu_{P,T}(\mu_{B_1,\dots,B_i} \otimes \mu_{B_{i+1},\dots,B_\ell})\Delta_B(x)\\ &= \mu_{P,T}(\mu_{B_1,\dots,B_i} \otimes \mu_{B_{i+1},\dots,B_\ell})\Delta_B(x) = \mu_B\Delta_B(x) = x_B.\end{aligned}$$

Using again (19) and the fact that $x_A = x_B$ we show the first equality in (18):

$$x_A = \mu_{P,T}\Delta_{P,T}(x_A) = \mu_{P,T}\Delta_{P,T}\mu_A\Delta_A(x). \tag{20}$$

Now let $P' = A_1 \cup \dots \cup A_{i-1}$, $Q' = \varnothing$, $R' = B_i$ and $T' = T$ in the compatibility relation (4):

$$\begin{aligned}\Delta_{P,T}\mu_A &= \Delta_{P'\cup R',T'}\mu_{P',R'\cup T'}(\mu_{A_1,\dots,A_{i-1}} \otimes \mu_{A_i,\dots,A_k})\\ &= (\mu_{P',R'} \otimes \mathbf{1}_{T'})(\mathbf{1}_{P'} \otimes \Delta_{R',T'})(\mu_{A_1,\dots,A_{i-1}} \otimes \mu_{A_i,\dots,A_k})\\ &= (\mu_{P',R'} \otimes \mathbf{1}_{T'})(\mu_{A_1,\dots,A_{i-1}} \otimes \mathbf{1}_{R'} \otimes \mathbf{1}_{T'})(\mathbf{1}_{A_1} \otimes \dots \otimes \mathbf{1}_{A_{i-1}} \otimes \Delta_{R',T'}\mu_{A_i,\dots,A_k}).\end{aligned}$$

We now expand $\Delta_{R',T'}\mu_{A_i,\dots,A_k}$ using similar manipulations. Let $T''=A_{i+1}\cup\cdots\cup A_k$,

$$\begin{aligned}\Delta_{R',T'}\mu_{A_i,\dots,A_k} &= \Delta_{B_i,U\cup T''}\mu_{B_i\cup U,T''}(\mathbf{1}_{A_i}\otimes\mu_{A_{i+1},\dots,A_k})\\ &= (\mathbf{1}_{B_i}\otimes\mu_{U,T''})(\Delta_{B_i,U}\otimes\mathbf{1}_{T''})(\mathbf{1}_{A_i}\otimes\mu_{A_{i+1},\dots,A_k})\\ &= (\mathbf{1}_{B_i}\otimes\mu_{U,T''})(\mathbf{1}_{B_i}\otimes\mathbf{1}_U\otimes\mu_{A_{i+1},\dots,A_k})(\Delta_{B_i,U}\otimes\mathbf{1}_{A_{i+1}}\otimes\cdots\otimes\mathbf{1}_{A_k})\\ &= (\mathbf{1}_{B_i}\otimes\mu_{U,A_{i+1},\dots,A_k})(\Delta_{B_i,U}\otimes\mathbf{1}_{A_{i+1}}\otimes\cdots\otimes\mathbf{1}_{A_k}).\end{aligned}$$

Remark that since $R'=B_i$,

$$\begin{aligned}\mu_C &= \mu_{P,T}(\mu_{A_1,\dots,A_{i-1},B_i}\otimes\mathbf{1}_{T'})(\mathbf{1}_{A_1}\otimes\cdots\otimes\mathbf{1}_{A_{i-1}}\otimes\mathbf{1}_{B_i}\otimes\mu_{U,A_{i+1},\dots,A_k})\\ &= \mu_{P,T}(\mu_{P',R'}\otimes\mathbf{1}_{T'})(\mu_{A_1,\dots,A_{i-1}}\otimes\mathbf{1}_{R'}\otimes\mathbf{1}_{T'})\\ &\qquad\qquad(\mathbf{1}_{A_1}\otimes\cdots\otimes\mathbf{1}_{A_{i-1}}\otimes\mathbf{1}_{B_i}\otimes\mu_{U,A_{i+1},\dots,A_k})\end{aligned}$$

and

$$\Delta_C=(\mathbf{1}_{A_1}\otimes\cdots\otimes\mathbf{1}_{A_{i-1}}\otimes\Delta_{B_i,U}\otimes\mathbf{1}_{A_{i+1}}\otimes\cdots\otimes\mathbf{1}_{A_k})\Delta_A.$$

Making use of the expressions given above for $\Delta_{P,T}\mu_A$, and comparing with $\mu_C\Delta_C$ we get

$$x_A=\mu_{P,T}(\Delta_{P,T}\mu_A)\Delta_A(x)=\mu_C\Delta_C(x)=x_C.$$

We conclude that the composition C satisfies (a), (b) and (c) contradicting the choice of A, hence we must have a unique minimal element in $(\mathcal{C}^{\beta,y}_{\alpha,x},\leqslant)$. □

For the rest of this section, let α,β,x and y be fixed and let $\Lambda=(\Lambda_1,\Lambda_2,\dots,\Lambda_m)$ be the minimum of $\mathcal{C}^{\beta,y}_{\alpha,x}\neq\varnothing$. For any $A\in\mathcal{C}^{\beta,y}_{\alpha,x}$ let $[\Lambda,A]$ denote the interval $\big\{B\models I:\Lambda\leqslant B\leqslant A\big\}\subseteq\mathcal{P}_I$. A priori, this interval does not need to be contained in $\mathcal{C}^{\beta,y}_{\alpha,x}$, but the following lemma shows that this is indeed the case.

Lemma 2.3. *If $\mathcal{C}^{\beta,y}_{\alpha,x}\neq\varnothing$, then for any $A\in\mathcal{C}^{\beta,y}_{\alpha,x}$ we have that $[\Lambda,A]\subseteq\mathcal{C}^{\beta,y}_{\alpha,x}$.*

Proof. Let $A=(A_1,A_2,\dots,A_k)\in\mathcal{C}^{\beta,y}_{\alpha,x}$. From Lemma 2.2 we know that $\Lambda\leqslant A$. We proceed by induction on $r=\ell(\Lambda)-\ell(A)$. If $r=0$, then we have that $A=\Lambda$ and the result follows. Suppose the result holds for $r>0$ and let A be such that $r+1=\ell(\Lambda)-\ell(A)$. Let $B=(B_1,\dots,B_{k+1})\in\mathcal{P}_I$ such that $\Lambda\leqslant B<A$ with $\ell(B)-\ell(A)=1$. Hence there is a unique i such that $A=(B_1,\dots,B_i\cup B_{i+1},B_{i+2},\dots,B_{k+1})$. We aim to show that $B\in\mathcal{C}^{\beta,y}_{\alpha,x}$, and then by induction hypothesis $[\Lambda,B]\subseteq\mathcal{C}^{\beta,y}_{\alpha,x}$.

Since $\Lambda\leqslant B$, there is a unique j such that

$$B_1\cup\cdots\cup B_i=\Lambda_1\cup\cdots\cup\Lambda_j.$$

Let $P=\Lambda_1\cup\cdots\cup\Lambda_j$ and $Q=\Lambda_{j+1}\cup\cdots\cup\Lambda_m$. Arguing as in equations (19) and (20) we have that

$$\begin{aligned}y=\mu_\Lambda\Delta_\Lambda(x) &= \mu_{P,Q}(\mu_{\Lambda_1,\dots,\Lambda_j}\otimes\mu_{\Lambda_{j+1},\dots,\Lambda_m})\Delta_\Lambda(x)\\ &= \mu_{P,Q}\Delta_{P,Q}\big(\mu_{P,Q}(\mu_{\Lambda_1,\dots,\Lambda_j}\otimes\mu_{\Lambda_{j+1},\dots,\Lambda_m})\Delta_\Lambda(x)\big)=\mu_{P,Q}\Delta_{P,Q}(y)\\ &= \mu_{P,Q}\Delta_{P,Q}(\mu_A\Delta_A(x))=\mu_B\Delta_B(x)=x_B.\end{aligned}$$

The same argument shows that $\beta=\mu_{P,Q}\Delta_{P,Q}\mu_A\Delta_A(\alpha)=\mu_B\Delta_B(\alpha)=\alpha_B$. Hence $B\in\mathcal{C}^{\beta,y}_{\alpha,x}$. We can now appeal to the induction hypothesis and conclude that for each such B the interval $[\Lambda,B]\subseteq\mathcal{C}^{\beta,y}_{\alpha,x}$ and thus the claim follows. Moreover, we conclude that $\mathcal{C}^{\beta,y}_{\alpha,x}$ is a lower ideal of the subposet $[\Lambda,(I)]=\{B\models I:\Lambda\leqslant B\}$. □

Lemma 2.4. *If $\mathcal{C}^{\beta,y}_{\alpha,x}\neq\varnothing$, then the minimal elements of $[\Lambda,(I)]\smallsetminus\mathcal{C}^{\beta,y}_{\alpha,x}$ are each of the form*

$$(\Lambda_1,\dots,\Lambda_{i-1},\Lambda_i\cup\Lambda_{i+1}\cup\cdots\cup\Lambda_j,\Lambda_{j+1},\dots,\Lambda_m)$$

for some $1\leqslant i<j\leqslant m$.

Proof. If $\mathcal{C}^{\beta,y}_{\alpha,x} \neq \varnothing$, let $B \in [\Lambda,(I)]$ be minimal such that $B \notin \mathcal{C}^{\beta,y}_{\alpha,x}$. That is, $\alpha_B \neq \beta$ or $x_B \neq y$. Let us first consider the case when $\alpha_B \neq \beta$. If $\alpha_B \neq \beta$ we must have at least one part of B that contains $\Lambda_i \cup \Lambda_{i+1}$ where the largest entry of $\alpha|_{\Lambda_i}$, say a, is such that $a >_\alpha b$, where b is the smallest entry of $\alpha|_{\Lambda_{i+1}}$. Hence,

$$\Lambda < B \leqslant (\Lambda_1, \dots, \Lambda_{i-1}, \Lambda_i \cup \Lambda_{i+1}, \Lambda_{i+2}, \dots, \Lambda_m),$$

hence $B = (\Lambda_1, \dots, \Lambda_{i-1}, \Lambda_i \cup \Lambda_{i+1}, \Lambda_{i+2}, \dots, \Lambda_m)$. Thus the claim follows when $\alpha_B \neq \beta$.

We now consider the case $x_B \neq y$. Assume that $B = (B_1, ..., B_k)$ has at least two parts that are unions of consecutive parts of Λ. Each part B_s of B is of the form $\Lambda_{a_s} \cup \cdots \cup \Lambda_{b_s}$, where $1 \leqslant a_s \leqslant b_s \leqslant m$. For each $1 \leqslant s \leqslant k$ consider the composition

$$C_{(s)} = (\Lambda_1, \dots, \Lambda_{a_s - 1}, B_s, \Lambda_{b_s+1}, \dots, \Lambda_m).$$

It follows that $C_{(s)}$ refines B (strictly) as there are at least two parts in B that are unions of consecutive parts of Λ. Hence $C_{(s)} \in \mathcal{C}^{\beta,y}_{\alpha,x}$ by the minimality of B, and thus $x_{C_{(s)}} = y$ for all $1 \leqslant s \leqslant k$. Hence,

$$\begin{aligned} x|_{\Lambda_1} \otimes \cdots \otimes x|_{\Lambda_{a_s-1}} \otimes x|_{B_s} \otimes x|_{\Lambda_{b_s+1}} \otimes \cdots \otimes x|_{\Lambda_m} \\ = \Delta_{C_{(s)}}(x_{C_{(s)}}) = \Delta_{C_{(s)}}(y) \\ = x|_{\Lambda_1} \otimes \cdots \otimes x|_{\Lambda_{a_s-1}} \otimes (x|_{\Lambda_{a_s}} \cdots x|_{\Lambda_{b_s}}) \otimes x|_{\Lambda_{b_s+1}} \otimes \cdots \otimes x|_{\Lambda_m} \end{aligned}$$

which gives us

$$x|_{B_s} = x|_{\Lambda_{a_s}} \cdots x|_{\Lambda_{b_s}}$$

for all $1 \leqslant s \leqslant k$. But this implies that

$$x_B = x|_{B_1} \cdots x|_{B_k} = (x_{\Lambda_{a_1}} \cdots x_{\Lambda_{b_1}}) \cdots (x_{\Lambda_{a_k}} \cdots x_{\Lambda_{b_k}}) = x_\Lambda = y,$$

which is a contradiction. Hence there is no more than one part of B that is not a single part of Λ. ☐

2.3. First Sign Reversing Involution on $c^{\beta,y}_{\alpha,x}$. Throughout this section recall that $\alpha, \beta \in \mathbf{l}[I]$ and $x, y \in \mathbf{h}[I]$ are fixed. If $\mathcal{C}^{\beta,y}_{\alpha,x} \neq \varnothing$, then we know that the subposet $(\mathcal{C}^{\beta,y}_{\alpha,x}, \leqslant)$ is a lower ideal with a unique minimum $\Lambda = (\Lambda_1, \Lambda_2, \dots, \Lambda_m)$. We define a sign reversing involution on the set $\mathcal{C}^{\beta,y}_{\alpha,x}$ that will cancel most of the terms in the signed sum

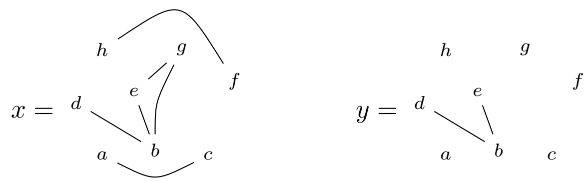
$$c^{\beta,y}_{\alpha,x} = \sum_{A \in \mathcal{C}^{\beta,y}_{\alpha,x}} (-1)^{\ell(A)}. \tag{21}$$

Using Lemma 2.4 we define an oriented graph $G^{\beta,y}_{\alpha,x}$ on the vertex set $[m]$ as follows. Two vertices a, b with $a < b$ form an oriented edge from a to b, denoted (a,b) or ab, for each minimal element of $[\Lambda,(I)] \backslash \mathcal{C}^{\beta,y}_{\alpha,x}$. More precisely, if $a < b$, then (a,b) is an edge in $G^{\beta,y}_{\alpha,x}$ if the following holds:

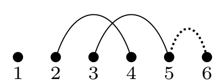
(1) Setting $B_{ab} = (\Lambda_1, \dots, \Lambda_{a-1}, \Lambda_a \cup \Lambda_{a+1} \cup \cdots \cup \Lambda_b, \Lambda_{b+1}, \dots, \Lambda_m)$, either $\alpha_{B_{ab}} \neq \beta$ or $x_{B_{ab}} \neq y$.
(2) For any $a < r < b$, we have $\alpha_{B_{ar}} = \beta = \alpha_{B_{rb}}$ and $x_{B_{ar}} = y = x_{B_{rb}}$.

We sometimes denote an edge (a,b) as ab to save space if the context is clear. Condition (1) guarantees that no element $A \in \mathcal{C}^{\beta,y}_{\alpha,x}$ induces an edge in $G^{\beta,y}_{\alpha,x}$. Condition (2) tells us that the edges in G are non-nesting, i.e., allows us to conclude that the graph G is *non-nesting*. That is, there are no pairs of edges ab and cd such that $a < c < d < b$.

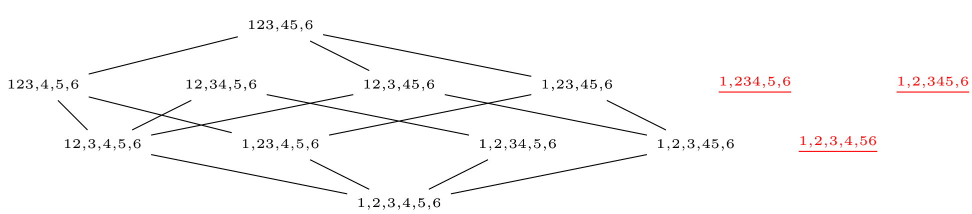
Example 2.5. Consider the Hopf monoid of graphs $\mathbf{G}$ linearized in the basis $\mathbf{g}$ as in Section 1.4. Let $I = \{a, b, c, d, e, f, g, h\}$, let $x, y \in \mathbf{g}[I]$ be the graphs



and let α, β be the orders $\alpha = abcdefgh$, $\beta = abdefghc$. The minimum element of $\mathcal{C}^{\beta,y}_{\alpha,x}$ is $\Lambda = (a, bde, f, g, h, c)$ (notice that indeed $x_\Lambda = y$ and $\alpha_\Lambda = \beta$). Since Λ has 6 parts, the graph $G^{\beta,y}_{\alpha,x}$ is build on the set $[6] = \{1, 2, \dots, 6\}$. We have

$G^{\beta,y}_{\alpha,x} =$  .

In particular, notice that $(1, 2)$ is not an edge as the element $B = (\Lambda_1 \cup \Lambda_2, \Lambda_3, \dots, \Lambda_6) = (abde, f, g, h, c) \in \mathcal{C}^{\beta,y}_{\alpha,x}$. The solid edges (i, j) indicate that $x_{B_{ij}} \neq y$, the dotted edge $(5, 6)$ indicates that $\alpha_{B_{56}} = abdefgch \neq \beta$. We now identify the set compositions in the interval $[\Lambda, (I)]$ with the set compositions of the interval $[(1, 2, \dots, m), (12 \cdots m)]$ and represent $\mathcal{C}^{\beta,y}_{\alpha,x}$ via the following poset



where the set compositions in red are the minimal compositions in $[\Lambda, (I)] \backslash \mathcal{C}^{\beta,y}_{\alpha,x}$ from Lemma 2.4.

Remark 2.6. As in the example above, from now on we will identify the set compositions in the interval $[\Lambda, (I)]$ with the set compositions in the interval $[(1, 2, \dots, m), (12 \cdots m)]$. Thus, each element $A \in \mathcal{C}^{\beta,y}_{\alpha,x}$ will be viewed as the corresponding set composition $A \models [m]$.

For any set composition B define its sign to be $sgn(B) := (-1)^{\ell(B)}$, where $\ell(B)$ is the length of B. We now define a sign reversing involution $\varphi\colon \mathcal{C}^{\beta,y}_{\alpha,x} \to \mathcal{C}^{\beta,y}_{\alpha,x}$, making use of auxiliary maps φ_i for each $1 \leqslant i < m$, as follows. Let $A = (A_1, \dots, A_k) \in \mathcal{C}^{\beta,y}_{\alpha,x}$ and let j be such that $i \in A_j$.

i-Merge. *If $A_j = \{i\}$ and (i, r) is not an edge of $G^{\beta,y}_{\alpha,x}$ for any $r \in A_{j+1}$, define*

$$\varphi_i(A) = (A_1, \dots, A_{j-1}, \{i\} \cup A_{j+1}, A_{j+2}, \dots, A_k).$$

i-Split. *If $|A_j| > 1$, $i = \min(A_j)$ and $\big(j = 1$ or $A_{j-1} \neq \{i - 1\}$ or $(i - 1, i) \in G^{\beta,y}_{\alpha,x}\big)$, then*

$$\varphi_i(A) = (A_1, \dots, A_{j-1}, \{i\}, A_j \smallsetminus \{i\}, A_{j+1}, \dots, A_k).$$

i-Fix. *If we do not have an i-merge or an i-split, then*

$$\varphi_i(A) = A.$$

Then the map φ is defined as

$$\varphi(A) := \begin{cases} A & \text{if } \varphi_i(A) = A \text{ for all } 1 \leqslant i < m, \\ \varphi_{i_0}(A) & \text{for } i_0 = \min\big\{i : \varphi_i(A) \neq A\big\}, \text{ otherwise.} \end{cases} \tag{22}$$

Lemma 2.7. *$\varphi\colon \mathcal{C}_{\alpha,x}^{\beta,y} \to \mathcal{C}_{\alpha,x}^{\beta,y}$ is an involution.*

Proof. Let $A \in \mathcal{C}_{\alpha,x}^{\beta,y}$. If $\varphi(A) = A$ the claim follows. Assume then that $\varphi(A) = A' \neq A$. Let $i_0 = \min\{i : \varphi_i(A) \neq A\}$, and thus $A' = \varphi_{i_0}(A)$. We first assume that A' is obtained from A by an i_0-split, then $A' < A$ and thus by Lemma 2.3, $A' \in \mathcal{C}_{\alpha,x}^{\beta,y}$. Moreover, applying an i_0-split to A guarantees that an (i_0-1)-merge can not be applied to A'. The minimality of i_0 guarantees that $\varphi_i(A') = A'$ for all $i < i_0$ and $\varphi(A') = \varphi_{i_0}(A') = A$ is obtained from A' by an i_0-merge as desired.

Now assume that A' is obtained by an i_0-merge. This implies that no part of A' contains (the vertices of) any edge of $G_{\alpha,x}^{\beta,y}$. Hence, $A' \in \mathcal{C}_{\alpha,x}^{\beta,y}$ by Lemma 2.4. Again, the minimality of i_0 guarantees that $\varphi_i(A') = A'$ for all $i < i_0$ and $\varphi(A') = \varphi_{i_0}(A') = A$ is obtained from A' by an i_0-split. Finally, notice that in either case, $sgn(\varphi(A)) \neq sgn(A)$ whenever $\varphi(A) \neq A$. ☐

2.4. Proof of Theorem 2.1. Lemma 2.7 tells us that every element A in the poset $\mathcal{C}_{\alpha,x}^{\beta,y}$ is either a fixed point, or is paired with a unique element $B \in \mathcal{C}_{\alpha,x}^{\beta,y}$ such that B is a covering of A or A covers it. Thus, equation (21) can be rewritten as:

$$c_{\alpha,x}^{\beta,y} = \sum_{\substack{A \in \mathcal{C}_{\alpha,x}^{\beta,y} \\ \varphi(A)=A}} (-1)^{\ell(A)}. \tag{23}$$

This depends only on the structure of the graph $G_{\alpha,x}^{\beta,y}$, which as remarked earlier, is non-nesting. In this section we let $G := G_{\alpha,x}^{\beta,y}$ be a non-nesting graph on the vertices $\{1, 2, \ldots, m\}$ and set $\mathcal{C}(G) := \mathcal{C}_{\alpha,x}^{\beta,y}$, $c(G) := c_{\alpha,x}^{\beta,y}$. Our next task is to describe the fixed points of $\varphi\colon \mathcal{C}(G) \to \mathcal{C}(G)$ in order to resolve equation (23). To this end, we now prove some auxiliary lemmas that show how $c(G)$ is affected by certain properties that the graph G may have.

Definition 2.8. *Let G be as above. We say that G is* decomposible *if there exists a vertex $1 \leqslant r < m$ such that there is no arc $(a, b) \in G$ with $a \in \{1, \ldots, r\}$ and $b \in \{r+1, \ldots, m\}$.*

Lemma 2.9. *If G is decomposible, then $c(G) = 0$.*

Proof. Let r be as in Definition 2.8. We construct a different sign reversing involution $\psi_r\colon \mathcal{C}(G) \to \mathcal{C}(G)$ with no fixed points, and thus the claim will follow. Let $A = (A_1, \ldots, A_k) \in \mathcal{C}(G)$ and let $r \in A_j$. If $r+1 \in A_j$ let

$$\psi_r(A_j) := (A_1, \ldots, A_{j-1}, \{\min(A_j), \ldots, r\}, \{r+1, \ldots, \max(A_j)\}, A_{j+1}, \ldots, A_k).$$

Thus by Lemma 2.3, $\psi_r(A_j) \in \mathcal{C}(G)$ since $\psi_r(A_j)$ refines A. If $r+1 \notin A_j$ then $r+1 = \min A_{j+1}$. In this case let

$$\psi_r(A) = (A_1, \ldots, A_{j-1}, A_j \cup A_{j+1}, A_{j+2}, \ldots, A_k).$$

Since G is decomposible at r we see that $\psi_r(A_j) \in \mathcal{C}(G)$, as desired. It is not difficult to check that in either case, $\psi_r(\psi_r(A_j)) = A_j$. This completes the proof. ☐

Lemma 2.10. *If $(i, i+1) \in G$ for some $1 \leqslant i < m$, then*

$$c(G) = c\big(G|_{\{1,\ldots,i\}}\big) \cdot c\big(G|_{\{i+1,\ldots,m\}}\big).$$

Proof. If $(i, i+1) \in G$ for some $1 \leqslant i < m$, then there is no other edge $(a, b) \in G$ with $a \leqslant i < b$ since G is non-nesting. Thus G is formed by the subgraphs $G' = G|_{\{1,\ldots,i\}}$ and $G'' = G|_{\{i+1,\ldots,m\}}$ together with the edge $(i, i+1)$ that connects G' and G''.

Moreover, in such case the set $\mathcal{C}(G)$ is isomorphic to $\mathcal{C}(G') \times \mathcal{C}(G'')$ since for any $A \in \mathcal{C}(G)$, i and $i+1$ must be separated in A. Hence

$$\begin{aligned} c(G) &= \sum_{A \in \mathcal{C}(G)} (-1)^{\ell(A)} \\ &= \sum_{(A',A'') \in \mathcal{C}(G') \times \mathcal{C}(G'')} (-1)^{\ell(A')+\ell(A'')} = c\big(G|_{\{1,\dots,i\}}\big) \cdot c\big(G|_{\{i+1,\dots,m\}}\big) \end{aligned}$$

as desired. $\square$

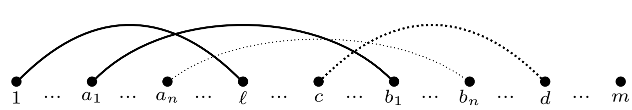
From Lemma 2.9 and Lemma 2.10, we can assume from now on that G is non-nesting, connected and with no short edges, i.e., edges of the form $(i, i+1)$. In particular, such G must contain an edge $(1, \ell) \in G$ with $2 < \ell \leqslant m$. Moreover, if $\ell = m$ it follows that $G = \{(1, m)\}$ and the only fixed point of φ is the set composition $A = (\{1\}, \{2, \dots, m\})$.

Now, assume that $2 < \ell < m$. Since G is connected, there must be an edge $(a, b) \in G$ such that $1 < a \leqslant \ell < b \leqslant m$. Consider the set of edges $\big\{(a_1, b_1), \dots, (a_n, b_n)\big\} \subseteq G$ such that

$$1 < a_1 < a_2 < \cdots < a_n \leqslant \ell < b_1 < b_2 < \cdots < b_n \leqslant m.$$

Lemma 2.11. *With $\big\{(a_1, b_1), \dots, (a_n, b_n)\big\}$ as above, we have that the fixed points of φ depend only on (a_1, b_1).*

Proof. Assume that $n > 1$ and that $A = (A_1, \dots, A_k) \in \mathcal{C}(G)$ is a fixed point of φ. We have that $A_1 = \{1\}$; otherwise we could perform a 1-split on A. Similarly, $A_2 = \{2, \dots, r\}$ and thus $\ell \leqslant r$; otherwise we could perform a 1-merge on A. Also, $\ell \leqslant r < b_1$ as the edge (a_1, b_1) can not be contained in A_2. Moreover, $|A_2| > 2$ and $\{r+1, \dots, m\}$ has at least two elements. Thus $A_3 = \{r+1, ...\}$ is nonempty. If $|A_3| > 1$, then we can perform an $r+1$-split which contradicts the choice of A. Hence, $A_3 = \{r+1\}$. Let $c = r+1$ and $A_4 = \{c+1, \dots, r'\}$. If there is no edge $(c, d) \in G$, then we would be allowed to do a c-merge on A, contradicting its choice. Thus such an edge (c, d) exists. Since G is non-nesting, we have $1 < a_1 \leqslant \ell < c \leqslant b_1 < b_n < d \leqslant m$. That is

(24) $G =$ 

Hence,

$$A = \big(\{1\}, \{2, \dots, c-1\}, \{c\}, \{c+1, \dots, r'\}, \dots\big) \tag{25}$$

where $r' \geqslant d$. Thus, the fixed point A does not depend on the edges $(a_2, b_2), \dots, (a_n, b_n)$, and the claim follows. $\square$

The proof of Lemma 2.11 gives us a necessary condition on the fixed points of φ.

Lemma 2.12. *Let G be connected with no small edges. If $A \in \mathcal{C}(G)$ is a fixed point of φ, then*

$$A = (\{1\}, \{2, \dots, x_2 - 1\}, \{x_2\}, \{x_2+1, \dots, x_4 - 1\}, \dots, \{x_{2k}\}, \{x_{2k}+1, \dots, m\})$$

when $\ell(A)$ is even, and

$$A = (\{1\}, \{2, \dots, x_2 - 1\}, \{x_2\}, \{x_2+1, \dots, x_4 - 1\}, \dots, \{x_{2k}\}, \{x_{2k}+1, \dots, m-1\}, \{m\})$$

when $\ell(A)$ is odd. In each case G contains, respectively, edges of the form

$$\big\{(x_0, y_0), (x_1, y_1), \dots, (x_{2k}, y_{2k})\big\} \text{ with } x_0 = 1 \text{ and } y_{2k} = m, \text{ or,}$$

$$\big\{(x_0, y_0), (x_1, y_1), \dots, (x_{2k}, y_{2k}), (x_{2k+1}, y_{2k+1})\big\} \text{ with } x_0 = 1 \text{ and } y_{2k+1} = m$$

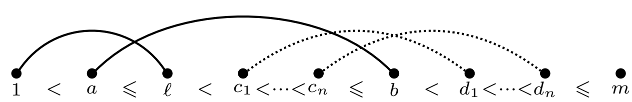
such that

(i) *for* $0 \leqslant i \leqslant k-1$ *we have* $x_{2i} < x_{2i+1} \leqslant y_{2i} < x_{2i+2} \leqslant y_{2i+1} < y_{2i+2}$,
(ii) *there is no edge* $(x,y) \in G$ *such that* $x_{2i} < x < x_{2i+1}$.

Proof. The case where G has only one edge was considered prior to Lemma 2.11. In this case, the unique fixed point is $A = (\{1\}, \{2, \ldots, m\})$.

If G has more than one edge, Lemma 2.11 tells us that the fixed points of φ depend only on edges of the form $(1,\ell)$, (a_1, b_1) and the possible (c,d) as in equation (24). If there is no such edge (c,d), then $G = \big\{(1,\ell), (a_1,b_1), \ldots, (a_n,b_n)\big\}$, where $1 < a_j \leqslant \ell < b_j \leqslant m$ and $b_n = m$. For $n > 1$, we have seen in the proof of Lemma 2.11 that if there is no arc $(c,d) \in G$ with $1 < a_1 < a_n \leqslant \ell < c \leqslant b_1 < b_n < d \leqslant m$, then there is no fixed point of φ. If $n = 1$, then $G = \big\{(1,\ell), (a,m)\big\}$ for $1 < a \leqslant \ell < m$. Our analysis shows that in this case there is a unique fixed point $A = (\{1\}, \{2, \ldots, m-1\}, \{m\})$. Here $\ell(A) = 3$ is odd, $k = 0$ and again all the conditions of the lemma are satisfied.

Assume now that G has an edge (c,d) as in equation (24). Since for $j > 1$, the edges (a_j, b_j) do not play a role in our analysis of the fixed point of φ, we can omit them. Let $(a,b) = (a_1,b_1)$ and consider the set of arcs $\big\{(c_1,d_1), \ldots, (c_n,d_n)\big\} \subseteq G$ such that $\ell < c_j \leqslant b < d_j \leqslant m$. We now have

(26) $G =$ 

For each $1 \leqslant j < n-1$, a potential fixed point according to Equation (25) would need to be of the form

$$A = (\{1\}, \{2, \ldots, c_j - 1\}, \{c_j\}, \{c_j+1, \ldots, r\}, \{r+1, \ldots\}, \ldots) \tag{27}$$

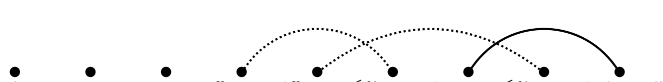
where $d_j \leqslant r < d_{j+1}$. The second inequality comes from the fact that we are not allowed to have c_{j+1} and d_{j+1} in the same part. Hence if A is a fixed point it must have the form described in Equation (27) and we must have

$$\big\{(x_0,y_0), (x_1,y_1), (x_2,y_2), \ldots\big\} = \big\{(1,\ell), (a,b), (c_j,d_j) \ldots\big\} \subseteq G,$$

where $1 < a \leqslant \ell < c_j \leqslant b < d_j$ and there is no edge $(x,y) \in G$ such that $1 < x < a$. The remaining structure of the fixed point in Equation (27) depends only on the structure of the smaller graph $G|_{\{c_j,\ldots,m\}}$. The result then follows by induction on the size of G. □

Now that we have a better understanding of the possible structure of the fixed points of φ, it may appear that there are many possibilities. It turns out that there could be at most two fixed points of different parity.

Proof of Theorem 2.1. Let A be a fixed point of φ. Assume first that $\ell(A)$ is even. Lemma 2.12 gives that we must have edges $\big\{(x_0,y_0), (x_1,y_1), \ldots, (x_{2k},y_{2k})\big\} \subseteq G$ satisfying the conditions (i) and (ii). If $k = 0$, then $G = \{(1,m)\}$ and there is a unique fixed point $A = (\{1\}, \{2, \ldots, m\})$. Now assume that $k > 0$, in which case $y_{2k} = m$ and the edge (x_{2k}, y_{2k}) is determined. With $i = k-1$ in condition (i) of Lemma 2.12 we have

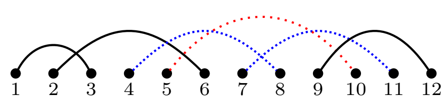
(28) $G =$  ,

and condition (ii) on the edges (x_{2k-2}, y_{2k-2}), (x_{2k-1}, y_{2k-1}) must also satisfy the condition (ii) of Lemma 2.12. Thus these edges (x_{2k-1}, y_{2k-1}) and (x_{2k-2}, y_{2k-2}) are uniquely determined and are such that they bound the vertex x_{2k} on the right and on the left, respectively, i.e. $y_{2k-2} < x_{2k} \leqslant y_{2k-1}$. In this way, $\{(x_{2k-2}, y_{2k-2}), (x_{2k-1}, y_{2k-1}), (x_{2k}, y_{2k})\}$ are uniquely determined. Now we can repeat the process with $i = k-2, k-3, \ldots, 0$ in condition (ii) of Lemma 2.12 to successively determine the edges $\{(x_0, y_0), (x_1, y_1), \ldots, (x_{2k}, y_{2k})\} \subseteq G$, and the partition A is given as in Lemma 2.12.

The case when the fixed point A has odd length is very similar. The condition (i) of Lemma 2.12 gives $y_{2k+1} = m$ and hence determines the edge (x_{2k+1}, y_{2k+1}) . Then condition (ii) of Lemma 2.12 with $i = k-1$ determines uniquely (if it exists) (x_{2k}, y_{2k}) as the rightmost edge of G such that $y_{2k} < y_{2k+1}$. Once x_{2k} is determined we continue the process as above with $i = k-2, k-3, \ldots, 0$ to determine uniquely, if possible, all the other edges. Again, if at any time in the process we fail, then there is no fixed point with $\ell(A)$ odd. If we do not fail, there is a unique fixed point with $\ell(A)$ odd.

In conclusion, there are four possibilities. We could have no fixed point and in this case $c(G) = 0$; we could have exactly one fixed point of odd length and $c(G) = -1$; we could have exactly one fixed point of even length and $c(G) = 1$; or we have exactly two fixed points of different parity each and $c(G) = 0$ in that case. In all cases Theorem 2.1 follows. □

Example 2.13. It is not hard to obtain examples with zero or one fixed point. The smallest example with two fixed points is for $n = 12$

(29) $G =$  .

An even fixed point is given by the arcs $\{(1,3), (2,6), (4,8), (7,11), (9,12)\}$ and the odd fixed point is given by $\{(1,3), (2,6), (5,10), (9,12)\}$. The arc $(9,12)$ is determined and the odd or even fixed points are determined from there.

Remark 2.14. Once a non-nesting graph G is given, the value of $c(G)$ is very efficient to compute. Lemma 2.9 gives us that $c(G) = 0$ if G is decomposible. Then we decompose G according to Lemma 2.10 into components G' with no short edges. For each component, we follow the procedure in the proof of Theorem 2.1 to determine if there is an even and/or an odd fixed point. This gives us quickly the value of $c(G')$ for each component G'.

Remark 2.15. The graph $G = G^{\beta,y}_{\alpha,x}$ is in fact the element $(12\cdots m, g)$ of $\mathbf{L} \times \mathbf{G}[m]$ where $12\cdots m$ is the natural order on $[m] = \{1, 2, \ldots, m\}$ and $g = G$. Let ϵ denote the graph with no edges. The reader can verify using the analysis above that $c^{\beta,y}_{\alpha,x} = c^{12\cdots m,\epsilon}_{12\cdots m,g}$ where $c^{\beta,y}_{\alpha,x}$ is the coefficient of (β, y) in $S(\alpha, x)$ for the antipode of $\mathbf{L} \times \mathbf{H}$ and $c^{12\cdots m,\epsilon}_{12\cdots m,g}$ is the coefficient of $(12\cdots m, \epsilon)$ in $S(12\cdots m, g)$ for the antipode of $\mathbf{L} \times \mathbf{G}$. The antipode of a general linearizable $\mathbf{L} \times \mathbf{H}$ can be compute from the Hopf monoid $\mathbf{L} \times \mathbf{G}$. This situation is analogous to Theorem 3.7 below.

3. Antipode for commutative linearized Hopf monoid H

In this section we show new antipode formulas for a commutative and cocommutative linearized Hopf monoid $\mathbf{H}$. Our formulas for the antipode of $\mathbf{H}$ lead to formulas for the antipode of the Hopf algebra $\overline{\mathcal{K}}(\mathbf{H})$. We also aim to introduce a geometric interpretation related to our antipode formula in terms of certain faces of a polytope in the spirit of the work of Aguiar and Ardila [1]. To achieve this, first we give a formula for the antipode in terms of orientations of hypergraphs in Section 3.4. The geometric interpretation related to our antipode formula appears in a sequel paper with J. Machacek [8].

3.1. Takeuchi's formula for $\mathbf{H}$. Let $\mathbf{H}$ be a Hopf monoid linearized in the basis $\mathbf{h}$. Again, we intend to resolve the cancellations in the Takeuchi formula for the antipode of $\mathbf{H}$. For a fixed finite set I let $x \in \mathbf{h}[I]$. From (5) we have

$$S_I(x) = \sum_{A \models I} (-1)^{\ell(A)} \mu_A \Delta_A(x) = \sum_{\substack{A \models I \\ \Delta_A(x) \neq 0}} (-1)^{\ell(A)} x_A, \tag{30}$$

where for $(A_1, \dots, A_k) \models I$ and $\Delta_A(x) \neq 0$ we write $x_A = \mu_A \Delta_A(x) \in \mathbf{h}[I]$. These elements are well-defined since $\mathbf{H}$ is linearized in the basis $\mathbf{h}$. We can thus rewrite equation (30) as

$$S_I(x) = \sum_{y \in \mathbf{h}[I]} \left(\sum_{\substack{A \models I \\ x_A = y}} (-1)^{\ell(A)} \right) y. \tag{31}$$

Let

$$\mathcal{C}_x^y = \big\{ A \models I : x_A = y \big\}.$$

So far we have not considered the commutativity of $\mathbf{H}$. In general we have no control on the set $\mathcal{C}_x^y$, but when $\mathbf{H}$ is commutative and cocommutative, our next theorem is a new formula for the antipode of $\mathbf{H}$. The result and its proof are very similar to analogous results in [10, 12]. In order to state this result, we need some more notation. Given $x, y \in \mathbf{h}[I]$ such that $\mathcal{C}_x^y \neq \varnothing$, choose a fixed minimal element $\Lambda = (\Lambda_1, \Lambda_2, \dots, \Lambda_m)$ in $\mathcal{C}_x^y$ under refinement. We will see in Lemma 3.3 that Λ is unique up to permutation of its parts, hence let $\Delta_\Lambda(x) = x_{\Lambda_1} \otimes \cdots \otimes x_{\Lambda_m} \neq 0$. The cocommutativity of $\mathbf{H}$ implies that for $P = \Lambda_i \subseteq I$ the element $x_P = x_{\Lambda_i}$ in $\Delta_\Lambda(x) = x_{\Lambda_1} \otimes \cdots \otimes x_{\Lambda_m}$ is well defined by the first component of the tensor $\Delta_{P, I \smallsetminus P}(x) = x_P \otimes x_{I \smallsetminus P}$. Recall that a *hypergraph* G on a vertex set V is a certain collection E of subsets of V. The elements of E are called *hyperedges* and the hypergraph G is *simple* if E is multiplicity free. We now define a simple hypergraph G_x^y on the vertex set $[m]$ such that $U \subseteq [m]$ is a hyperedge of G_x^y if and only if

$$\prod_{i \in U} x_{\Lambda_i} \neq x_{\cup_{i \in U} \Lambda_i} \qquad \text{and} \qquad \forall (P \subset U) \quad \prod_{i \in P} x_{\Lambda_i} = x_{\cup_{i \in P} \Lambda_i}. \tag{32}$$

Up to reordering of the vertices $\{1, 2, \dots, m\}$, commutativity, cocommutativity and Lemma 3.3 will guarantee that G_x^y does not depend on our choice of Λ.

Theorem 3.1. *For $\mathbf{H}$ a commutative and cocommutative Hopf monoid linearized in the basis $\mathbf{h}$, we have*

$$S_I(x) = \sum_{y \in \mathbf{h}[I]} a(G_x^y) y, \tag{33}$$

where $a(G_x^y)$ is a signed sum of acyclic orientations of the hypergraph G_x^y defined in Section 3.4*.*

Remark 3.2. If G_x^y is a graph, that is, any hyperedge $U \in G_x^y$ is such that $|U| = 2$, then every acyclic orientation will have the same sign, as seen in Example 4.2. Hence the theorem above gives a cancellation free formula similar to the antipode as shown in [16]. In general it will not be cancellation free but it is the best generalization, to our knowledge, for hypergraphs and to a large class of Hopf monoids and Hopf algebras.

3.2. Structure of $\mathcal{C}_x^y$ and its hypergraph G_x^y. Before we prove Theorem 3.1 we need to establish some properties of $\mathcal{C}_x^y = \{A \models I : x_A = y\}$. This will allow us to determine the coefficient of y in $S(x)$ given by

$$c_x^y = \sum_{A\in\mathcal{C}_x^y} (-1)^{\ell(A)}. \tag{34}$$

Lemma 3.3. *If A and Λ in $\mathcal{C}_x^y$ are two minimal set compositions under refinement, then A is a permutation of the parts of Λ. Conversely, any set composition obtained by a permutation of the parts of Λ belongs to $\mathcal{C}_x^y$ and is minimal.*

Proof. Given any $B = (B_1, B_2, \dots, B_k) \in \mathcal{C}_x^y$ and any permutation $\sigma\colon [k] \to [k]$, we have that $\sigma(B) := (B_{\sigma(1)}, \dots, B_{\sigma(k)}) \in \mathcal{C}_x^y$. Indeed, this follows from commutativity and cocommutativity since $x_{\sigma(B)} = x_B = y$. Now if $\Lambda = (\Lambda_1, \dots, \Lambda_m) \in \mathcal{C}_x^y$ is a minimal set composition under refinement, then $\sigma(\Lambda)$ is in $\mathcal{C}_x^y$ for any permutation $\sigma\colon [m] \to [m]$. Furthermore $\sigma(\Lambda)$ must be minimal under refinement for if $B < \sigma(\Lambda)$ such that $B \in \mathcal{C}_x^y$, then we can find a permutation τ such that $\tau(B) < \Lambda$ and $\tau(B) \in \mathcal{C}_x^y$. This would contradict the minimality of Λ. This shows the second part of the lemma.

Now suppose $A = (A_1, \dots, A_\ell) \in \mathcal{C}_x^y$ is another minimal set composition under refinement. Assume that $A \neq \sigma(\Lambda)$ for any σ. We claim that there is a rearrangement of the parts of Λ and A such that $\varnothing \neq U_1 := A_1 \cap \Lambda_1 \neq \Lambda_1$. If not, then for all i, j such that $A_i \cap \Lambda_j \neq \varnothing$ we would have $A_i \cap \Lambda_j = \Lambda_j$ and this would imply that a permutation of Λ is a refinement of A, a contradiction. We can further rearrange the parts of Λ such that $U_i := A_1 \cap \Lambda_i \neq \varnothing$ for $1 \leqslant i \leqslant r$ and $A_1 \cap \Lambda_i = \varnothing$ for $r < i \leqslant m$. As in Equation (19), for $T = A_2 \cup \cdots \cup A_\ell$ we have

$$y = x_A = \mu_{A_1,T}\Delta_{A_1,T}(x_A). \tag{35}$$

Let $V_i = \Lambda_i \smallsetminus U_i = \Lambda_i \cap T$ for $1 \leqslant i \leqslant r$. As in the proof of Lemma 2.2, we claim that the set composition

$$C = (U_1, V_1, \dots, U_r, V_r, \Lambda_{r+1} \dots, \Lambda_m) < \Lambda,$$

(we remove any occurrence of $\varnothing$ parts), and C belong to $\mathcal{C}_x^y$. The refinement is strict since $U_1 \neq \Lambda_1$ and this contradicts the minimality of Λ, hence no such A exists.

To show our last claim, we apply Equation (35) to $x_\Lambda = y = x_A$. Let $\Lambda_{i\cdot\cdot m} = \Lambda_i \cup \Lambda_{i+1} \cup \cdots \cup \Lambda_m$, $U_{i\cdot\cdot r} = A_1 \cap \Lambda_{i\cdot\cdot m}$ and $T_{i\cdot\cdot m} = T \cap \Lambda_{i\cdot\cdot m}$ we have

$$y = \mu_{A_1,T}\Delta_{A_1,T}(x_\Lambda) = \mu_{A_1,T}\Delta_{A_1,T}\mu_{\Lambda_1,\Lambda_{2\cdot\cdot m}}(\mathbf{1}_{\Lambda_1} \otimes \mu_{(\Lambda_2,\dots,\Lambda_m)})\Delta_\Lambda(x). \tag{36}$$

We now apply the compatibility (4), associativity and commutativity to obtain

$$\begin{aligned}
&\mu_{A_1,T}\Delta_{A_1,T}\mu_{\Lambda_1,\Lambda_{2\cdot\cdot m}} \\
&\qquad = \mu_{A_1,T}(\mu_{U_1,U_{2\cdot\cdot r}} \otimes \mu_{V_1,T_{2\cdot\cdot m}})(\mathbf{1}_{U_1} \otimes \tau_{V_1,U_{2\cdot\cdot r}} \otimes \mathbf{1}_{T_{2\cdot\cdot m}})(\Delta_{U_1,V_1} \otimes \Delta_{U_{2\cdot\cdot r},T_{2\cdot\cdot m}}) \\
&\qquad = \mu_{U_1,V_1,U_{2\cdot\cdot r},T_{2\cdot\cdot m}}(\Delta_{U_1,V_1} \otimes \Delta_{U_{2\cdot\cdot r},T_{2\cdot\cdot m}}) \\
&\qquad = \mu_{\Lambda_1,\Lambda_{2\cdot\cdot m}}\big(\mu_{U_1,V_1}\Delta_{U_1,V_1} \otimes \mu_{U_{2\cdot\cdot r},T_{2\cdot\cdot m}}\Delta_{U_{2\cdot\cdot r},T_{2\cdot\cdot m}}\big).
\end{aligned}$$

Putting this back in Equation (36) we get

$$\begin{aligned}
y &= \mu_{\Lambda_1,\Lambda_{2\cdot\cdot m}}\big(\mu_{U_1,V_1}\Delta_{U_1,V_1} \otimes \mu_{U_{2\cdot\cdot r},T_{2\cdot\cdot m}}\Delta_{U_{2\cdot\cdot r},T_{2\cdot\cdot m}}\big)(\mathbf{1}_{\Lambda_1} \otimes \mu_{(\Lambda_2,\dots,\Lambda_m)})\Delta_\Lambda(x) \\
&= \mu_{\Lambda_1,\Lambda_{2\cdot\cdot m}}\big(\mu_{U_1,V_1}\Delta_{U_1,V_1} \otimes \mu_{U_{2\cdot\cdot r},T_{2\cdot\cdot m}}\Delta_{U_{2\cdot\cdot r},T_{2\cdot\cdot m}}\mu_{(\Lambda_2,\dots,\Lambda_m)}\big)\Delta_\Lambda(x).
\end{aligned}$$

If $r = 1$, then $U_{2\cdot\cdot r} = \varnothing$ and $\mu_{U_{2\cdot\cdot r},T_{2\cdot\cdot m}} = \Delta_{U_{2\cdot\cdot r},T_{2\cdot\cdot m}} = \mathbf{1}_{T_{2\cdot\cdot m}}$. In this case we get

$$\begin{aligned}
y &= \mu_{\Lambda_1,\Lambda_{2\cdot\cdot m}}\big(\mu_{U_1,V_1}\Delta_{U_1,V_1} \otimes \mu_{(\Lambda_2,\dots,\Lambda_m)}\big)\Delta_\Lambda(x) \\
&= \mu_{\Lambda_1,\Lambda_{2\cdot\cdot m}}(\mu_{U_1,V_1} \otimes \mu_{(\Lambda_2,\dots,\Lambda_m)})(\Delta_{U_1,V_1} \otimes \mathbf{1}_{\Lambda_2} \otimes \cdots \otimes \mathbf{1}_{\Lambda_m})\Delta_\Lambda(x) \\
&= \mu_C\Delta_C(x) = x_C.
\end{aligned}$$

If $r>1$, then we repeat the process above with $\mu_{U_{i\cdots r},T_{i\cdots m}}\Delta_{U_{i\cdots r},T_{i\cdots m}}\mu_{(\Lambda_i,\dots,\Lambda_m)}$ for $2\leqslant i\leqslant r$ and we obtain $y=x_C$ in all cases. This shows that $C\in\mathcal{C}_x^y$ contradicting the minimality of Λ. $\square$

We now consider the analogue to Lemmas 2.3 and 2.4 for $\mathbf{H}$.

Lemma 3.4. *If $\mathcal{C}_x^y\neq\varnothing$, then for any $A\in\mathcal{C}_x^y$ and $\Lambda\in\mathcal{C}_x^y$ minimal, we have that $[\Lambda,A]\subseteq\mathcal{C}_x^y$.*

Proof sketch. If $[\Lambda,A]=\varnothing$, then the lemma is trivially true. If $\Lambda\leqslant A$, then the proof is exactly as in Lemma 2.3. This shows that $\mathcal{C}_x^y$ is a lower ideal in the order $\bigcup_{\sigma\in S_m}[\sigma\Lambda,(I)]$. $\square$

The next lemma shows us how $\mathcal{C}_x^y$ cuts off from $\bigcup_{\sigma\in S_m}[\sigma\Lambda,(I)]$.

Lemma 3.5. *The minimal elements of $\big(\bigcup_{\sigma\in S_m}[\sigma\Lambda,(I)]\big)\smallsetminus\mathcal{C}_x^y$ are all the permutations of set compositions of the form*

$$\Big(\bigcup_{i\in U}\Lambda_i,\Lambda_{v_1},\Lambda_{v_2},\dots,\Lambda_{v_r}\Big)$$

for some $U\in\{1,2,\dots,m\}$, where $r=m-|U|$ and $\{v_1,\dots,v_r\}=I\smallsetminus U$.

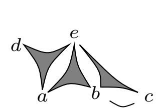
Proof sketch. This proof is a direct adaptation of the proof of Lemma 2.4. It is clear that the upper ideal $\big(\bigcup_{\sigma\in S_m}[\sigma\Lambda,(I)]\big)\smallsetminus\mathcal{C}_x^y$ is invariant under permutations. Let $B\in\big(\bigcup_{\sigma\in S_m}[\sigma\Lambda,(I)]\big)\smallsetminus\mathcal{C}_x^y$ be minimal. If B has more than two parts that are not single parts of Λ, then let $\sigma\in S_m$ be such that $\sigma\Lambda<B$ and proceed as in the second part of the proof on Lemma 2.4 to reach a contradiction. $\square$

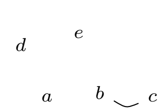
We now define the hypergraph G_x^y associated with $\mathcal{C}_x^y$. For fixed x and y, Lemma 3.5 gives us a set of subsets $U\subseteq I$ defining $\mathcal{C}_x^y$. Let

$$G_x^y=\{U\subseteq I\ :\ U\text{ minimal},\ \prod_{i\in U}x_{\Lambda_i}\neq x_{\bigcup_{i\in U}\Lambda_i}\}.$$

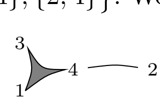
In general, G_x^y is such that all of its hyperedges have cardinality at least 2 and if $U\in G_x^y$ then for all $U\subset V\subseteq I$, we have $V\notin G_x^y$. this second property follows from the minimality of the element of $\big(\bigcup_{\sigma\in S_m}[\sigma\Lambda,(I)]\big)\smallsetminus\mathcal{C}_x^y$. The hypergraph G_x^y is thus, as defined in Equation (32).

Example 3.6. Let $\mathbf{HG}$ be as in Section 1.5. Consider $I=\{a,b,c,d,e\}$ and pick $x=\big\{\{b,c\},\{a,b,e\},\{a,d,e\},\{b,c,e\}\big\}$ and $y=\big\{\{b,c\}\big\}$ in $\mathbf{hg}[I]$. We can represent x and y as follows:

$x=$ 

$y=$ 

Up to permutation, the minimum refinement of $\mathcal{C}_x^y$ is $\Lambda=(a,bc,d,e)$. Since Λ has 4 parts, the hypergraph G_x^y is build on the set $\{1,2,3,4\}$. We have that $x_{bc}x_e\neq x_{bce}$ and $x_ax_dx_e\neq x_{ade}$. Those are the only minimal coarsening of parts of Λ that yield such inequalities. Hence $G_x^y=\big\{\{1,3,4\},\{2,4\}\big\}$. We represent this as follows:

$G_x^y=$ 

We now identify the set compositions in $\bigcup_{\sigma\in S_m}[\sigma\Lambda,(I)]$ with the set compositions in $\bigcup_{\sigma\in S_m}[(\sigma(1),\dots,\sigma(m)),(12\cdots m)]$. There are 4! minimal elements with four parts. There are 30 compositions with 3 parts, namely all the permutation of $(12,3,4),(13,2,4),(14,2,3),(23,1,4),(34,1,2)$. We have removed here all the permutations of $(24,1,3)$. With 2 parts we have all the permutations of $(123,4),(12,34),(14,23)$ for a total of 6. We have removed the permutations of $(134,2)$ and all the coarsenings of permutations of $(24,1,3)$. Here

$$c_x^y = 24 - 30 + 6 = 0.$$

The identification between $\bigcup_{\sigma\in S_m}[\sigma\Lambda,(I)]$ and $\bigcup_{\sigma\in S_m}[(\sigma(1),\ldots,\sigma(m)),(12\cdots m)]$ shows that computing c_x^y is equivalent to computing the coefficient of ϵ, the hypergraph on $[m]$ with no edges, in the antipode of G_x^y in the Hopf monoid of hypergraphs. This implies the following theorem.

Theorem 3.7. *Given* $\mathbf{H}$ *a commutative and cocommutative Hopf monoid linearized in the basis* $\mathbf{h}$*, let* $x,y\in\mathbf{h}[I]$*. We have that*[(1)]

$$c_x^y = c_{x/y}^{\epsilon}$$

where ϵ *is the hypergraph on* $[m]$ *with no edges and* $x/y = G_x^y$ *is the hypergraph given in* (32)*.*

Remark 3.8. In [1], the authors also consider a Hopf monoid of hypergraphs. In is important to notice that our Hopf monoid **HG** is cocommutative whereas the Hopf monoid considered in [1] is not. In particular, the antipode formula for hypergraphs in [1] differs from ours.

3.3. A different formula for c_x^y. Although the results in this section are not needed for the proof of Theorem 3.1, we present them as an application of Section 2. Using Remark 2.14 we now give a more efficient formula to compute c_x^y. In order to do this we decompose the poset $\mathcal{C}_x^y$ into disjoint suborders, one for each permutation of S_m, where as before, we identify $\mathcal{C}_x^y$ with a lower ideal of $\bigcup_{\sigma\in S_m}[\sigma,(12\cdots m)]$. Given $A=(A_1,A_2,\ldots,A_\ell)\in\mathcal{C}_x^y$, we obtain a unique refinement $\sigma(A)<A$ by ordering increasingly each of the parts A_i and then splitting them into singletons. For example if $A=(\{2,5,7\},\{1\},\{3,4,9\},\{6,8\})$, then $\sigma(A)=(2,5,7,1,3,4,9,6,8)$. Let

$$\mathcal{C}_{x,\tau}^y = \big\{A\in\mathcal{C}_x^y : \sigma(A)=\tau\big\}.$$

With these definitions in mind we state the following proposition.

Proposition 3.9. *For any* $x,y\in\mathbf{h}[I]$ *such that* $\mathcal{C}_x^y\neq\varnothing$*, let* $\Lambda=(\Lambda_1,\ldots,\Lambda_m)\in\mathcal{C}_x^y$ *be a fixed minimal element. We have that*

$$c_x^y = \sum_{\tau\in S_m} c(G_{12\cdots m,x/y}^{\tau,\epsilon})$$

where ϵ *is the hypergraph on the set* $[m]$ *with no edges and* $x/y=G_x^y$ *is the hypergraph given in Equation* (32)*.*

Proof. From the definition of $\mathcal{C}_x^y$ it is clear that $\mathcal{C}_x^y=\biguplus_{\tau\in S_m}\mathcal{C}_{x,\tau}^y$. For a fixed τ, we have that $A\in\mathcal{C}_{x,\tau}^y$ if and only if

$$\tau=(12\cdots m)|_A \qquad \text{and} \qquad G_x^y\big|_A=\epsilon.$$

This gives

$$\sum_{A\in\mathcal{C}_{x,\tau}^y}(-1)^{\ell(A)} = c(G_{12\cdots m,x/y}^{\tau,\epsilon})$$

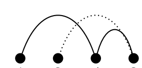
where $c(G_{12\cdots m,x/y}^{\tau,\epsilon})$ is the coefficient of (τ,ϵ) in the expansion of $S(12\cdots m,x/y)$ for the Hopf monoid $\mathbf{L}\times\mathbf{HG}$ with $\mathbf{HG}$ as defined in Section 1.5. □

(1) The reader should be aware of the abuse of notation here: on one hand c_x^y is an antipode coefficient in the Hopf monoid $\mathbf{H}$, on the other hand $c_{x/y}^{\epsilon}$ is an antipode coefficient in the Hopf monoid $\mathbf{HG}$.

Theorem 2.1 tells us that $c(G^{\tau,\epsilon}_{12\cdots m,x/y})$ is 0 or ± 1. Proposition 3.9 gives us an interesting new way to compute antipodes, as a sum over permutations instead of a sum of set compositions.

Example 3.10. We compute the coefficient of the hypergraph ϵ in the antipode $S(x)$ of the hypergraph $x = \big\{\{1,2,4\},\{2,3,4\}\big\} \in \mathbf{HG}[\mathbf{4}]$. Let $\tau = 1243$ and recall the construction of the graph $G = G^{\tau,\epsilon}_{1234,x/\epsilon}$ as in Example 2.5. This is a graph on the ordered vertex set 1243 such that there is an arc $(i, i+1)$ for each descent $\tau(i) > \tau(i+1)$. Also, we draw an arc (i,j) for each hyperedge $U \in G$ where $i = \min_\tau(U)$ and $j = \max_\tau(U)$ are the minimum and maximum values of U according to the order τ. Then we erase all drawn arcs that contain a nested arc. With τ as above, we have the arc $(4,3)$ from the descent of τ and the arcs $(1,4)$ and $(2,3)$ for the hyperedges $\{1,2,4\}$ and $\{2,3,4\}$ respectively. Then we erase the arc $(2,3)$ since it contains the nested arc $(4,3)$. The resulting graph is

$$G = G^{1243,\epsilon}_{1234,x/\epsilon} =$$

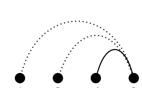


where the dotted arcs correspond to the removed edges. Then we get

$$c(G) = c(G|_{124}) \cdot c(G|_3) = (1) \cdot (-1)$$

where the first equality comes from Lemma 2.10 and the second equality follows by Lemma 2.12 since the only fixed point adding up to $c(G|_{124})$ is the composition $(1,24)$, which contributes to 1; similarly, the only fixed point adding up to $c(G|_3)$ is the composition (3) which contributes to (-1). For different τ's in this example, we get a decomposible graph and Lemma 2.9 gives us $c(G^{\tau,\epsilon}_{1234,x/\epsilon}) = 0$ in those cases. For example,

$$G^{1342,\epsilon}_{1234,x/\epsilon} =$$



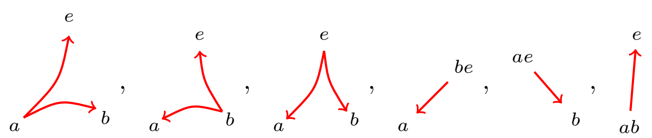
.

Removing the dotted arcs produce a decomposible graph; hence the result is zero. The only permutations τ that will contribute non-trivially are $1243, 2341, 3124, 4321$ with signs $-1, -1, -1, 1$ respectively. Hence the coefficient of ϵ in $S(x)$ is $-1-1-1+1 = -2$.

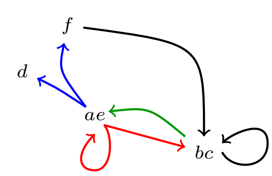
3.4. c^y_x as a signed sum of acyclic orientations of simple hypergraphs. We now turn to Theorem 3.1 to get an antipode formula for c^y_x as a signed sum of acyclic orientations of the hypergraph G^y_x. When G^y_x is a graph, then we will recover the formula of Humpert and Martin [16]. If G^y_x is an arbutrary hypergraph, then the antipode formula may still have cancellation but, in sequel work [8], we make sense of this formula geometrically. Recall that G^y_x is a hypergraph on the vertex set $[m]$ as defined in Equation (32). The ordering of the vertex set depends on a fixed choice of minimal element in $\mathcal{C}^y_x$.

Definition 3.11 (Orientation). *Given a hypergraph G an* orientation $(\mathfrak{a},\mathfrak{b})$ *of a hyperedge $U \in G$ is a choice of two nonempty subsets $\mathfrak{a},\mathfrak{b}$ of U such that $U = \mathfrak{a} \cup \mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b} = \varnothing$. We can think of the orientation of a hyperedge U as current or flow on U from a single vertex of $\mathfrak{a}$ to the vertices in $\mathfrak{b}$ in which case we say that $\mathfrak{a}$ is the* head *of the orientation $\mathfrak{a} \to \mathfrak{b}$ of U. If $|U| = n$, then there are a total of $2^n - 2$ possible orientations. An* orientation of G *is an orientation of all its hyperedges. Given an orientation $\mathcal{O}$ on G, we say that $(\mathfrak{a},\mathfrak{b}) \in \mathcal{O}$ if it is the orientation of a hyperedge U in G.*

Example 3.12. With $G = \big\{\{b,c\}, {\color{red}\{a,b,e\}}, {\color{blue}\{a,d,e,f\}}, {\color{green}\{b,c,e\}}, \{f,c\}\big\}$, we can orient the edge $U = \{a,b,e\}$ in $2^3 - 2 = 6$ different ways; three with a head of size 1: $(\{a\},\{b,e\})$, $(\{b\},\{a,e\})$, $(\{e\},\{a,b\})$, and three with a head of size 2: $(\{b,e\},\{a\})$, $(\{a,e\},\{b\})$, $(\{a,b\},\{e\})$. We represent this graphically as follows:



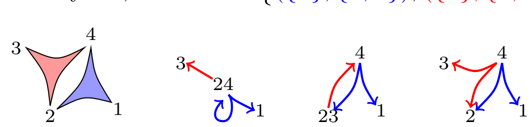
To orient G, we have to make a choice of orientation for each hyperedge. For example we can choose $\mathcal{O} = \big\{(\{b\},\{c\}), {\color{red}(\{a\},\{b,e\})}, {\color{blue}(\{a,e\},\{d,f\})}, {\color{green}(\{b,c\},,\{e\})}, (\{f\},\{c\})\big\}$ and we represent this as

$G/\mathcal{O} =$ 

In general, given a hypergraph G on the vertex set V and an orientation $\mathcal{O}$ of G, we construct an oriented (not necessarily simple) graph $G/\mathcal{O}$ as follows. We let $V/\mathcal{O}$ be the finest equivalence class of elements of V defined by the heads of $\mathcal{O}$. That is, the transitive closure of the relation $a \sim a'$ if $a, a' \in \mathfrak{a}$ for some head $\mathfrak{a}$ of $\mathcal{O}$. For each oriented hyperedge $(\mathfrak{a},\mathfrak{b})$ of $\mathcal{O}$, we have $|\mathfrak{b}|$ oriented edges $([\mathfrak{a}],[b])$ in $G/\mathcal{O}$ where $[\mathfrak{a}],[b] \in V/\mathcal{O}$ are equivalence classes and $b \in \mathfrak{b}$.

Definition 3.13 (Acyclic orientation). *An orientation $\mathcal{O}$ of G is* acyclic *if the oriented graph $G/\mathcal{O}$ has no cycles.*

Example 3.14. Let $G = \big\{{\color{blue}\{1,2,4\}}, {\color{red}\{2,3,4\}}\big\}$ be a hypergraph on $V = \{1,2,3,4\}$. As we can see the orientations $\mathcal{O} = \big\{{\color{blue}(\{4\},\{1,2\})}, {\color{red}(\{2,4\},\{3\})}\big\}$ and $\mathcal{O}' = \big\{{\color{blue}(\{4\},\{1,2\})}, {\color{red}(\{2,3\},\{4\})}\big\}$ are not acyclic, but $\mathcal{O}'' = \big\{{\color{blue}(\{4\},\{1,2\})}, {\color{red}(\{4\},\{2,3\})}\big\}$ is acyclic:



$G \qquad G/\mathcal{O} \qquad G/\mathcal{O}' \qquad G/\mathcal{O}''$

Out of the possible 36 orientations of G only 20 are acyclic:

$$\begin{array}{ll}
\{(\{4\},\{1,2\}),(\{4\},\{2,3\})\}; & \{(\{4\},\{1,2\}),(\{3\},\{2,4\})\};\\
\{(\{2\},\{1,4\}),(\{2\},\{3,4\})\}; & \{(\{2\},\{1,4\}),(\{2,3\},\{4\})\};\\
\{(\{1\},\{2,4\}),(\{2\},\{3,4\})\}; & \{(\{1\},\{2,4\}),(\{2,3\},\{4\})\};\\
\{(\{1,2\},\{4\}),(\{3\},\{2,4\})\}; & \{(\{1,2\},\{4\}),(\{2\},\{3,4\})\};\\
\{(\{1,4\},\{2\}),(\{3\},\{2,4\})\}; & \{(\{1,4\},\{2\}),(\{3,4\},\{2\})\};\\
\\
\{(\{4\},\{1,2\}),(\{3,4\},\{2\})\}; & \{(\{2\},\{1,4\}),(\{3\},\{2,4\})\};\\
\{(\{1\},\{2,4\}),(\{4\},\{2,3\})\}; & \{(\{1\},\{2,4\}),(\{3\},\{2,4\})\};\\
\{(\{1\},\{2,4\}),(\{2,4\},\{3\})\}; & \{(\{1\},\{2,4\}),(\{3,4\},\{2\})\};\\
\{(\{1,2\},\{4\}),(\{2,3\},\{4\})\}; & \{(\{1,4\},\{2\}),(\{4\},\{2,3\})\};\\
\{(\{2,4\},\{1\}),(\{3\},\{2,4\})\}; & \{(\{2,4\},\{1\}),(\{2,4\},\{3\})\}.
\end{array}$$

Our next lemma will show that for every set composition $A \in \mathcal{C}_x^y$ there is a unique acyclic orientation of G_x^y. Conversely for any acyclic orientation there is a least one $A = (A_1, A_2, \ldots, A_\ell) \in \mathcal{C}_x^y$ that gives that orientation. Denote by $\mathfrak{O}_x^y$ the set of acyclic orientations of G^y_x, and consider the following surjective map $\Omega\colon \mathcal{C}^y_x \to \mathfrak{O}^y_x$. For any $1 \leqslant i \leqslant \ell$, let $A_{i,\ell} = A_i \cup A_{i+1} \cup \cdots \cup A_\ell$ and let $G/\mathcal{O}_{i,\ell}$ be the restriction of $G/\mathcal{O}$ to the set $A_{i,\ell}$.

Lemma 3.15. *Let $x, y \in \mathbf{h}[I]$ and let G^y_x be the hypergraph on $V = [m]$ as defined before. The map $\Omega\colon \mathcal{C}^y_x \to \mathfrak{O}^y_x$ defined as follows is surjective:*

(a) *For any $A = (A_1, A_2, \dots, A_\ell) \in \mathcal{C}^y_x$ let $\Omega(A)$ be the unique element in $\mathfrak{O}^y_x$ such that for any $U \in G^y_x$ the orientation of U is given by $(U \cap A_i, U \smallsetminus A_i)$ where $i = \min\{j : A_j \cap U \neq \varnothing\}$.*

(b) *For any $\mathcal{O} \in \mathfrak{O}^y_x$, there is a unique $A_{\mathcal{O}} = (A_1, A_2, \dots, A_\ell) \in \mathcal{C}^y_x$ such that $\{A_1, A_2, \dots, A_\ell\} = V/\mathcal{O}$ and A_i is the unique source of the restriction $G/\mathcal{O}_{i,\ell}$ such that $\min(A_i)$ is maximal among the sources of $G/\mathcal{O}_{i,\ell}$. Thus, $\Omega(A_{\mathcal{O}}) = \mathcal{O}$.*

Moreover, $V/\Omega(A)$ is a refinement of $\{A_1, A_2, \dots, A_\ell\}$.

Proof. For part (a), let $A = (A_1, A_2, \dots, A_\ell) \in \mathcal{C}^y_x$. From Theorem 3.7, we have that A must break every hyperedge of G^y_x. In particular, for any part A_i of A and $U \in G^y_x$, we always have $A_i \cap U \neq U$. Hence $(U \cap A_i, U \smallsetminus A_i)$ for $i = \min\{j : A_j \cap U \neq \varnothing\}$ defines a proper orientation of each edge of G^y_x. Thus we obtain an orientation $\mathcal{O}$ of G^y_x. By construction, each head $\mathfrak{a}$ of $\mathcal{O}$ is completely included within a part A_i for a unique part $1 \leqslant i \leqslant \ell$. This implies that $V/\mathcal{O}$ refines $\{A_1, \dots, A_\ell\}$ and it allows us to define a function $f\colon V/\mathcal{O} \to \{1, 2, \dots, \ell\}$ where $f([a]) = i$ if and only if $[a] \subseteq A_i$. By the way $\mathcal{O}$ is constructed we have that for any $([a], [b]) \in G^y_x/\mathcal{O}$ the function f is such that $f([a]) < f([b])$. This implies that $G^y_x/\mathcal{O}$ has no cycles. Hence $\mathcal{O}$ is acyclic.

For part (b), let $\mathcal{O}$ be an acyclic orientation on G^y_x. It is clear that the set composition $A_{\mathcal{O}}$ is well defined for $G/\mathcal{O}$. We need to show that part (a) applied to $A_{\mathcal{O}}$ gives back $\mathcal{O}$. We have that $\{A_1, \dots, A_\ell\} = V/\mathcal{O}$. Hence for any $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{O}$ we must have $\mathfrak{a} \subseteq A_i$ for some unique $1 \leqslant i \leqslant \ell$. We claim that

$$A_j \cap \mathfrak{b} \neq \varnothing \qquad \Longrightarrow \qquad j > i\,.$$

If this were not the case, then there would be $j < i$ such that $A_j \cap \mathfrak{b} \neq \varnothing$. This means there is an edge from A_i to A_j in $G/\mathcal{O}_{j,\ell}$, which contradicts the fact that A_j is a source of $G/\mathcal{O}_{j,\ell}$; hence j must be such that $j > i$. ☐

Theorem 3.16. *For any $x, y \in \mathbf{h}[I]$ such that $\mathcal{C}^y_x \neq \varnothing$ we have*

$$c^y_x = \sum_{\mathcal{O} \in \mathfrak{O}^y_x} (-1)^{\ell(A_{\mathcal{O}})}\,.$$

Proof. Our proof will be similar to the one appearing in [12]. First we use the surjective map Ω from Lemma 3.15 to decompose the formula (34)

$$c^y_x = \sum_{B \in \mathcal{C}^y_x} (-1)^{\ell(B)} = \sum_{\mathcal{O} \in \mathfrak{O}^y_x} \left(\sum_{\substack{B \in \mathcal{C}^y_x \\ \Omega(B) = \mathcal{O}}} (-1)^{\ell(B)} \right).$$

For any fixed orientation $\mathcal{O}$, we thus have to show

$$\sum_{\substack{B \in \mathcal{C}^y_x \\ \Omega(B) = \mathcal{O}}} (-1)^{\ell(B)} = (-1)^{\ell(A_{\mathcal{O}})}\,.$$

Let $\mathcal{C}^y_{x,\mathcal{O}} = \{B \in \mathcal{C}^y_x : \Omega(B) = \mathcal{O}\}$. As in [10, 12], we construct a sign reversing involution $\varphi\colon \mathcal{C}^y_{x,\mathcal{O}} \to \mathcal{C}^y_{x,\mathcal{O}}$ such that

(A) $\varphi(A_{\mathcal{O}}) = A_{\mathcal{O}}$ is the only fixed point,

(B) for $B \neq A_{\mathcal{O}}$, we have $\ell(\varphi(B)) = \ell(B) \pm 1$.

If $B \neq A_{\mathcal{O}}$, then we have from Lemma 3.15 that each part of $A_{\mathcal{O}}$ is included in a part of A. Let $A_{\mathcal{O}} = (A_1, A_2, \dots, A_\ell)$ and $B = (B_1, B_2, \dots, B_k)$. We define

$$f_B \colon \{A_1, A_2, \dots, A_\ell\} \to \{B_1, B_2, \dots, B_k\}$$

as the function such that $A_i \subseteq f(A_i)$. Since $B \neq A_{\mathcal{O}}$, we have that $f_B \neq Id$. Find the smallest i such that $f_B^{-1}(B_i) \neq \{A_i\}$. Let $G/\mathcal{O}_{i,\ell}$ be the restriction of $G/\mathcal{O}$ to the set $A_{i,\ell}$. All the elements in $f_B^{-1}(B_i)$ are sources in the graph $G/\mathcal{O}_{i,\ell}$. By Lemma 3.15(b), we have that $\min(A_i)$ is the largest among the sources of $G/\mathcal{O}_{i,\ell}$. Since $f_B^{-1}(B_i) \neq \{A_i\}$, there must be a source $A_r \in f_B^{-1}(i)$ such that $\min(A_r) < \min(A_i)$. Let $X \in f_B^{-1}(B_i)$ be such that $\min(X) < \min(A_r)$ for all $A_r \in f_B^{-1}(B_i)$. We then find the smallest $j \geqslant i$ such that $A_r \in f_B^{-1}(B_j)$ is a source of $G/\mathcal{O}_{i,\ell}$ and $\min(A_r) > \min(X)$. Such j exists since $G/\mathcal{O}_{i,\ell}$ contain at least one source, namely A_i, such that $\min(A_i) > \min(X)$. We let

$$U = \left\{ \begin{array}{l} Z \in f_B^{-1}(B_j) : \exists Y \text{ a source of } G/\mathcal{O}_{i,\ell}, \text{ a path from } Y \text{ to } X \\ \hfill \text{and } \min(Y) \leqslant \min(X) \end{array} \right\}.$$

If $U = \varnothing$, then $j > i$ since $X \notin U$. In this case we remark that our choice of j implies that for all $A_r \in f_B^{-1}(B_j)$, the element A_r is connected to a source Y where $\min(Y) \leqslant \min(X)$. Hence, there is no edge (Y, X) in $G/\mathcal{O}_{i,\ell}$ where $Y \in f_B^{-1}(B_{j-1})$ and $X \in f_B^{-1}(B_j)$. If $U = \varnothing$, then we define

$$\varphi(B) = (B_1, \dots, B_{j-2}, B_{j-1} \cup B_j, B_{j+1}, \dots, B_k). \tag{37}$$

Let $B' = \varphi(B)$. It is clear that $\varphi(B) = B' \in \mathcal{C}^y_{x,\mathcal{O}}$ with $\ell(B') = \ell(B) - 1$. Moreover

$$f_{B'}^{-1}(B'_r) = \begin{cases} f_B^{-1}(B_r) & \text{if } r < j-1, \\ f_B^{-1}(B_{j-1}) \cup f_B^{-1}(B_j) & \text{if } r = j-1, \\ f_B^{-1}(B_{r+1}) & \text{if } r > j-1. \end{cases}$$

Repeating the procedure above for B' we will obtain i', X', j', U' in such a way that $i' = i$, $X' = X$, $j' = j-1$ and $U' = f_B^{-1}(B_{j-1}) \neq \varnothing$. Now we consider the case when $U \neq \varnothing$. Reversing what we did, let $U^c = f_B^{-1}(B_j) \smallsetminus U$. All the $Z \in U$ are connected to a source Y of $G/\mathcal{O}_{i,\ell}$ with value $\min(Y) \leqslant \min(X)$. Since there is no edge e of $G/\mathcal{O}_{i,\ell}$ such that e is incident to a vertex in U and a vertex in U^c, then

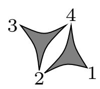
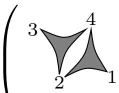
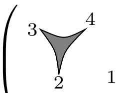
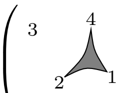
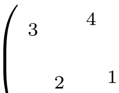
$$\varphi(B) = B' = (B_1, \dots, B_{j-1}, \bigcup_{Z \in U} Z, \bigcup_{Z' \in U^c} Z', B_{j+1}, \dots, B_k). \tag{38}$$

Remark that now $\ell(B') = \ell(B) + 1$. Moreover

$$f_{B'}^{-1}(B_r) = \begin{cases} f_B^{-1}(B_r) & \text{if } r < j-1, \\ U & \text{if } r = j, \\ U^c & \text{if } r = j+1, \\ f_B^{-1}(B_{r-1}) & \text{if } r > j+1 \end{cases}$$

and for this B' we will obtain i', X', j', U' in such a way that $i' = i$, $X' = X$, $j' = j+1$ and $U' = \varnothing$. The map φ is thus the desired involution. $\square$

Example 3.17. Let us revisit Example 3.10 in the Hopf monoid of hypergraphs. Let $x = \big\{\{1,2,4\},\{2,3,4\}\big\}$ be a hypergraph on the vertices $\{1,2,3,4\}$. A full computation of the antipode gives us

$S\Bigg($  $\Bigg) = -\Bigg($  $\Bigg) + 2\Bigg($  $\Bigg) + 2\Bigg($  $\Bigg) - 2\Bigg($  $\Bigg).$

The coefficient -2 in front of the empty hypergraph ϵ was computed in Example 3.10 using 4! permutations. Here we do so by means of Theorem 3.1 and the 20 acyclic orientations of Example 3.14. Lemma 3.15 (b) tells us that each of those orientations is paired with one of the following 20 set compositions (respectively)

$$\begin{array}{llll}
(4,3,2,1); & (3,4,2,1); & (34,2,1); & (3,2,4,1); \\
(2,4,3,1); & (23,4,1); & (1,4,3,2); & (3,1,4,2); \\
(1,2,4,3); & (1,23,4); & (1,24,3); & (1,34,2); \\
(3,12,4); & (12,4,3); & (123,4); & (14,3,2); \\
(3,14,2); & (134,2); & (3,24,1); & (24,3,1).
\end{array}$$

There are 9 even length set compositions in this list and 11 odd length. The coefficient is indeed $9-11=-2$. For the coefficient of x in $S(x)$, we remark that x/x is a single point with no edges. There is a unique orientation of x/x and it is represented by a set composition with a single part. Thus the coefficient is -1. For $y = \big\{\{1,2,4\}\big\}$, x/y is a graph on two vertices, say u and v, with a single edge between them and thus it has two acyclic orientations, which correspond to the set compositions (u,v), (v,u). Hence the coefficient is 2. The same argument applies for $y' = \big\{\{2,3,4\}\big\}$.

4. Some applications with Hopf algebras

In this section, we will consider some examples of antipodes corresponding to some combinatorial Hopf algebras. We recover results from [1, 7, 9, 10, 16], and derive some new formulas.

4.1. Antipode in the commutative case $H = \overline{\mathcal{K}}(\mathbf{H})$.

We now consider some commutative and cocommutative Hopf monoid $\mathbf{H}$ and look at the antipode of $H = \overline{\mathcal{K}}(\mathbf{H})$.

Example 4.1. Consider the Hopf monoid $\boldsymbol{\pi}$ in Section 1.3 and the basis $\boldsymbol{\pi}$. Given a set partition $X \in \boldsymbol{\pi}[I]$ and any set composition $A \models I$ we have that $X_A = X$ if a permutation of X refines A, and $X_A = 0$ otherwise. This means that the only term in $S(X)$ is X and its coefficient is c_X^X. A minimal Λ in $\mathcal{C}_X^X$ is X with some ordering of its parts. The hypergraph G_X^X has $m = |X|$ vertices and no hyperedges. If we use Theorem 3.16, there is a unique orientation of G_X^X and its sign is $(-1)^m$.

If instead we use Proposition 3.9, we sum over the permutations τ of m where $G_{12\cdots m,\epsilon}^{\tau,\epsilon}$ has only short edges $(i,i+1)$ for each descent $\tau(i) > \tau(i+1)$ of τ. This graph is decomposible unless $\tau = (m, m-1, \dots, 2, 1)$ for which $c(G_{12\cdots m,\epsilon}^{\tau,\epsilon}) = (-1)^m$.

The Hopf algebra $\overline{\mathcal{K}}(\boldsymbol{\pi})$ is the space of symmetric functions Sym and the basis element $\overline{\mathcal{K}}(X) = p_{type(X)}$ is the power sum basis where $type(X) = (|X_1|, |X_2|, \dots, |X_m|)$ written in non-increasing order. This gives the well known antipode formula $S(p_\lambda) = (-1)^{\ell(\lambda)} p_\lambda$.

Example 4.2. Consider the Hopf monoid $\mathbf{G}$ from Section 1.4 with basis $\mathbf{g}$. Given a graph $x \in \mathbf{g}[I]$ and $A \models I$ we have that $x_A = y$ is a subgraph of x and in fact, subgraphs y obtained in this way are also known as flats of x. A minimal element Λ in $\mathcal{C}_x^y$ is given by any ordering of the equivalence relation I/y where $a, b \in I$ are equivalent whenever there is a path in y connecting them. The hypergraph G_x^y is the simple graph x/y, obtained by contracting x along the edges of y. It is a graph on the vertex set $V = I/y$ and edges $\{[a],[b]\}$ whenever $[a] \neq [b]$ and there is an edge $\{a',b'\}$ in x such that $a' \in [a]$ and $b' \in [b]$. Since G_x^y has no hyperedges U such that $|U| > 2$, all orientations $\mathcal{O}$ of G_x^y are such that $V/\mathcal{O} = V$, since the head of each edge has cardinality 1. Hence in Theorem 3.16 we have that $\ell(A_{\mathcal{O}}) = |V| = |I/y|$ for each $\mathcal{O}$. No cancellation occurs and we recover the formula in [10, 16].

Example 4.3. We can extend the previous example to a Hopf monoid **SC** of abstract simplicial complexes. A simplicial complex on a set I is a collection $x \in 2^I$ such that

$$V \in x \implies U \in x, \quad \forall U \subseteq V,\ |U| > 1.$$

In this way, simplicial complexes extend the notion of graphs and it is a subfamily of hypergraphs. Now let $\mathbf{sc}[I]$ be the linear span of all simplicial complexes on I. The product and coproduct of **HG**, as defined in 1.5, restricts well to **SC**. Hence, **SC** is a monoid of abstract simplicial complexes with basis **sc**.

Given $x \in \mathbf{sc}[I]$ and any set composition $A \models I$ we have that $x_A = y$ is a simplicial subcomplex of x. A minimal element Λ in $\mathcal{C}_x^y$ is given by any ordering of the equivalence relation I/y where $a, b \in I$ are equivalent whenever there is a path in y connecting them. The hypergraph G_x^y is the simple graph given by the 1-skeleton of x/y, where x/y is obtained by contracting x along the all hyperedeges of y. It is a graph on the vertex set $V = I/y$ and edges $\{[a],[b]\}$ whenever $[a] \neq [b]$ and there is an edge $\{a',b'\}$ in x such that $a' \in [a]$ and $b' \in [b]$. Since G_x^y is a simple graph, all orientations $\mathcal{O}$ of G_x^y are such that $V/\mathcal{O} = V$. Hence Theorem 3.16 gives $\ell(A_{\mathcal{O}}) = |V| = |I/y|$ for each $\mathcal{O}$. No cancellation occurs and we recover the formula of [9].

Remark 4.4. As seen in Examples 4.2 and 4.3. the antipode formula in the Hopf monoid **SC** is a lifting of the antipode of **G**. Thus, it is natural to ask if such a lifting can be done to find an antipode formula in **HG**. This case, however, is more intricate as lots of cancellation occur in Theorem 3.16. Many of these cancellations are resolved in [9].

Example 4.5. (suggested to us by J. Machacek) Given a hypergraph G, we say that $a_0 \xrightarrow{U_1} a_1 \xrightarrow{U_2} \cdots \xrightarrow{U_\ell} a_\ell$ is a *path* of G if $a_{i-1} \neq a_i$ and $\{a_{i-1}, a_i\} \subset U_i \in G$ for each $1 \leqslant i \leqslant \ell$. We say that a path is *proper* if all the hyperedges U_i are distinct. A proper cycle in G is a proper path such that $a_0 = a_\ell$. A hypergraph is a *hyperforest* if it does not contain proper cycles. Let us consider the family of *hyperforests*. Let $\mathbf{hf}[I]$ be the set of hyperforest on I. It is not hard to check that the operations of **HG** restrict properly in the subset of hyperforests. Hence we have **HF** the hopf submonoid of hyperforests of **HG** with basis **hf**.

Given a $x \in \mathbf{h}f[I]$ and any set composition $A \models I$ we have that $f_A = h$ is a subforest of f. A minimal element Λ in $\mathcal{C}_f^h$ is given by any ordering of the equivalence relation I/h where $a, b \in I$ are equivalent whenever there is a path in h connecting them. The hypergraph G_f^h is the hyperforest given by f/h, the contraction of f along all the hyperedeges of h. Any two vertices of G_f^h that are connected, will be so via a unique proper path. Since it is a hyperforest, any orientation of G_f^h is acyclic and will contribute to the computation of the coefficient in Theorem 3.16.

Proposition 4.6. *Let $k = |G_f^h|$ and ℓ be the number of connected components of G_f^h. Then*

$$c_f^h = \begin{cases} (-1)^\ell (-2)^k & \text{if } \forall U \in G_f^h \text{ we have } |U| \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We give a proof based on a sign reversing involution on acyclic orientation of G^h_f. As in Theorem 3.16 let $\mathfrak{O}^h_f$ denote the set of acyclic orientation of G^h_f. As we remarked above, for a hyperforest, these are all the orientations of G^h_f.

Recall from Section 3.4 that G^h_f is thought of as a hypergraph (here a hyperforest) on $V = [m]$. We now define a sign reversing involution $\Phi\colon \mathfrak{O}^h_f \to \mathfrak{O}^h_f$. Given an orientation $\mathcal{O}$ of G^h_f, if possible, find the largest element $z \in V$ such that for some $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{O}$ we have $z = \max(\mathfrak{a} \cup \mathfrak{b})$ and

$$(z \in \mathfrak{a}, \quad |\mathfrak{a}| > 1) \quad \text{or} \quad (z \in \mathfrak{b}, \quad |\mathfrak{b}| > 1). \tag{39}$$

Then choose $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{O}$ such that $\mathfrak{a} \cup \mathfrak{b}$ is lexicographically maximal among the hyperedges that satisfy (39). We then define $\Phi(\mathcal{O}) = \mathcal{O}'$ where $\mathcal{O}'$ is obtained from $\mathcal{O}$ after replacing $(\mathfrak{a}, \mathfrak{b})$ by $(\mathfrak{a} \smallsetminus \{z\}, \mathfrak{b} \cup \{z\})$ if $z \in \mathfrak{a}$, or $(\mathfrak{a} \cup \{z\}, \mathfrak{b} \smallsetminus \{z\})$ otherwise. It is clear that Φ is an involution that toggles the maximal element of the orientation of a hyperedge between the two situations in (39). If no such z exists, then define $\Phi(\mathcal{O}) = \mathcal{O}$.

We now show that Φ reverses sign, except in its fixed points. First recall from Lemma 3.15 that $\ell(A_{\mathcal{O}}) = |V/\mathcal{O}|$. Assume $\Phi(\mathcal{O}) = \mathcal{O}' \neq \mathcal{O}$ and let z and $(\mathfrak{a}, \mathfrak{b})$ be as above. In the situation where $z \in \mathfrak{a}$, we now have $(\mathfrak{a} \smallsetminus \{z\}, \mathfrak{b} \cup \{z\}) \in \mathcal{O}'$ and the rest of the orientations are the same as in $\mathcal{O}$. Since there exists a unique proper path between any two equivalent vertices in the equivalent classes $[\mathfrak{a}]_{\mathcal{O}}$ containing z in $V/\mathcal{O}$, this class will break in exactly two classes $[\mathfrak{a} \smallsetminus \{z\}]_{\mathcal{O}'}$ and $[\{z\}]_{\mathcal{O}'}$ in $V/\mathcal{O}'$. All the other classes of $V/\mathcal{O}$ and $V/\mathcal{O}'$ remain the same. Hence $(-1)^{\ell(A_{\mathcal{O}})} = -(-1)^{\ell(A_{\mathcal{O}'})}$ and Φ is sign reversing in this case. The argument in the other case is similar.

The involution Φ reduces the identity in Theorem 3.16 to

$$c^h_f = \sum_{\substack{\mathcal{O} \in \mathfrak{O}^{\mathfrak{h}}_{\mathfrak{f}} \\ \Phi(\mathcal{O}) = \mathcal{O}}} (-1)^{\ell(A_{\mathcal{O}})}.$$

To finish the proof, we need to describe the fixed points of Φ. If there is no z satisfying equation (39), then for all $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{O}$ and $z = \max(\mathfrak{a} \cup \mathfrak{b})$ we have

$$\mathfrak{a} = \{z\} \quad \text{or} \quad \mathfrak{b} = \{z\}, \tag{40}$$

and the orientations $\mathcal{O}$ that satisfy (40) are the fixed points of Φ. If $|G^h_f| = 0$, so that G^h_f has $m = \ell$ vertices and no hyperedges, then $c^h_f = (-1)^\ell$ as desired. If $|G^h_f| > 0$, then let $U \in G^h_f$ be any fixed hyperedge. For instance, pick U to be lexicographically maximal in G^h_f and let $z = \max(U)$. In any orientation of G^h_f fixed by Φ we can toggle the orientation of U between the two situations in (40) and still get a fixed point of Φ. That is, we can pair all the fixed point of Φ as $(\mathcal{O}, \mathcal{O}')$ where $\mathcal{O} \neq \mathcal{O}'$ and they differ only by the orientation of $U = \mathfrak{c} \cup \{z\}$ with $(\mathfrak{c}, \{z\}) \in \mathcal{O}$ and $(\{z\}, \mathfrak{c}) \in \mathcal{O}'$. Using again the fact that there is at most a unique proper path between any vertices in G^h_f, the elements of $\mathfrak{c}$ are in a single equivalence class in $V/\mathcal{O}$ and in distinct equivalent classes in $V/\mathcal{O}'$. Hence $|V/\mathcal{O}'| = |V/\mathcal{O}| + |U| - 2$. We now have

$$c^h_f = \sum_{\substack{\mathcal{O} \in \mathfrak{O}^{\mathfrak{h}}_{\mathfrak{f}} \\ \Phi(\mathcal{O}) = \mathcal{O}}} (-1)^{\ell(A_{\mathcal{O}})} = \sum_{(\mathcal{O}, \mathcal{O}')} (-1)^{\ell(A_{\mathcal{O}})}(1 + (-1)^{|U|}).$$

Let us denote by $G^{h \cup U}_f$ the hyperforest obtained by contracting the hyperedge U in G^h_f. There is a clear correspondence between the orientation $\mathcal{O}$ of G^h_f and the orientation $\mathcal{O}''$ of $G^{h \cup U}_f$ together with an orientation of U. This is true only for hyperforest as there is a unique proper path between any two vertices. We thus have

$$c_f^h = \sum_{(\mathcal{O},\mathcal{O}')} (-1)^{\ell(A_{\mathcal{O}})}(1+(-1)^{|U|}) = -(1+(-1)^{|U|}) \sum_{\substack{\mathcal{O}''\in\mathfrak{O}_{\mathfrak{f}}^{\mathfrak{h}\cup\mathfrak{U}} \\ \Phi(\mathcal{O}'')=\mathcal{O}''}} (-1)^{\ell(A_{\mathcal{O}''})}.$$

The negative sign in the second equality comes from the fact that contracting U joins together the classes $[z]_{\mathcal{O}}$ and $[\mathfrak{c}]_{\mathcal{O}}$. We now have that $c_f^h = -(1+(-1)^{|U|})c_f^{h\cup U}$. The proposition follows by induction. If $|U|$ is odd, then we get zero. If $|U|$ is even, then we get a contribution of -2 for that edge and the induction ends with an empty hypergraph with the same number of connected component as G_f^h. ☐

4.2. Antipode of $\overline{\mathcal{K}}(\mathbf{L}\times\mathbf{H}) \cong \mathcal{K}(\mathbf{H})$ for linearized $\mathbf{H}$. As we noticed in Section 1.8 we have that $\overline{\mathcal{K}}(\mathbf{L}\times\mathbf{H}) \cong \mathcal{K}(\mathbf{H})$ for any Hopf monoid $\mathbf{H}$. Given $(\alpha,x)\in(\mathbf{L}\times\mathbf{H})[n]_{S_n}$, the isomorphism is explicitly given by the map $(\alpha,x)\mapsto \mathbf{H}[\alpha^{-1}](x)$ where $\alpha^{-1}\colon [n]\to[n]$ is the unique bijection such that $\alpha^{-1}(\alpha) = 12\cdots n$ and $\mathbf{H}[\alpha^{-1}](x)\in\mathbf{H}[n]$ is the image of x under the bijection $\mathbf{H}[\alpha^{-1}]\colon \mathbf{H}[n]\to\mathbf{H}[n]$ obtained via the functor $\mathbf{H}$. Since $\overline{\mathcal{K}}$ preserves antipodes (see [4, Section 15.2]), in the case where $\mathbf{H}$ is linearized, Theorem 2.1 gives us the following formula. For $x\in\mathbf{H}[n]$

$$S(x) = \sum_{(\beta,y)\in(\mathbf{l}\times\mathbf{h})[n]} c_{12\cdots n,x}^{\beta,y}\mathbf{H}[\beta^{-1}](y) = \sum_{z\in\mathbf{h}[n]}\Big(\sum_{\beta\in\mathbf{l}[n]} c_{12\cdots n,x}^{\beta,\mathbf{H}[\beta](z)}\Big)z\,. \tag{41}$$

Here we have identified the linear order $\beta\in\mathbf{l}[n]$ and the bijection $\beta=(\beta^{-1})^{-1}\colon [n]\to[n]$ in the notation $\mathbf{H}[\beta](z)$. From Theorem 2.1 we have that the coefficients $c_{12\cdots n,x}^{\beta,\mathbf{H}[\beta](z)}$ are ±1, but further cancellation may occur in Equation (41). It is not a cancellation free formula in most cases but it is definitely improves the computation compared to Takeuchi's formula.

Example 4.7. Consider the Hopf monoid $\boldsymbol{\pi}$ from Section 1.3. As seen in [7], the Hopf algebra $\mathcal{K}(\boldsymbol{\pi})$ is the space of symmetric functions in non-commutative variables. Our formula (41) is cancellation free in this case as all the non-zero terms have the same sign (see Corollary 4.9 of [7] for more details).

Example 4.8. Consider now the Hopf monoid $\mathbf{L}$ in Section 1.2. The Hopf algebra $PR=\mathcal{K}(\mathbf{L})$ was introduced by Patras-Reutenauer [19] and is also studied under the name $R\Pi$ in [4]. The antipode formula (41) for PR gives us that for $\alpha\in\mathbf{l}[n]$:

$$S(\alpha) = \sum_{\gamma\in\mathbf{l}[n]}\Big(\sum_{\beta\in\mathbf{l}[n]} c_{\epsilon,\alpha}^{\beta,\beta\circ\gamma}\Big)\gamma \tag{42}$$

where $\epsilon = 12\ldots n$ is the identity permutation. In this example, $\beta=(\beta_1,\ldots,\beta_n)\in\mathbf{l}[n]$ can be encoding three different objects depending on the context. It is first the total order $\beta_1<\beta_2<\cdots<\beta_n$ on the points $1,2,\ldots,n$. In (42), when we write $\beta\circ\gamma$, we consider β as the permutation defined by $\beta(i)=\beta_i$. Hence $\beta\circ\gamma = (\beta(\gamma_1),\beta(\gamma_2),\ldots,\beta(\gamma_n))$. Bellow we will consider β and $\beta\circ\gamma$ as encoding the set composition $(\{\beta_1\},\ldots,\{\beta_n\})$ and $(\{\beta(\gamma_1)\},\ldots,\{\beta(\gamma_n)\})$. These conventions should be clear from the context. We now need a complete description of $c_{\epsilon,\alpha}^{\beta,\beta\circ\gamma}$ in order to understand (42). The set $\mathcal{C}_{\epsilon,\alpha}^{\beta,\beta\circ\gamma}\neq\varnothing$ if and only if the minimal element Λ of $\mathcal{C}_{\epsilon,\alpha}^{\beta,\beta\circ\gamma}$ exists and it is the finest set composition such that

(1) $\beta\leqslant\Lambda$ and β is increasing with respect to ϵ within each part of Λ,
(2) $\beta\circ\gamma\leqslant\Lambda$ and $\beta\circ\gamma$ is increasing with respect to α within each part of Λ.

These conditions follow from the proof of Lemma 2.2 in the case of $\mathbf{L}\times\mathbf{L}$. Let $A=\beta\vee(\beta\circ\gamma)$ be the finest set composition such that $\beta\leqslant A$ and $\beta\circ\gamma\leqslant A$. We must have that $A\leqslant\Lambda$. Now, if β is not increasing with respect to ϵ within each part of

A, then it would not be increasing with respect to ϵ within each part of Λ either. Similarly if $\beta \circ \gamma$ is not increasing with respect to α within each part of A, then Λ would not be increasing with respect to ϵ within each part of A either. Hence we have that $\mathcal{C}_{\epsilon,\alpha}^{\beta,\beta\circ\gamma} \neq \varnothing$ if and only if $\Lambda = \beta \vee (\beta \circ \gamma)$ is such that β is increasing with respect to ϵ within each part of Λ and $\beta \circ \gamma$ is increasing with respect to α within each part of Λ.

For instance, if $\alpha = (5,2,1,3,4)$, $\beta = (2,1,3,5,4)$ and $\beta \circ \gamma = (2,5,1,3,4)$, then we see that $\Lambda = \beta \vee (\beta \circ \gamma) = (2, 135, 4)$. The elements $1,3,5$ of β are increasing with respect to $\epsilon = (1,2,3,4,5)$ within the part 135 of Λ. In $\beta \circ \gamma$ these elements are in the order $5,1,3$ which is increasing with respect to α. Hence in this little example $\mathcal{C}_{\epsilon,\alpha}^{\beta,\beta\circ\gamma} \neq \varnothing$ and $\Lambda = (2,135,4)$ is the minimum. If we take a different γ' so that $\beta \circ \gamma' = (2,5,3,1,4)$, then $\Lambda = \beta \vee (\beta \circ \gamma') = (2,135,4)$ but the element $5,3,1$ are not in increasing order with respect to α, hence $\mathcal{C}_{\epsilon,\alpha}^{\beta,\beta\circ\gamma'} = \varnothing$.

Now we remark that the number of parts of $\Lambda = \beta \vee (\beta \circ \gamma)$ depends only on γ and not β. This follows from the simple fact that

$$\beta \vee (\beta \circ \gamma) = \beta \circ \big(\epsilon \vee \gamma\big).$$

Hence $\ell\big(\beta \vee (\beta\circ\gamma)\big) = \ell\big(\epsilon \vee \gamma\big) = m$ and if $\mathcal{C}_{\epsilon,\alpha}^{\beta,\beta\circ\gamma} \neq \varnothing$, then the graph $G_{\epsilon,\alpha}^{\beta,\beta\circ\gamma}$ is a graph on the vertex set $[m]$ with an edge $(i,i+1)$ if and only if $\max_\epsilon(\Lambda_i) >_\epsilon \min_\epsilon(\Lambda_{i+1})$ or $\max_\alpha(\Lambda_i) >_\alpha \min_\alpha(\Lambda_{i+1})$, using the order ϵ and α respectively. We remark that $G_{\epsilon,\alpha}^{\beta,\beta\circ\gamma}$ contains only short edges. Hence $c_{\epsilon,\alpha}^{\beta,\beta\circ\gamma} = (-1)^m$ if $(i,i+1) \in G_{\epsilon,\alpha}^{\beta,\beta\circ\gamma}$ for all $1 \leqslant i < m$; otherwise the graph is decomposible and $c_{\epsilon,\alpha}^{\beta,\beta\circ\gamma} = 0$. We summarize our discussion in the following theorem.

Theorem 4.9. *Given $\alpha \in \mathbf{l}[n]$, in the Hopf algebra PR we have*

$$S(\alpha) = \sum_{\gamma \in \mathbf{l}[n]} (-1)^m d_{\alpha,\gamma}\, \gamma$$

where $m = \ell\big(\epsilon \vee \gamma\big)$ and $d_{\alpha,\gamma}$ is the number of $\beta \in \mathbf{l}[n]$ such that for $\Lambda = \beta \vee (\beta \circ \gamma)$ we have

(i) *β is increasing with respect to ϵ within each part of Λ,*
(ii) *$\beta \circ \gamma$ is increasing with respect to α within each part of Λ, and*
(iii) *$\max_\epsilon(\Lambda_i) >_\epsilon \min_\epsilon(\Lambda_{i+1})$ or $\max_\alpha(\Lambda_i) >_\alpha \min_\alpha(\Lambda_{i+1})$ for all $1 \leqslant i < m$.*

To our knowledge this theorem is new and provides a cancellation free formula in PR.

Example 4.10. For the monoid $\mathbf{G}$ with basis $\mathbf{g}$ in Section 1.4 the formula (41) is not cancellation free. However we can find another basis $\overline{\mathbf{g}}$ that linearizes $\mathbf{G}$ such that the formula (41) is cancellation free. More specifically for a connected graph $x \in \mathbf{g}[I]$ let

$$\overline{x} = \sum_{\Phi \in \boldsymbol{\pi}[I]} (-1)^{|\Phi|-1}\big(|\Phi|-1\big)!\; x_\Phi,$$

where $\boldsymbol{\pi}[I]$ is the set of set partitions of I and for $\Phi = \{A_1, A_2, \dots, A_\ell\}$ we define $x_\Phi = x|_{A_1} x|_{A_2} \cdots x|_{A_\ell}$. The product x_Φ is well defined since $\mathbf{G}$ is commutative.

When x is connected, we have that

$$\overline{x} = x + (\textit{terms with more than 2 connected components}).$$

This is not true if x is not connected. We leave to the reader the exercise of showing that when x is connected, we have

$$\Delta_{A_1,A_2}(\overline{x}) = 0$$

for any non-trivial decomposition $(A_1, A_2) \models I$.[(2)] That is to say, $\overline{x}$ is primitive.

If $x \in \mathbf{g}[I]$ is not connected, then it decomposes uniquely into connected components $x = x_1 x_2 \dots x_m$ where x_i is a connected subgraph on the vertex set $I_i \subseteq I$. Here $\{I_1, \dots, I_m\}$ is a set partition of I. For such x, let us define $\overline{x} = \overline{x}_1 \overline{x}_2 \dots \overline{x}_m$. Now, we obtain that

$$\overline{x} = x + (\textit{terms with more than } m+1 \textit{ connected components}).$$

Hence the set $\{\overline{x} : x \in \mathbf{l}[I]\}$ forms a basis of $\mathbf{G}[I]$. In this basis, the multiplication is the same as before but the comultiplication is now

$$\Delta_{A_1,A_2}(\overline{x}) = \begin{cases} \overline{x}|_{A_1} \otimes \overline{x}|_{A_2} & \text{if } A_1 \text{ is the union of some of the parts of } \{I_1, \dots, I_m\}, \\ 0 & \text{otherwise.} \end{cases}$$

With this in hand, we now have a different basis $\overline{\mathbf{g}}$ that linearizes $\mathbf{G}$ with a different comultiplication behavior. With a reasonable amount of work similar to Examples 4.8 and 4.7, the reader will find that formula (41) is also cancellation free in this case. The reader should also compare this to [18, Theorem 4.7].

4.3. Using Antipodes to derive new identities. As we have seen in the introduction, any multiplicative morphism $\zeta\colon H \to \Bbbk$ gives rise to a combinatorial invariant $\chi = \phi_t \circ \Psi$ on H. For $x \in H_n$, the polynomials $\chi_x(t)$ encode combinatorial information about x which depends on our choice of ζ. Also, the combinatorial reciprocity $\chi_x(-1) = (\zeta \circ S)(x)$ is easily verified.

Example 4.11. For $H = \overline{\mathcal{K}}(\mathbf{G})$ and for $x \in \mathbf{g}[n]$, let $\zeta(x) = 1$ if x is edgeless, zero otherwise. In this case $\chi_x(t)$ is the chromatic polynomial of the graph x. Stanley's (-1)-color theorem [20] follows as $\pm\chi_x(-1)$ is the number of acyclic orientation of x.

The following example suggests a new venue to explore combinatorial identities using permutations.

Example 4.12. Consider the Hopf algebra $PR = \mathcal{K}(\mathbf{L})$ as studied in example 4.8. Define $\zeta(x) = 1$ if $x = \epsilon$, and zero otherwise and extend linearly. We have that ζ is indeed multiplicative. Since PR is cocommutative, $\Psi\colon PR \to QSym$ will in fact be a symmetric function (see [2] for details). Here for $\alpha \in \mathbf{l}[n]$ we have

$$\Psi(\alpha) = \sum_{a \models n} c_a(\alpha) M_a,$$

where $a = (a_1, \dots, a_\ell) \models n$ is an integer composition of n, and $c_a(\alpha)$ is the number of ways to decompose α into increasing subsequences of type a. More precisely

$$c_a(\alpha) = \big|\{A \models [n] : \text{for } 1 \leqslant i \leqslant \ell,\ |A_i| = a_i \text{ and } \alpha|_{A_i} \text{ is increasing}\}\big|.$$

These numbers are studied in various place in mathematics and computer science. In particular Robinson-Schensted-Knuth(RSK) insertion shows that the coarsest possible a for which $c_a(\alpha) \neq 0$ is a permutation of the shapes obtain via RSK (see [21]).

The *chromatic polynomial* $\chi_\alpha(t)$ is then

$$\chi_\alpha(t) = \sum_{a \models n} c_a(\alpha) \binom{t}{\ell}.$$

(2) one needs to show and use the identity $\displaystyle\sum_{k=0}^{\min(n,m)} (-1)^{n+m-k-1}(n+m-k-1)!k!\binom{n}{k}\binom{m}{k} = 0.$

This polynomial, when evaluated at $t = m$, counts the number of ways to color the entries of the permutation α with at most m distinct colors such that α restricted to a single color is increasing. Using Theorem 4.9 we get the identity

$$\sum_{a\models n}(-1)^{\ell(a)}c_a(\alpha) = \chi_\alpha(-1) = \zeta\circ S(\alpha) = (-1)^n d_{\alpha,\epsilon}. \tag{43}$$

For any $\beta \in \mathbf{l}[n]$ and $\gamma = \epsilon$ in Theorem 4.9, we have $\Lambda = \beta$ and the conditions (i) and (ii) are automatically satisfied. Hence

$$d_{\alpha,\epsilon} = \big|\{\beta\in\mathbf{l}[n] : \beta_i > \beta_{i+1} \text{ or } \alpha^{-1}(\beta_i) > \alpha^{-1}(\beta_{i+1})\}\big|. \tag{44}$$

The identity in Equation (43) relates combinatorial invariants for permutation that looks a priori unrelated. We summarize this in the following theorem

Theorem 4.13. *For $\alpha \in \mathbf{l}[n]$, the chromatic polynomial $\chi_\alpha(t)$ counts the number of ways to color increasing sequences of α with at most t distinct colors. We have the identity*

$$\chi_\alpha(-1) = (-1)^n d_{\alpha,\epsilon},$$

where $d_{\alpha,\epsilon}$ is the number of α-decreasing orders (as defined in Equation (44)).

Remark 4.14. Given $\alpha \in \mathbf{l}[n]$, one can associate a partial order P_α where $\alpha_i \prec \alpha_j$ if $i < j$ and $\alpha_i > \alpha_j$. As in [20] we can construct the incomparable graph G_α associated to P_α. The symmetric function $\Psi(\alpha)$ above is in fact the Stanley chromatic symmetric function of the Graph G_α. A famous conjecture of Stanley and Stembridge [20] states that $\Psi(\alpha)$ is e-positive if P_α is $(\mathbf{3}+\mathbf{1})$-avoiding. In the language of permutations this is equivalent to say that α is 4123 and 2341 avoiding [5]. From the Hopf structure, one can see that it is natural to describe e positivity in terms of pattern-avoiding sequences. What is surprising here is the fact that there should be finitely many and very simple patterns to avoid. We also point out that here $d_{\alpha,\epsilon}$ also counts the number of acyclic orientations of G_α.

The Hopf algebra PR is free and generated by total orders that do not have any global ascent. The free generators are $\{1, 21, 321, 231, 312, \ldots\}$. In the example above, we choose ζ to be 1 on the generator 1 and zero for all other generators. One can construct different ζ's by choosing any subset of generators. This would lead to different coloring schemes and new identities with permutations.

Example 4.15. Let us consider the case where $\zeta_{21}\colon PR \to \Bbbk$ is defined to be 1 on the (free) generator 21 and zero on the other. That is, define $\zeta_{21}(x) = 1$ if $x = 2143\ldots(2n)(2n-1)$, zero otherwise. This defines a symmetric function valued morphism $\Psi_{21}\colon PR \to QSym$ such that for $\alpha \in \mathbf{l}[n]$ we have

$$\Psi_{21}(\alpha) = \sum_{a\models n} c'_a(\alpha) M_a,$$

where $a = (2a_1, \ldots, 2a_\ell) \models n$ is an integer composition of n with even parts, and $c_a(\alpha)$ is the number of ways to decompose α into 21*-subsequences of type a. More precisely

$$c'_a(\alpha) = \big|\{A \models [n] : \text{for } 1\leqslant i\leqslant \ell,\ |A_i| = 2a_i \text{ and } st(\alpha|_{A_i}) = 2143\ldots(2a_i)(2a_i-1)\}\big|,$$

where $st(-)$ denotes the standardization map via the functor PR as defined in [4, Notation 2.5]. It would be interesting to understand the properties of the numbers $c'_a(\alpha)$.

The *chromatic* polynomial $\chi^{21}_\alpha(t)$ is then given by

$$\chi^{21}_\alpha(t) = \sum_{a\models n} c'_a(\alpha)\binom{t}{\ell(a)}.$$

This polynomial, when evaluated at $t = m$ counts the number of ways to color the entries of α with at most m distinct colors such that α restricted to a single color is a 21^*-sequence. Using Theorem 4.9 we get the identity

$$\sum_{a \models n} (-1)^{\ell(a)} c'_a(\alpha) = (-1)^{n/2} d_{\alpha,2143\ldots(2n)(2n-1)}. \tag{45}$$

Conjecture 4.16. *There is a finite set of permutations $\mathcal{A}$ such that for $\alpha \in \mathbf{l}[n]$*

$$(-1)^{n/2}\Psi_{21}(\alpha)(-h_1, -h_2, \ldots)$$

is h-positive for any α that is $\mathcal{A}$-avoiding. So far, our computer evidence suggests that $\mathcal{A} = \varnothing$. We also conjecture that Stanley chromatic symmetric functions satisfy this property.

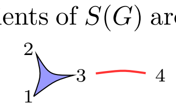
In a forthcoming paper [6], Aval, Bergeron and Machacek will present a proof of Conjecture 4.16.

To conclude this paper, we remark that the Hopf monoid of hypergraphs **HG** as defined in Section 1.5 plays a central role in the computation of antipodes for commutative and cocommutative Hopf monoid. This Hopf monoid is different in nature from the Hopf algebra of hypergraphs in [1] which is not cocommutative. In a forthcoming paper [8] we show that given a hypergraph G on the vertex set $V = [n]$, the coefficient of the edgeless hypergraph in the antipode $S(G)$ is the homology of the complex labelled by the acyclic orientations of G. This can be understood from facets of the *Hypergraphic polytope* P_G associated to G. That is the polytope in $\mathbb{R}^n = \mathbb{R}\{e_1, e_2, \ldots, e_n\}$ defined by the Minkowsk sum

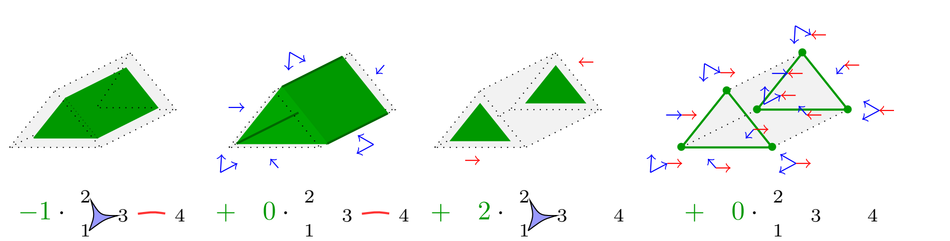
$$P_G = \sum_{U \in G} \mathbf{\Delta}_U,$$

where $\mathbf{\Delta}_U$ is the simplex given by the convex hull of the points $\{e_i : i \in U\}$.

The acyclic orientations of G actually label certain *exterior* faces of P_G. Hence the coefficient of the discrete hypergraph in $S(G)$ is the homology of the complex labelled by the acyclic orientations of G. The other coefficients of $S(G)$ are also encoded in P_G.

For example, consider the hypergraph $G =$ 

, the flats of G are $G, \{\{3,4\}\}, \{\{1,2,3\}\}$ and $\varnothing$. The coefficient of each flat F in $S(G)$ is, up to a sign, given by the Euler characteristic of the union of some faces of $P_G = \mathbf{\Delta}_{123} + \mathbf{\Delta}_{34}$ indexed by acyclic orientations of G/F:



Remark 4.17. It is interesting to see that hypergraphs play a crutial role in the study of square free ideals [17]. As an open question, how could one use the invariants studied in this paper for hypergraphs to derive properties of the associated ideal?

Acknowledgements. We are very grateful to the referees for their useful comments that helped us improving our exposition.

References

[1] Marcelo Aguiar and Federico Ardila, *Hopf monoid of generalized permutahedra*, `https://arxiv.org/abs/1709.07504`, 2017.

[2] Marcelo Aguiar, Nantel Bergeron, and Frank Sottile, *Combinatorial Hopf algebras and generalized Dehn–Sommerville relations*, Compositio Mathematica **142** (2006), 1–30.

[3] Marcelo Aguiar, Nantel Bergeron, and Nathaniel Thiem, *Hopf monoids from class functions on unitriangular matrices*, Algebra and Number Theory **7** (2013), no. 7, 1743–1779.

[4] Marcelo Aguiar and Swapneel Mahajan, *Monoidal functors, species and Hopf algebras*, CRM Monograph Series, vol. 29, American Mathematical Society, Providence, RI, 2010.

[5] M. D. Atkinson, Bruce E. Sagan, and Vincent Vatter, *Counting* $(\mathbf{3}+\mathbf{1})$*-avoiding permutations*, European Journal of Combinatorics **33** (2012), no. 1, 49–61.

[6] Jean-Christophe Aval, Nantel Bergeron, and John Machacek, *New invariants for permutations, orders and graphs*, 2018, draft.

[7] Duff Baker-Jarvis, Nantel Bergeron, and Nathaniel Thiem, *The antipode and primitive elements in the Hopf monoid of supercharacters*, Journal of Algebraic Combinatorics **40** (2014), no. 4, 903–938.

[8] Carolina Benedetti, Nantel Bergeron, and John Machacek, *Hypergraphic polytopes: combinatorial properties and antipode*, `https://arxiv.org/abs/1712.08848`, 2017.

[9] Carolina Benedetti, Joshua Hallam, and John Machacek, *Combinatorial Hopf Algebras of Simplicial Complexes*, SIAM Journal on Discrete Mathematics **30** (2016), no. 3, 1737–1757.

[10] Carolina Benedetti and Bruce E. Sagan, *Antipodes and involutions*, J. Combin. Theory Ser. A **148** (2017), 275–315.

[11] François Bergeron, Gilbert Labelle, and Pierre Leroux, *Combinatorial species and tree-like structures*, Encyclopedia of Mathematics and its Applications, vol. 67, Cambridge University Press, Cambridge, 1998, Translated from the 1994 French original by Margaret Readdy, With a foreword by Gian-Carlo Rota.

[12] Nantel Bergeron and Cesar Ceballos, *A Hopf algebra of subword complexes*, Advances in Mathematics **305** (2017), 1163–1201.

[13] Nantel Bergeron and Mike Zabrocki, *The Hopf algebras of symmetric functions and quasi-symmetric functions in non-commutative variables are free and co-free*, J. Algebra Appl. **8** (2009), no. 4, 581–600.

[14] Hector Figueroa and Jose M. Gracia-Bondía, *Combinatorial Hopf algebras in quantum field theory. I*, Rev. Math. Phys. **17** (2005), no. 8, 881–976.

[15] Darij Grinberg and Victor Reiner, *Hopf algebras in combinatorics*, preprint of notes: `https://arxiv.org/abs/1409.8356`, 2014.

[16] Brandon Humpert and Jeremy L. Martin, *The incidence Hopf algebra of graphs*, SIAM Journal on Discrete Mathematics **26** (2012), no. 2, 555–570.

[17] Kuei-Nuan Lin and Jason McCullough, *Hypergraphs and regularity of square-free monomial ideals*, Internat. J. Algebra Comput. **23** (2013), no. 7, 1573–1590.

[18] Eric Marberg, *Strong forms of linearization for Hopf monoids in species*, Journal of Algebraic Combinatorics **42** (2015), no. 2, 391–428.

[19] Frédéric Patras and Christophe Reutenauer, *On descent algebras and twisted bialgebras*, Mosc. Math. J. **4** (2004), no. 1, 199–216.

[20] Richard P. Stanley, *A Symmetric Function Generalization of the Chromatic Polynomial of a Graph*, Advances in Mathematics **111** (1995), no. 1, 166–194.

[21] ———, *Increasing and decreasing subsequences and their variants*, Proceedings of the International Congress of Mathematicians, Madrid, Spain, 2006-2007 European Mathematical Society (Providence, RI), Amer. Math. Soc., 2007.

Carolina Benedetti, Departamento de Matemáticas, Universidad de los Andes, Bogotá, Colombia
E-mail : `c.benedetti@uniandes.edu.co`
Url : `https://sites.google.com/site/carobenedettimath/`

Nantel Bergeron, Department of Mathematics and Statistics, York University, Toronto, Ontario M3J 1P3, Canada
E-mail : `bergeron@mathstat.yorku.ca`
Url : `http://www.math.yorku.ca/bergeron`